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# YEAR 12 *Trial Exam Paper*

# 2019

# MATHEMATICAL METHODS

## Written examination 2

## *Worked solutions*

**This book presents:**

- worked solutions
- mark allocations
- tips.

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# SECTION A – Multiple-choice questions

## Question 1

***Answer: D***

**Explanatory notes**

The period is $2\pi / \dfrac{2\pi}{5} = 5$.

The range of $3\cos\left(\dfrac{2\pi x}{5}\right)$ is $[-3, 3]$. Therefore the range of $3\cos\left(\dfrac{2\pi x}{5}\right) - 2$ is $[-5, 1]$.

## Question 2

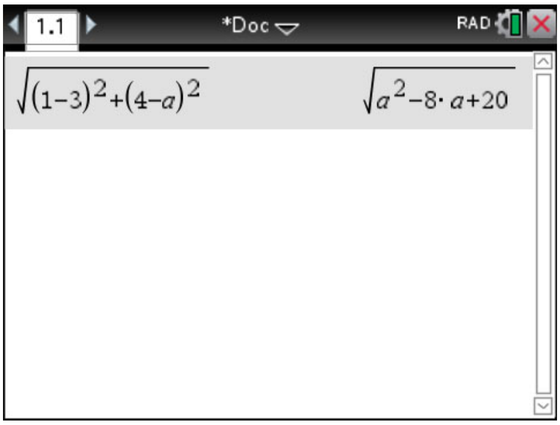
***Answer: B***

**Explanatory notes**

Use the formula for the distance between two points:

$$\sqrt{(1-3)^2 + (4-a)^2} = \sqrt{a^2 - 8a + 20}$$

Use CAS to do the algebra and avoid common algebraic errors.



**Question 3**

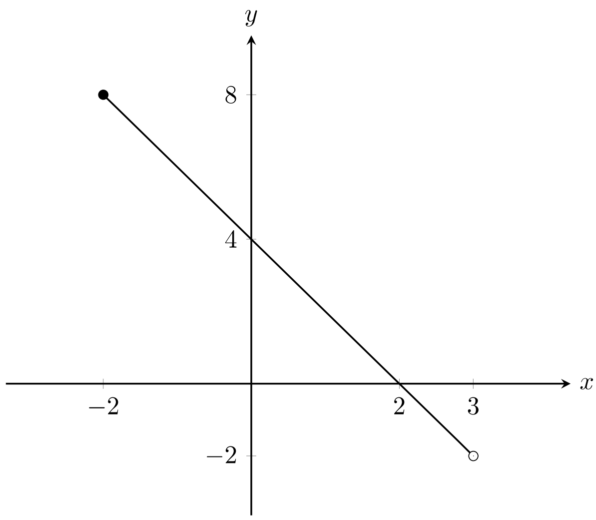
***Answer: A***

**Explanatory notes**

Since $f(-2)=8$ is included and $f(3)=-2$ is excluded, the range of $f$ is $(-2,8]$.

**Tip**

- *It may be helpful for you to draw a sketch of the function on the domain given:*

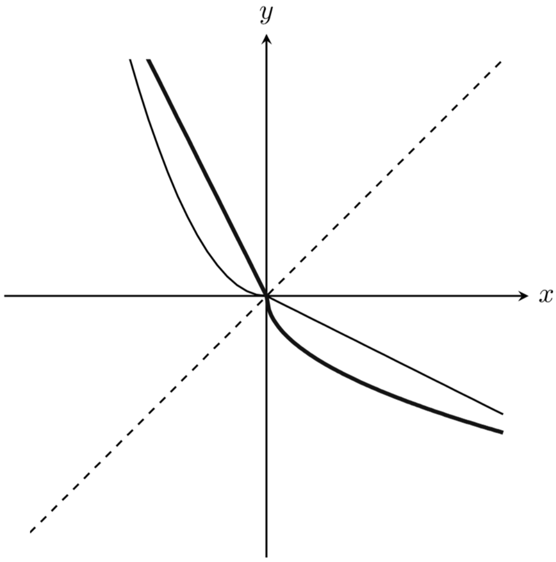


**Question 4**

***Answer: B***

**Explanatory notes**

The inverse may be found by reflecting the graph of function $f$ around the line $y = x$ as shown in the diagram below.




**Note:** The function with the thickest line is the inverse function.

**Tip**

- *You could eliminate some options in this question using the fact that points in the 2nd quadrant are always reflected to/from the 4th quadrant, while points in the 1st and 3rd quadrants don't change quadrants.*

### Question 5

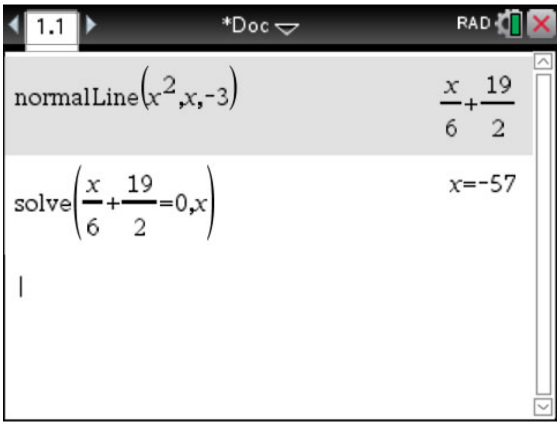
***Answer: C***

**Explanatory notes**

$\frac{dy}{dx} = 2x$. When $x = -3$ , $\frac{dy}{dx} = -6$. So the gradient of the perpendicular line is $\frac{1}{6}$ and the equation of this line is $y - 9 = \frac{1}{6}(x+3)$.

When $y = 0$, $-9 = \frac{1}{6}(x+3)$ and so $-54 = x + 3 \Rightarrow x = -57$.

Alternatively, CAS may be used. The equation of the perpendicular line of $y = x^2$ at the point $(-3,9)$ is $y = \frac{x}{6} + \frac{19}{2}$. This linear function passes through the $x$-axis when $x = -57$.



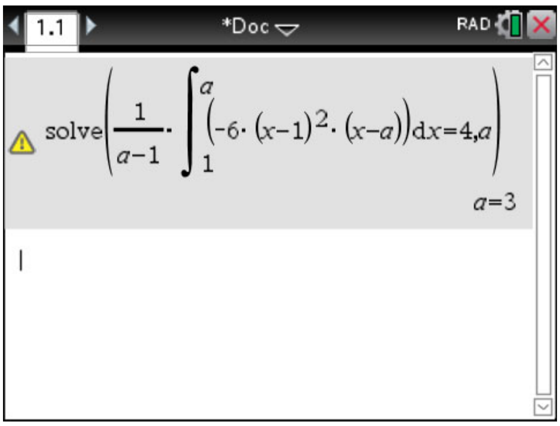

### Question 6

***Answer: E***

**Explanatory notes**

We need to solve the equation $\frac{1}{a-1}\int_1^a -6(x-1)^2(x-a)dx = 4$ for $a$.

Using CAS, we find that $a = 3$.



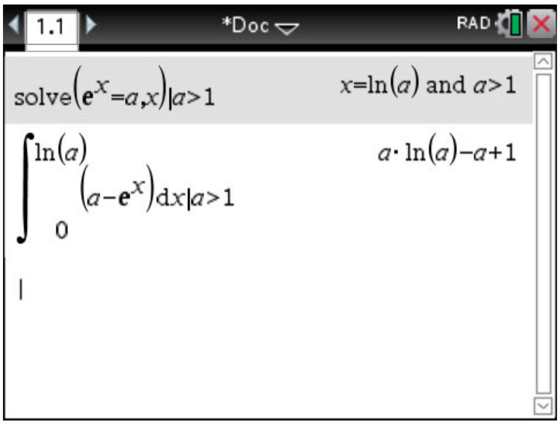
### Question 7

***Answer: C***

**Explanatory notes**

The graphs of $y = e^x$ and $y = a$ meet when $x = \log_e(a)$.

Use CAS to find the required area (in terms of $a$).

