2019 PRACTICE EXAM

MATHS METHODS UNITS 3&4

Exam 1 Question Booklet
Exam 2 Question Booklet
Worked Solution Booklet

ATARNotes

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ATARNotes

STUDENT NUMBER | | | | | | | | | Letter | |

MATHEMATICAL METHODS

Written examination 2

2019

Reading time: 9:00 a.m. to 9:15 a.m. (15 minutes)
Writing time: 9:15 a.m. to 11:15 a.m. (2 hour)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 19 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
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Question 1

Let $f: R \to R, f(x) = -3\sin(2x) + 2$.

The period and range of this function are respectively

A. π and $[-1,5]$
B. π and $[-3,2]$
C. 2π and $[-1,5]$
D. 2π and $[-3,2]$
E. 4π and $[1,5]$

Question 2

Let $f: R \to R, f(x) = 2x^3 - 3x^2 - ax + b$. When $f(x)$ is divided by $(x+1)$, the remainder is 5.

The sum of the values of a and b is

A. -5
B. -1
C. 1
D. 5
E. 10

Question 3

The midpoint of the points $(4,5)$ and $P(a,b)$ is $(1,1)$.

The coordinates of the point P are

A. $(7,9)$
B. $\left(\frac{3}{2},3\right)$
C. $(5,6)$
D. $(-2,3)$
E. $(-2,-3)$

Question 4

The function $g: R \to R, g(x) = x^2 + bx + 2$, where $b > 0$, has a range of $[-14, \infty)$.

The value of b is

A. 2
B. 3
C. $11/2$
D. 6
E. 8

Question 5

The function f is strictly decreasing over the interval (a, b).

If $a, b > 0$, then which of the following must be true?

A. $f(b - a) > 0$
B. $f'(a) > 0$
C. $f'(b) < 0$
D. $f'(a - b) = 0$
E. $f(a) - f(b) > 0$

Question 6

A large random sample of visitors to the Wonder Wharf amusement park were surveyed and asked if they enjoyed their visit to the park. The survey results state, with a 95% confidence interval, that between 85% and 93% of visitors enjoyed their visit to the park.

The number of surveyed visitors who enjoyed their visit to the Wonder Wharf amusement park is closest to

A. 89
B. 148
C. 167
D. 209
E. 235

Question 7

The function f has an average value of k over the interval $[a, b]$.

The value of $\int_a^b \left(2 - f(x)\right) dx$ is equal to

A. $2b - 2a + k$
B. $2b - 2a - k$
C. $k + 2a - 2b$
D. $(a - b)(k - 2)$
E. $(b - a)(k + 2)$

Question 8

The average rate of change of the function with the rule $f(x) = (x^2 - 1)^2$ over the interval $[-2, b]$, where $b > -2$, is -3.

The value of b is

A. 1
B. $\frac{3}{4}$
C. -1
D. $\frac{1}{2}$
E. $-\frac{1}{2}$

Question 9

The function g has rule $g(x) = \sqrt{\log_e(kx)}$, where k is a real positive constant.

The maximal domain of g is

A. $\left[\frac{1}{k}, \infty\right)$
B. $\left(\frac{1}{k}, \infty\right)$
C. $\left(-\infty, -\frac{1}{k}\right) \cup \left(\frac{1}{k}, \infty\right)$
D. $\left(-\infty, -\frac{1}{k}\right)$
E. $(-\infty, k]$

Question 10

Consider the transformation T_1, defined as

$$T_1: R^2 \to R^2, T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The transformation T_1 maps the graph of $y = f(x)$ onto the graph of $y = g(x)$, where $f(x) = \sqrt{x^3}$.

The transformation $T_2: R^2 \to R^2$ also maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

The transformation T_2 could be given by

A. $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
B. $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
C. $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
D. $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
E. $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Question 11

The continuous random variable X has a normal distribution with mean 1 and standard deviation 2.
The continuous random variable Z has the standard normal distribution.

The probability that Z is between -1 and 2 is equal to

A. $\Pr(0 < X < 2)$
B. $\Pr(0 < X < 3)$
C. $\Pr(-1 < X < 3)$
D. $\Pr(X < -1) + \Pr(X > 3)$
E. $\Pr(-3 < X < 3)$

Question 12

The number of pets, X, owned by a random individual is a discrete random variable with the following probability distribution.

x	0	1	2	3	4
$\Pr(X = x)$	0.2	0.34	0.24	0.12	0.1

If two random individuals are chosen, the probability that they own a total of 2 pets is

A. 0.0576
B. 0.2116
C. 0.096
D. 0.2376
E. 0.1636

Question 13

The continuous random variable Z has the standard normal distribution.

If $\Pr(Z > a) = p$, where $a > 0$, then $\Pr(-a < z < 0 | Z < a)$ is

A. $\frac{1-p}{2}$
B. $\frac{p}{1-p}$
C. $\frac{1}{2p}$
D. $\frac{2p-1}{p-1}$
E. $\frac{2p-1}{2p-2}$

Question 14

In a particular scoring game, players attempt to throw a small hoop onto a pole. Players score a point each time they throw the hoop onto the pole. For each throw, the probability of throwing the hoop onto the pole is p.

If a player takes two throws, the expected number of points that this player will score is

A. $2p$
B. p^2
C. $2p^2$
D. $2p(1-p)$
E. $p(2-p)$

Question 15

Consider the functions $f\colon [a, b]$ and $g\colon [c, d]$.

If $g'(x) > 0$ over the interval $[c, d]$, then $f\big(g(x)\big)$ exists if

A. $b > d > c > a$
B. $b > c > d > a$
C. $b < d < c < a$
D. $b > g(d) > g(c) > a$
E. $a < g(d) < g(c) < b$

Question 16

A box contains three red balls and four green balls. Two balls are drawn at random from the box without replacement.

The probability that the marbles are of the same colour is

A. $\frac{1}{7}$
B. $\frac{25}{42}$
C. $\frac{25}{49}$
D. $\frac{3}{7}$
E. $\frac{4}{7}$

Question 17

The graphs of $y = 3ax^2$ and $y = mx + c$, where $m \neq 0$, will have two points of intersections if

A. $m > 0$
B. $m < 0$
C. $a > 0$ and $c < 0$
D. $a < 0$ and $c < 0$
E. $m > 2\sqrt{3ac}$

Question 18

Let $f(x) = xe^{ax} + a$, where $a \geq 0$.

The stationary point of f is closest to the origin when a is closest to

A. 0
B. 0.69
C. 1
D. 1.03
E. 1.20

Question 19

A probability density function f is given by

$$f(x) = \begin{cases} \frac{1}{m}\sin\left(\frac{x}{m}\right) & 0 < x < b \\ 0 & \text{elsewhere} \end{cases}$$

where $m > 0$.