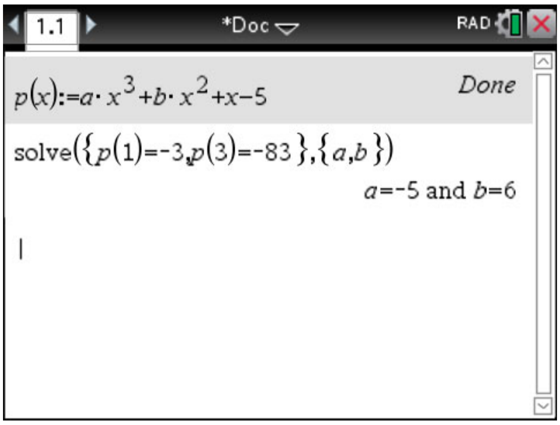


**Question 8**

***Answer: C***

**Explanatory notes**

Use CAS to determine the values of $a$ and $b$ by first defining the function, then substituting in the two points and finally solving for the parameters.




The values of $a$ and $b$ are $-5$ and $6$ respectively. The sum of $a$ and $b$ is 1.

## Question 9

***Answer: C***

**Explanatory notes**

First the value of $a$ must be found. Since $X$ is a probability distribution, it must be the case that

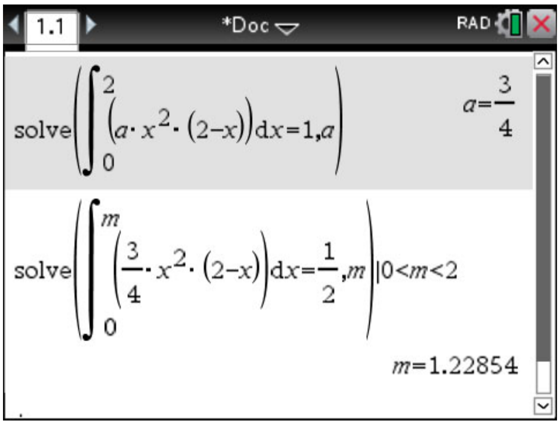
$$\int_{-\infty}^{\infty} f(x)dx = \int_0^2 ax^2(2-x)dx = 1.$$

Use CAS to find that $a = \frac{3}{4}$.

Now find the median by solving

$\int_0^m \frac{3}{4}x^2(2-x)dx = \frac{1}{2}$ for $m$.

It is found that $m \approx 1.23$.




## Question 10

***Answer: A***

**Explanatory notes**

The average value of $f$ on the interval $[0,2]$ is $\frac{1}{2}\int_0^2 f(x)dx = \frac{5}{2}$. Therefore, it must be the case that $\frac{1}{4}\int_{-4}^{0} f\left(-\frac{x}{2}\right)dx = \frac{5}{2}$, since dilation from or reflection in the $y$-axis does not change a function's average value.

Therefore, $\int_{-4}^{0} 3f\left(-\frac{x}{2}\right)dx = 4 \times 3 \times \frac{5}{2} = 30.$

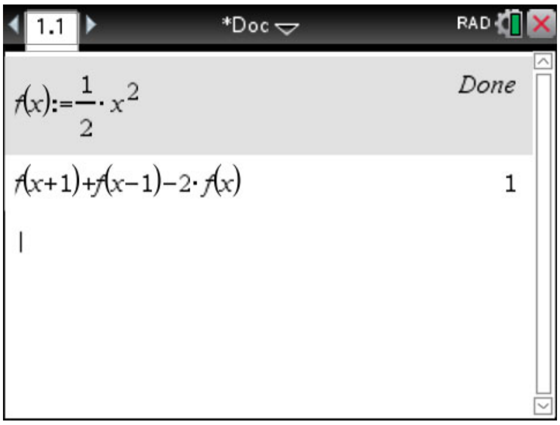
 

### Question 11

***Answer: E***

**Explanatory notes**

If $f(x)=\frac{1}{2}x^2$ then $f(x+1)+f(x-1)-2f(x)=1$. This can be checked using CAS.




**Tip**

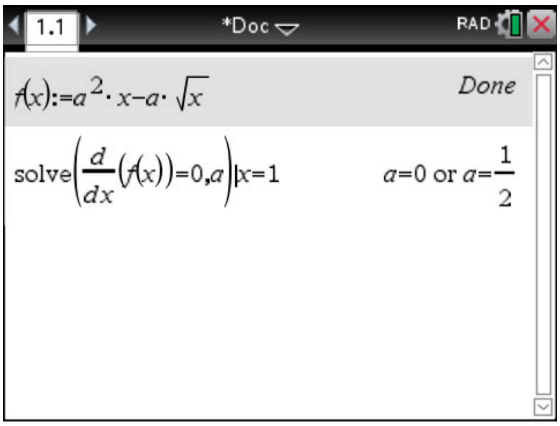
- *In the absence of a better method, you may use trial and error. In this case, starting from the last option and working upwards can be a good strategy.*

### Question 12

***Answer: B***

**Explanatory notes**

Use CAS to solve $f'(x)=0$ for $a$ when $x=1$.



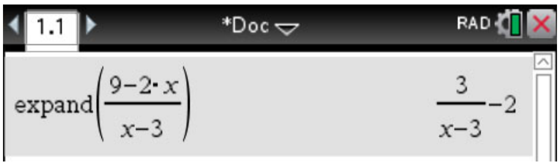

Since $a>0$ the trivial solution is rejected.

**Question 13**

***Answer: D***

**Explanatory notes**

Express $f$ in the form $a+\dfrac{b}{x-3}$ to find the vertical and horizontal asymptotes.




So the graph of $f$ has a vertical asymptote of $x = 3$ and a horizontal asymptote of $y = -2$.

**Tip**

- *While you could do this by hand, having a good knowledge of the CAS calculator allows the computation to be performed more easily.*

**Question 14**

***Answer: E***

**Explanatory notes**

Equate the functions, rearrange and multiply through by $e^x$ to obtain

$ke^{2x}-3e^x+1=0$.

This is a quadratic in $e^x$ and the discriminant is $9-4k$. From the discriminant there can be two solutions if $9-4k>0 \Rightarrow k<\dfrac{9}{4}$. Solving the equation $ky^2-3y+1=0$ gives

$y=\dfrac{3\pm\sqrt{9-4k}}{2k}$. Since $y=e^x$ it is necessary for both these solutions to be positive in order for the original equation to have two solutions. If $k$ is negative or zero, then $\dfrac{3+\sqrt{9-4k}}{2k}\le 0$ so at most one solution is possible, whereas for $0<k<\dfrac{9}{4}$ both solutions are positive as required.

**Tip**

- *Use sliders on your CAS graphing screen to quickly visualise the effect when changing the parameters.*

## Question 15

***Answer: B***

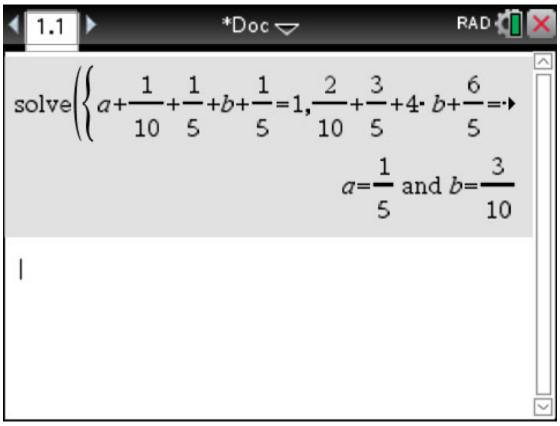
**Explanatory notes**

From the information provided, we can write down two simultaneous equations:

$$a+\frac{1}{10}+\frac{1}{5}+b+\frac{1}{5}=1$$

$$\frac{2}{10}+\frac{3}{5}+4b+\frac{6}{5}=\frac{32}{10}$$

These are solved to give $a=\frac{1}{5}$ and $b=\frac{3}{10}$.




**Tip**

- *You should use CAS to solve simultaneous equations in Examination 2.*

**Question 16**

***Answer: E***

**Explanatory notes**

Use the standard normal $Z \sim N(0,1)$:

$$\Pr(Z > 120) = 0.2$$

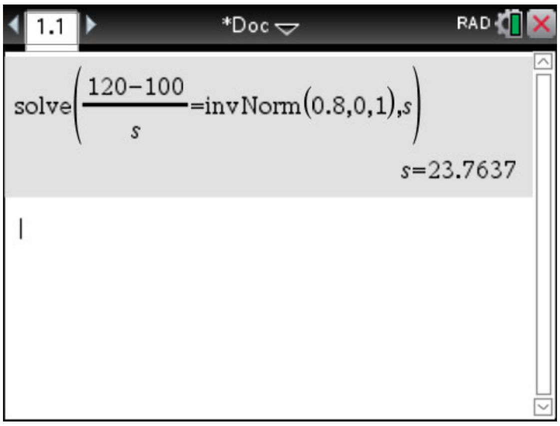
$$\Pr\left(Z > \frac{120-100}{\sigma}\right) = 0.2$$

$$\frac{120-100}{\sigma} = 0.8416$$

$$\sigma = 23.7637$$

So $\sigma$ is closest to 24.

This can be done in a single line using CAS.



## Question 17

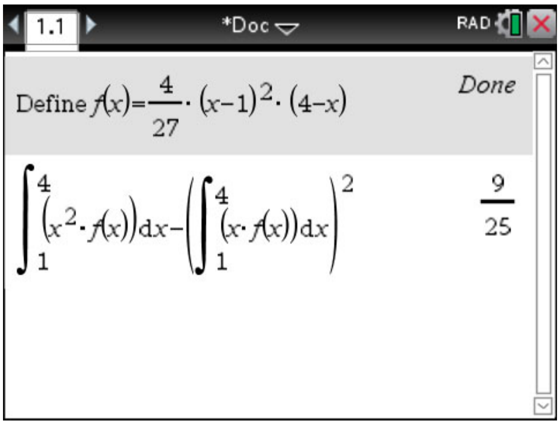
***Answer: A***

**Explanatory notes**

Use the formula for the variance of a random variable $X$:

$\text{Var}(X) = E\left(X^2\right) - \left(E(X)\right)^2$

This is best done using CAS.




So $\text{Var}(X) = \frac{9}{25} \Rightarrow \sigma = \frac{3}{5}$.

## Question 18

***Answer: B***

**Explanatory notes**

Since $\Pr(B) = 0.3$ it follows that $\Pr(B') = 1 - 0.3 = 0.7$ and since $A$ and $B$ are independent events, $\Pr(A \cap B') = 0.28$.

So

$$\begin{aligned}\Pr(A \cup B') &= \Pr(A) + \Pr(B') - \Pr(A \cap B') \\ &= 0.4 + 0.7 - 0.28 \\ &= 0.82\end{aligned}$$

**Question 19**

***Answer: A***

**Explanatory notes**

The approximate area is

$$\left(5-\frac{5}{4}\right)+(5-2)+\left(5-\frac{13}{4}\right)=\frac{17}{2}$$

The actual area is

$$\int_1^5 2\sqrt{x-1}\,dx=\frac{32}{3}$$

The error is

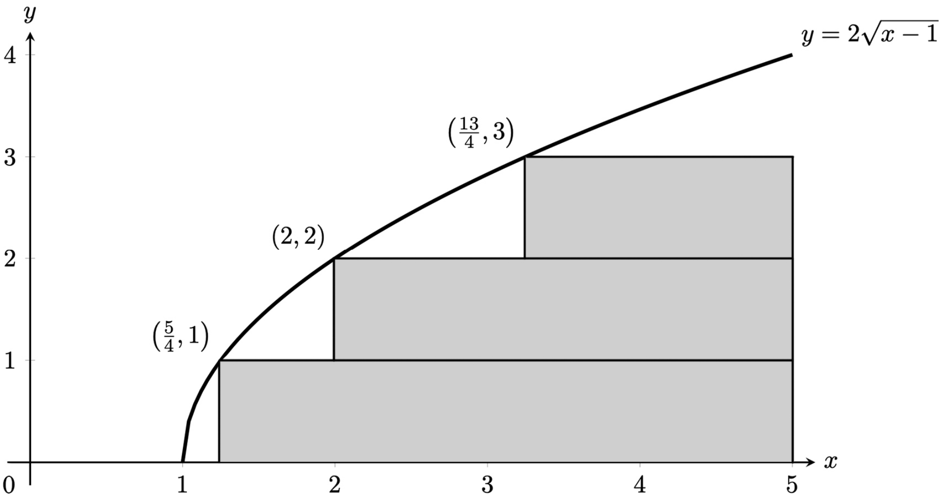
$$\text{actual} - \text{approximate}=\frac{32}{3}-\frac{17}{2}$$
$$=\frac{13}{6}$$

Therefore, the percentage error is

$$\frac{\frac{13}{6}}{\frac{32}{3}}\times 100\approx 20.3\%$$

**Tip**

- *The inverse function of* $y=2\sqrt{x-1}$ *is* $y=\frac{x^2}{4}+1$. *This will assist you in finding the length of the rectangles, for when* $x=1$, $y=\frac{5}{4}$, *when* $x=2$, $y=2$ *and when* $x=3$, $y=\frac{13}{4}$. *This is shown below:*



**Question 20**

***Answer: B***

**Explanatory notes**

Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$ and so $x' = -2x+1$, $y' = 3y$.

Since the original function needs to be found from the transformed function, substitute $x'$ and $y'$ into the equation $y' = 3\sqrt{4-x'}$ to obtain

$$3y = 3\sqrt{4-(-2x+1)}$$
$$y = \sqrt{3+2x}$$

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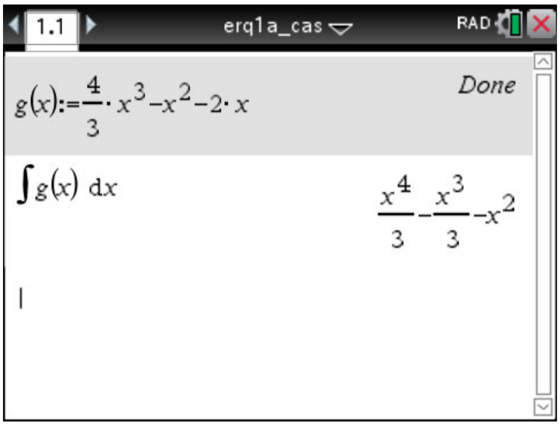
# SECTION B

### Question 1a.i.

**Worked solution**

Integrate by hand or use CAS to obtain $f(x)=\frac{1}{3}x^4-\frac{1}{3}x^3-x^2+c$, and apply the condition that $f(0)=-1$ to find the value of $c$.




Answer: $\frac{1}{3}x^4-\frac{1}{3}x^3-x^2-1$

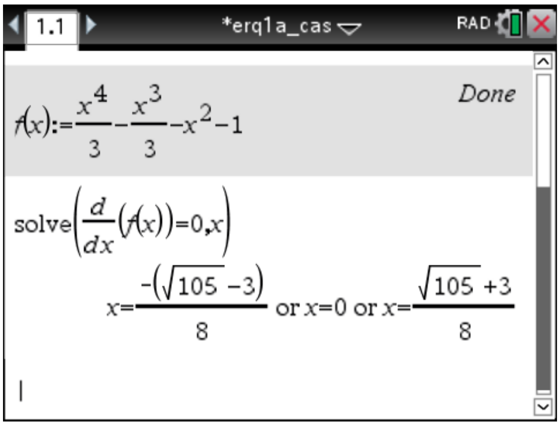
***Mark allocation: 1 mark***

- 1 mark for finding $f(x)=\frac{1}{3}x^4-\frac{1}{3}x^3-x^2+c$ and applying the condition to give the required answer

**Question 1a.ii.**

**Worked solution**

Use CAS to solve $f'(x)=0$ to find the stationary points.




Answer: $x=\dfrac{3\pm\sqrt{105}}{8}$

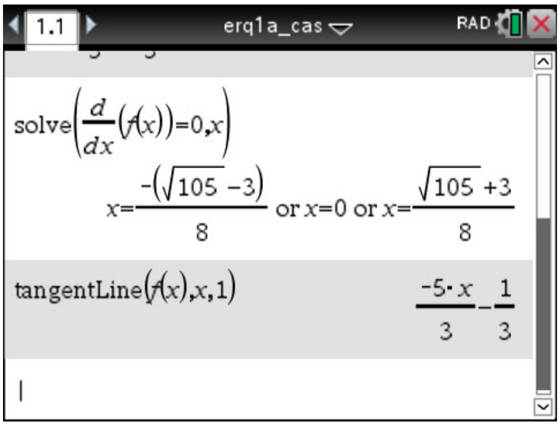
***Mark allocation: 2 marks***

- 1 mark for solving $f'(x)=g(x)=\dfrac{4}{3}x^3-x^2-2x=0$
- 1 mark for the correct answer

**Question 1b.i.**

**Worked solution**

Use CAS to find the equation of the tangent to $f\left(x\right)$ at $x=1$.




Answer: $y=-\frac{5}{3}x-\frac{1}{3}$

***Mark allocation: 1 mark***

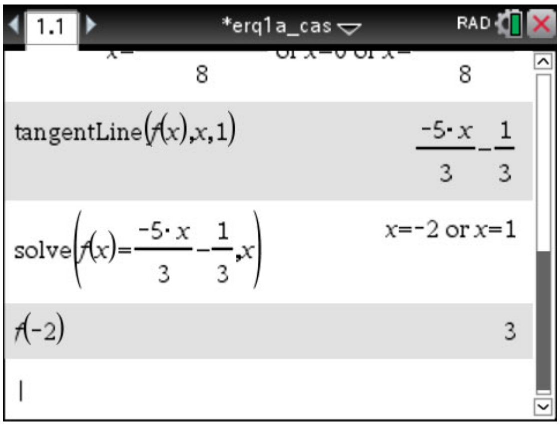
- 1 mark for the correct answer

**Question 1b.ii.**

**Worked solution**

Use CAS to solve $f(x)=-\frac{5}{3}x-\frac{1}{3}$ for $x$. It is found that $x=-2$ and $x=1$.

Since $f(-2)=3$, the coordinates of $A$ are $(-2,3)$.




Answer: $\left(-2,3\right)$.

***Mark allocation: 1 mark***

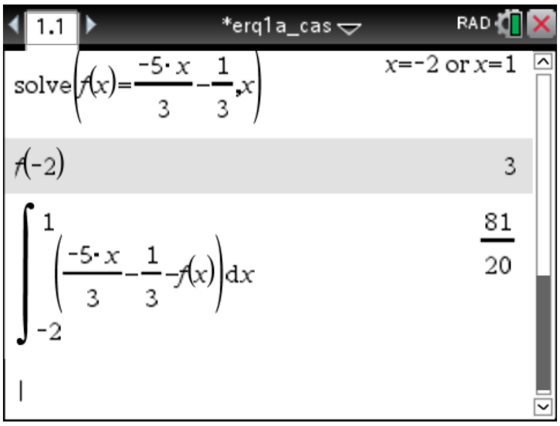
- 1 mark for the correct answer

**Question 1b.iii.**

**Worked solution**

The area bounded by the tangent $l$ and the graph of $f(x)$ is equal to

$$\int_{-2}^{1}\left(-\frac{5}{3}x-\frac{1}{3}-f(x)\right)dx=\frac{81}{20}.$$




Answer: $\frac{81}{20}$

***Mark allocation: 2 marks***

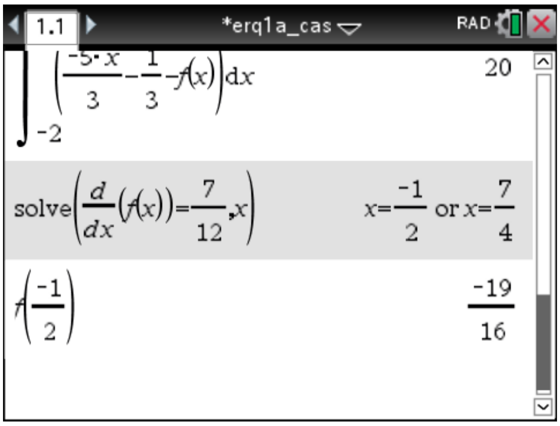
- 1 mark for setting up appropriate integral
- 1 mark for the correct answer

**Question 1c.i.**

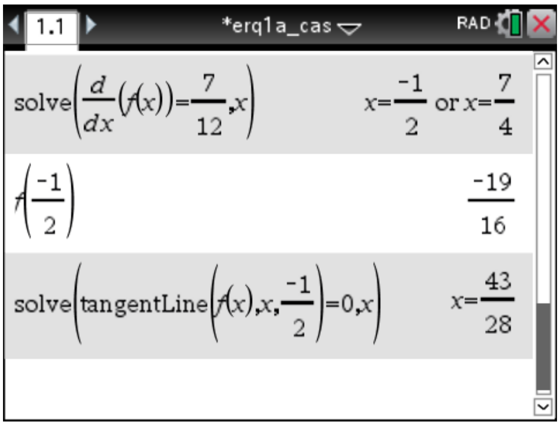
**Worked solution**

Using CAS to solve $f'(x)=\frac{7}{12}$ gives $x=-\frac{1}{2}$ and $\frac{7}{4}$

$f\left(-\frac{1}{2}\right)=-\frac{19}{16}$ and so the coordinates of $B$ are $\left(-\frac{1}{2},-\frac{19}{16}\right)$.




Now use CAS to determine when the tangent line to the graph of $f$ crosses the $x$-axis. The coordinates of $C$ are $\left(\frac{43}{28},0\right)$.




Answer: $\left(-\frac{1}{2},-\frac{19}{16}\right)$ and $\left(\frac{43}{28},0\right)$

***Mark allocation: 3 marks***

- 1 mark for solving $f'(x)=\frac{7}{12}$ and finding $x=-\frac{1}{2}$
- 1 mark deriving the coordinates of $B$
- 1 mark deriving the coordinates of $C$

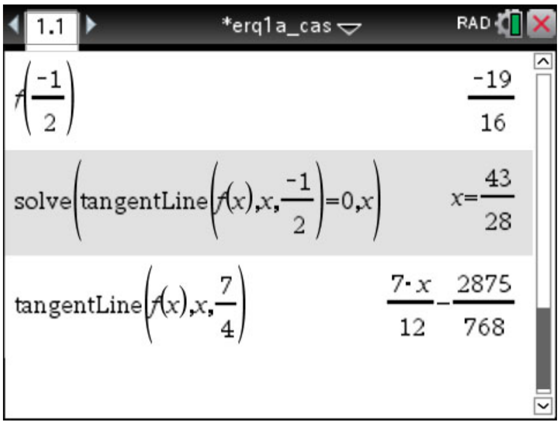
 

**Question 1c.ii.**

**Worked solution**

The other tangent with gradient $\frac{7}{12}$ meets the graph of $f$ when $x = \frac{7}{4}$.

Use CAS to find the equation of this tangent.




Answer: $y = \frac{7}{12}x - \frac{2875}{768}$

***Mark allocation: 1 mark***

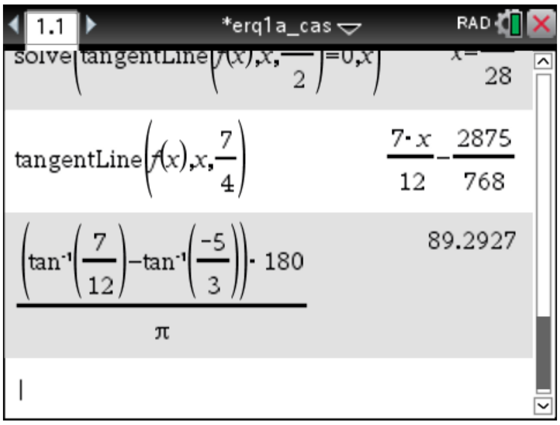
- 1 mark for the correct answer

**Question 1d.**

**Worked solution**

The angle $\theta$ between the two tangents is

$$\theta = \tan^{-1}\left(\frac{7}{12}\right) - \tan^{-1}\left(-\frac{5}{3}\right) = 89.29^{\circ}.$$




Answer: $89.29^{\circ}$

***Mark allocation: 2 marks***

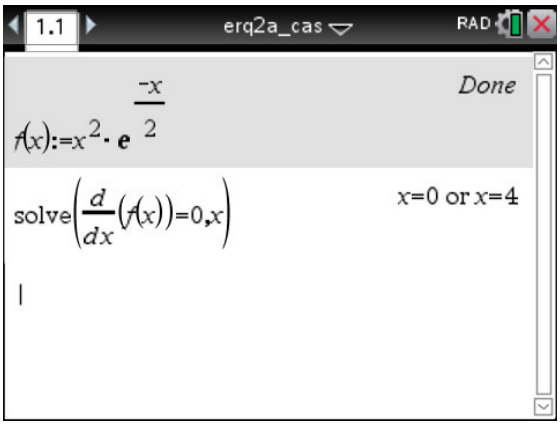
- 1 mark for correct use of inverse tan formula to find the angle
- 1 mark for the correct answer

**Question 2a.**

**Worked solution**

Use CAS to solve $f'(x)=0$ and find that the turning points are $x=0$ and $x=4$.




Answer: $[0,4]$

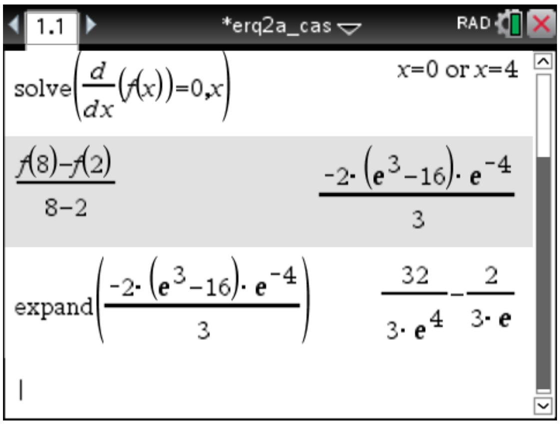
***Mark allocation: 1 mark***

- 1 mark for the correct answer

**Question 2b.**

**Worked solution**

Use CAS to perform computation.




$$\frac{f(8)-f(2)}{8-2}=\frac{32}{3e^4}-\frac{2}{3e}$$

Answer: $\frac{32}{3e^4}-\frac{2}{3e}$

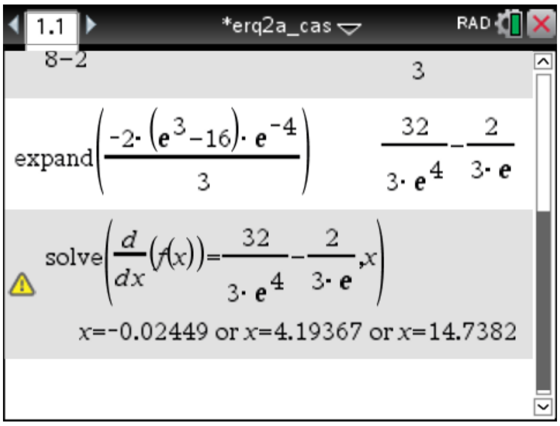
***Mark allocation: 2 marks***

- 1 mark for using the gradient formula
- 1 mark for the correct answer

**Question 2c.**

**Worked solution**

Use CAS to solve $f'(x)=\dfrac{32}{3e^4}-\dfrac{2}{3e}$.




Answer: $x=4.194$

***Mark allocation: 2 marks***

- 1 mark for solving $f'(x)=\dfrac{32}{3e^4}-\dfrac{2}{3e}$
- 1 mark for the correct answer

**Question 2d.i.**

**Worked solution**

In order for $f(g(x))$ to be defined we require rang $\subseteq$ dom$f=[0,10]$. Since $g(x)=x^2$ it must be the case that $D=\left[-\sqrt{10},\sqrt{10}\right]$.

Therefore, the domain of $h'$ is $\left(-\sqrt{10},\sqrt{10}\right)$ as the derivative is not defined at the endpoints.

Answer: $\left(-\sqrt{10},\sqrt{10}\right)$

***Mark allocation: 2 marks***

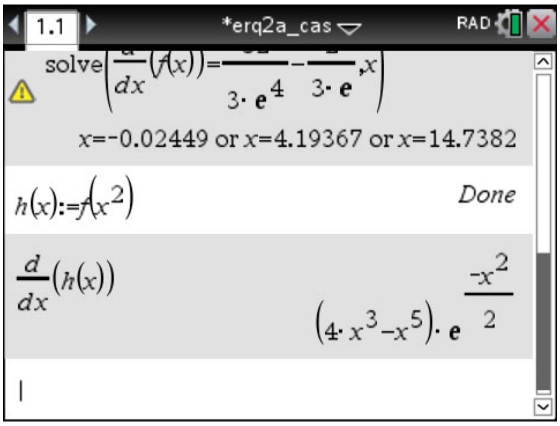
- 1 mark for finding $D=\left[-\sqrt{10},\sqrt{10}\right]$
- 1 mark for the correct answer

**Question 2d.ii.**

**Worked solution**

Use CAS to find $h'(x) = \left(4x^3 - x^5\right)e^{-\frac{x^2}{2}}$.




Answer: $h'(x) = \left(4x^3 - x^5\right)e^{-\frac{x^2}{2}}$

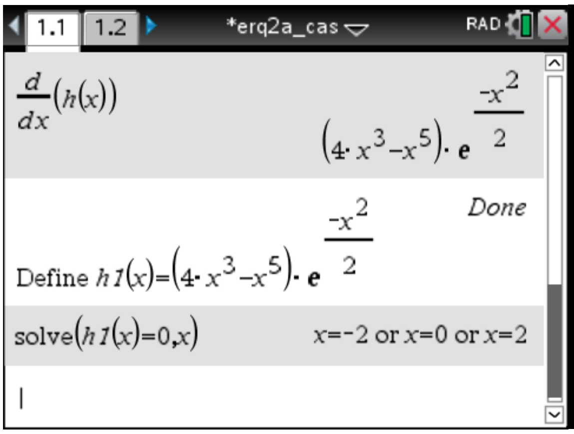
***Mark allocation: 1 mark***

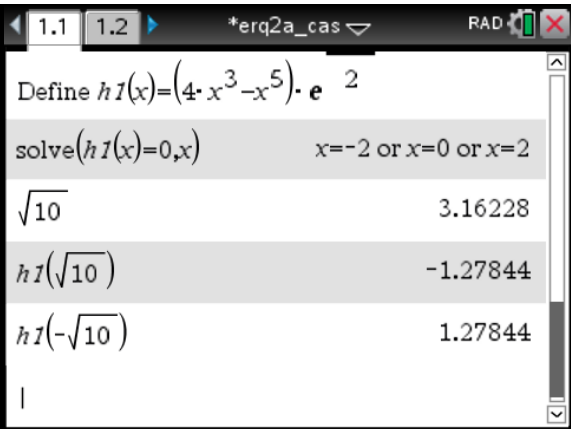
- 1 mark for the correct answer

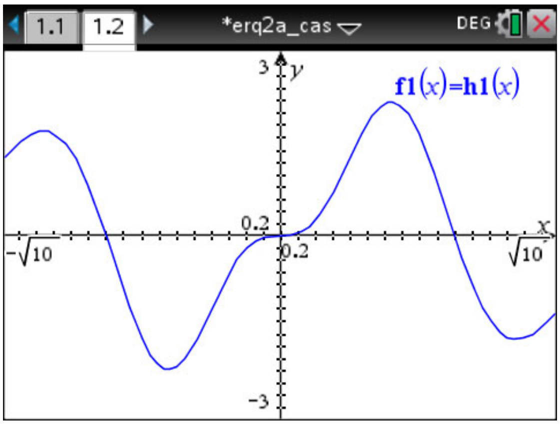
**Question 2d.iii.**

**Worked solution**

Use CAS to assist in graphing and to help find the coordinates of the endpoints.



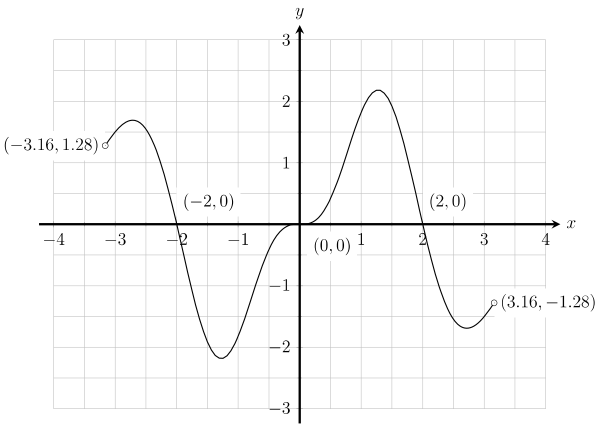




The $x$-intercepts are at (–2, 0) and (2, 0)

The endpoint coordinates are at (–3.16, 1.28) and (3.16, –1.28)

Answer:




***Mark allocation: 3 marks***

- 1 mark for a graph with the correct shape
- 1 mark for the endpoints correct
- 1 mark for the correct coordinates of axis intercepts

**Question 3a.**

**Worked solution**

The amplitude of $f$ is 2, so the length of $OC$ is $12\text{ cm}$. The area $OABC$ is $15\times 12=180\,\text{cm}^2$

Answer: $180\text{ cm}^2$

***Mark allocation: 1 mark***

- 1 mark for the correct answer

**Question 3b.**

**Worked solution**

Note that the $y$-intercept of the parabola is $(0,8)$. Therefore

$$g(0)=225a=8\Rightarrow a=\frac{8}{225}.$$

Answer: $\dfrac{8}{225}$

***Mark allocation: 1 mark***

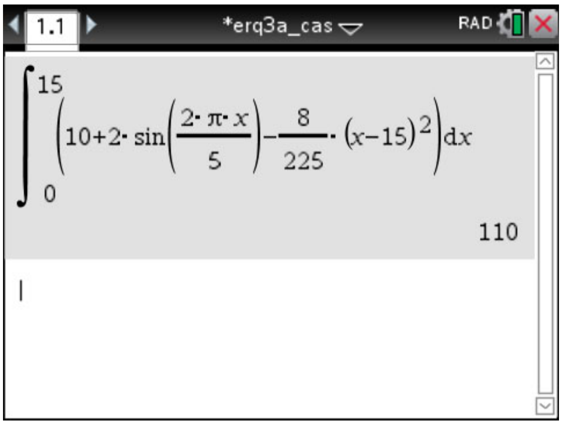
- 1 mark for the correct answer

**Question 3c.**

**Worked solution**

The area of the component is given by

$$\int_0^{15}\left(10+2\sin\left(\frac{2\pi x}{5}\right)-\frac{8}{225}(x-15)^2\right)dx=110\,\text{cm}^2$$




Answer: $110\,\text{cm}^2$

***Mark allocation: 2 marks***

- 1 mark for an appropriate integral
- 1 mark for the correct answer

**Question 3d.**

**Worked solution**

The distance between the two functions is $d(x) = 10 + 2\sin\left(\frac{2\pi x}{5}\right) - \frac{8}{225}(x-15)^2$

Answer: $d(x) = 10 + 2\sin\left(\frac{2\pi x}{5}\right) - \frac{8}{225}(x-15)^2 = 2\sin\left(\frac{2\pi x}{5}\right) - \frac{8}{225}x^2 + \frac{16}{15}x + 2$

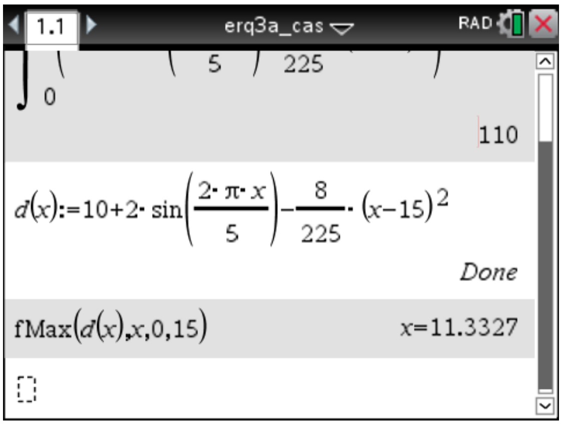
***Mark allocation: 1 mark***

- 1 mark for the correct answer

**Question 3e.**

**Worked solution**

Use CAS to find the maximum value of $d(x)$ on the interval $[0,15]$. It is found that $x = 11.333$ to three decimal places.




Answer: $x = 11.333$

***Mark allocation: 1 mark***

- 1 mark for the correct answer

**Question 3f.**

**Worked solution**

The average value is $\frac{1}{15}\int_0^{15}\left(10 + 2\sin\left(\frac{2\pi x}{5}\right) - \frac{8}{225}(x-15)^2\right)dx = \frac{110}{15} = \frac{22}{3}$

Answer: $\frac{22}{3}$

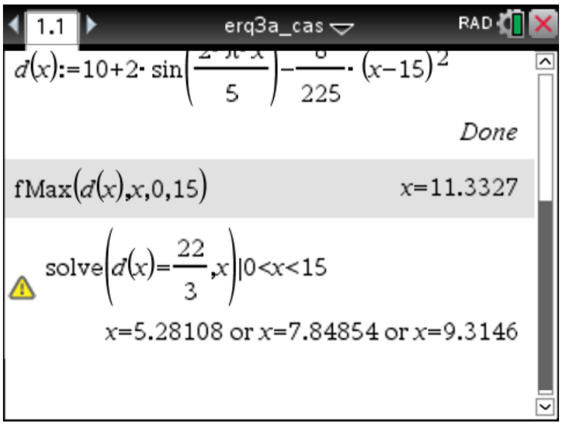
***Mark allocation: 1 mark***

- 1 mark for the correct answer

**Question 3g.**

**Worked solution**

Use CAS to solve the equation $d(x)=\dfrac{22}{3}$.




Answer: $x = 5.281$, $x = 7.849$ and $x = 9.315$

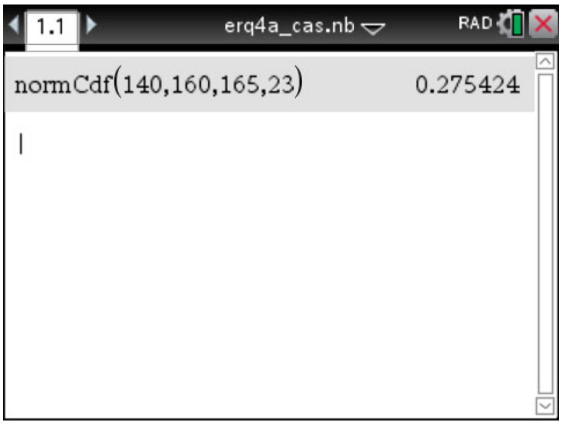
***Mark allocation: 3 marks***

- 1 mark for solving equation $d(x)=\dfrac{22}{3}$
- 2 marks for all three correct answers

**Question 4a.**

**Worked solution**

Use CAS to find the required probability.




Answer: 0.2754

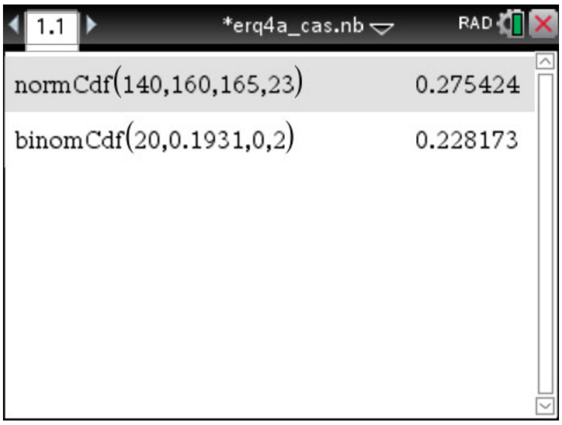
***Mark allocation: 1 mark***

- 1 mark for the correct answer

**Question 4b.i.**

**Worked solution**

Let $Y \sim Bi(20, 0.1931)$. Then $\Pr(Y \leq 2) = 0.228$.




Answer: 0.228

***Mark allocation: 2 marks***

- 1 mark for the correct binomial distribution
- 1 mark for the correct answer

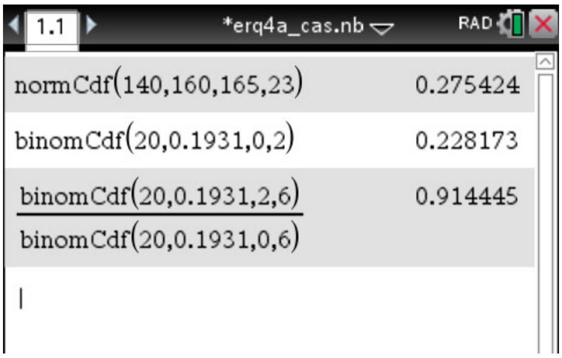
**Question 4b.ii**

**Worked solution**

$$\Pr\left(\hat{P} \geq 0.1 \mid \hat{P} \leq 0.3\right) = \frac{\Pr(0.1 \leq \hat{P} \leq 0.3)}{\Pr(\hat{P} \leq 0.3)}$$

$$= \frac{\Pr(2 \leq Y \leq 6)}{\Pr(Y \leq 6)}$$

$$= 0.914$$




Answer: 0.914.

***Mark allocation: 3 marks***

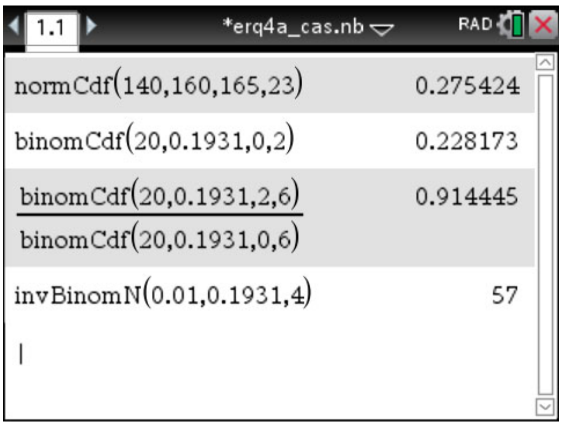
- 1 mark for the correct conditional probability (first line)
- 1 mark for the conversion to $Y$
- 1 mark for the correct answer

**Question 4b.iii.**

**Worked solution**

Let $W \sim Bi(n, 0.1931)$. We want to find $n$ such that $\Pr(W \geq 5) > 0.99$. This is equivalent to $\Pr(W < 5) < 0.01$.

Use the inverse binomial function of the CAS calculator to find that $n = 57$.




Answer: 57

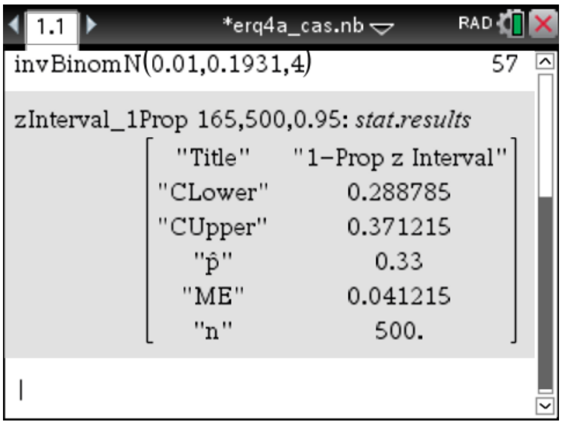
***Mark allocation: 2 marks***

- 1 mark for the statement equivalent to $W \sim Bi(n, 0.1931)$
- 1 mark for the correct answer

**Question 4c.**

**Worked solution**

Use the CAS calculator to determine the confidence interval.




Answer: $(0.289, 0.371)$

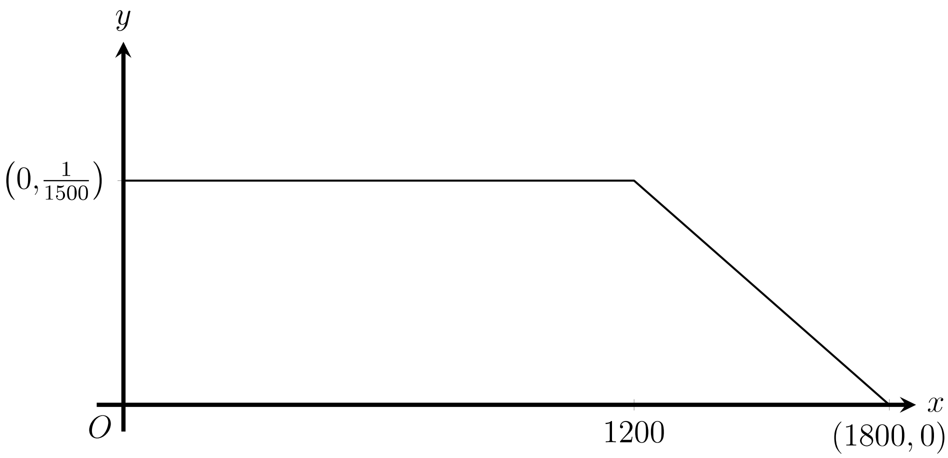
***Mark allocation: 1 mark***

- 1 mark for correct answer

**Question 4d.i**

**Worked solution**

The graph is plotted below.




***Mark allocation: 2 marks***

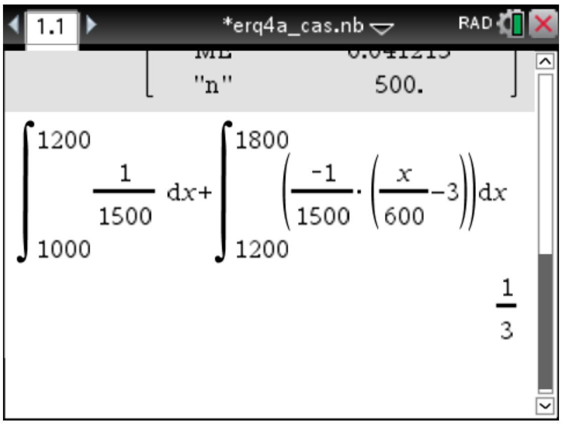
- 1 mark for the correct shape (two straight lines)
- 1 mark for the correct axis coordinates

**Question 4d.ii.**

**Worked solution**

Use CAS to evaluate:

$$\Pr(X > 1000) = \int_{1000}^{1200} \frac{1}{1500}\,dx + \int_{1200}^{1800} -\frac{1}{1500}\left(\frac{x}{600} - 3\right)dx = \frac{1}{3}$$




Alternatively,

$$\begin{aligned}\Pr(X > 1000) &= 1 - \Pr(X < 1000)\\ &= 1 - \int_{0}^{1000} \frac{1}{1500}\,dx\\ &= 1 - \frac{1000}{1500}\\ &= 1 - \frac{2}{3}\\ &= \frac{1}{3}\end{aligned}$$

Answer: $\frac{1}{3}$

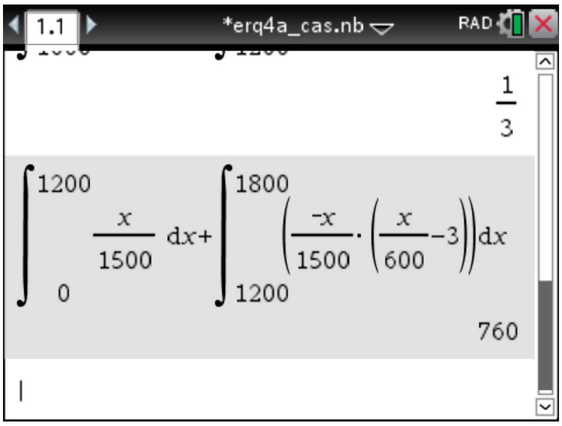
***Mark allocation: 1 mark***

- 1 mark for the correct answer

**Question 4d.iii.**

**Worked solution**

$$\mathrm{E}(X)=\int_0^{1200}\frac{x}{1500}\,dx+\int_{1200}^{1800}-\frac{x}{1500}\left(\frac{x}{600}-3\right)dx=760$$




Answer: 760 m

***Mark allocation: 2 marks***

- 1 mark for the integrals
- 1 mark for the correct answer

**Question 4e.**

**Worked solution**

Let $H$ be the event that a randomly selected apple is hybrid and $A$ be the event that a randomly selected apple is grown at an altitude greater than 1000 m.

Then $\Pr(H)=0.4$ and $\Pr(A)=\frac{1}{3}$ and so $\Pr(A')=1-\frac{1}{3}=\frac{2}{3}$.

Now $\Pr(H\mid A)=0.15\Rightarrow\Pr(H\cap A)=0.15\times\frac{1}{3}=0.05$.

Therefore, $\Pr(H\cap A')=0.4-0.05=0.35$.

Finally, $\Pr(H\mid A')=\frac{\Pr(H\cap A')}{\Pr(A')}=\frac{0.35}{2/3}=0.525$.

Answer: 0.525

***Mark allocation: 2 marks***

- 1 mark for recognising conditional probability and finding $\Pr(H\cap A)=0.05$
- 1 mark for the correct answer

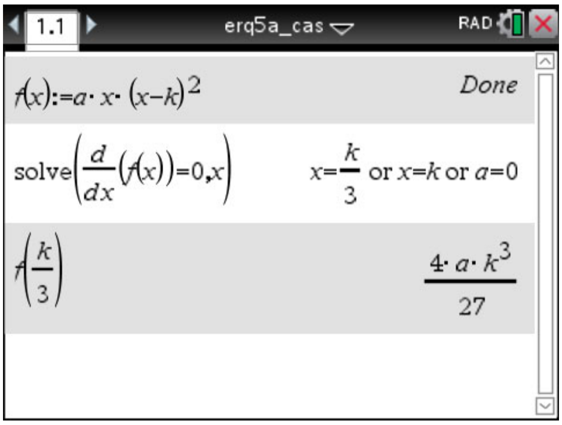
**Question 5a.**

**Worked solution**

Use CAS to find that $f'(x) = 0 \Rightarrow x = \frac{k}{3}$ or $x = k$.

The turning point at $x = k$ is the local minimum.

Use CAS to evaluate $f\left(\frac{k}{3}\right) = \frac{4ak^3}{27}$.




Answer: $\left(\frac{k}{3}, \frac{4ak^3}{27}\right)$

***Mark allocation: 2 marks***

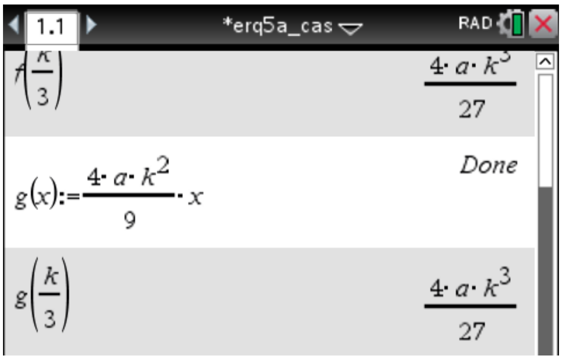
- 1 mark for stating solving $f'(x) = 0$
- 1 mark for the correct coordinates

**Question 5b.**

**Worked solution**

Use CAS to confirm that $g\left(\frac{k}{3}\right)=\frac{4ak^3}{27}$.




Answer: $g\left(\frac{k}{3}\right)=\frac{4ak^3}{27}$

***Mark allocation: 1 mark***

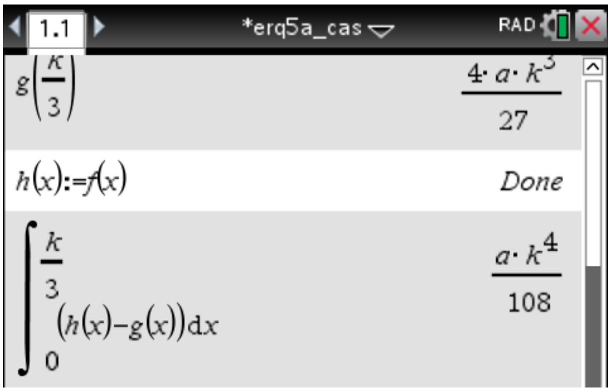
- 1 mark for confirming that $g\left(\frac{k}{3}\right)=\frac{4ak^3}{27}$

**Question 5c.**

**Worked solution**

Use CAS to find that the area bounded by the graphs of $h$ and $g$ is

$\int_0^{\frac{k}{3}}\left(h(x)-g(x)\right)dx=\frac{ak^4}{108}$.




Answer: $\frac{ak^4}{108}$

***Mark allocation: 2 marks***

- 1 mark for the integral $\int_0^{\frac{k}{3}}\left(h(x)-g(x)\right)dx$
- 1 mark for the answer $\frac{ak^4}{108}$

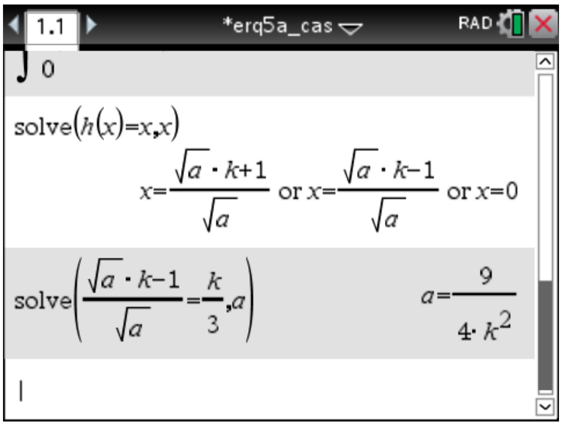
**Question 5d.**

**Worked solution**

The graphs of $y = h(x)$ and $y = x$ meet when $x = 0$ and when $x = \dfrac{\sqrt{a}k \pm 1}{\sqrt{a}}$.

For the graphs of $y = h(x)$ and $y = x$ to meet exactly twice on the interval $\left[0, \dfrac{k}{3}\right]$ with the value of $a$ as large as possible, it must be the case that $\dfrac{\sqrt{a}k - 1}{\sqrt{a}} = \dfrac{k}{3}$.

Therefore $a = \dfrac{9}{4k^2}$




***Mark allocation: 2 marks***

- 1 mark for finding $x = 0$ and $x = \dfrac{\sqrt{a}k \pm 1}{\sqrt{a}}$
- 1 mark for solving $\dfrac{\sqrt{a}k - 1}{\sqrt{a}} = \dfrac{k}{3}$

### Question 5e.

**Worked solution**

If $h$ and $h^{-1}$ meet at $O$ when $x=\dfrac{k}{3}$, then the area bounded by the graphs is

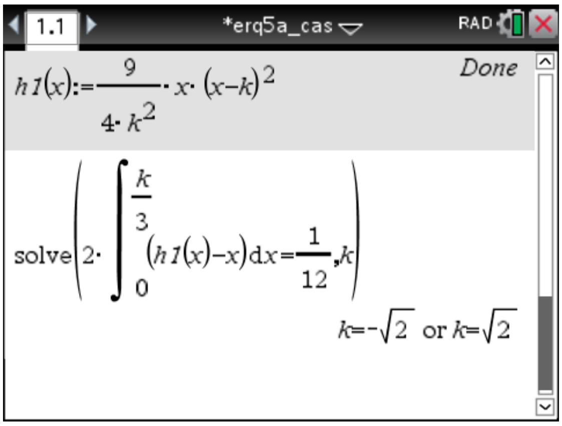
$$\int_0^{\frac{k}{3}}\left(h(x)-h^{-1}(x)\right)dx=2\int_0^{\frac{k}{3}}\left(h(x)-x\right)dx$$

So solve

$$2\int_0^{\frac{k}{3}}\left(h(x)-x\right)dx=\frac{1}{12}$$

with $a=\dfrac{9}{4k^2}$.

It is found that $k=\sqrt{2}$.




Answer: $k=\sqrt{2}$

***Mark allocation: 3 marks***

- 1 mark for recognising that $\int_0^{\frac{k}{3}}\left(h(x)-h^{-1}(x)\right)dx=2\int_0^{\frac{k}{3}}\left(h(x)-x\right)dx$
- 1 mark for solving $2\int_0^{\frac{k}{3}}\left(h(x)-x\right)dx=\dfrac{1}{12}$
- 1 mark for the answer $k=\sqrt{2}$

**END OF WORKED SOLUTIONS**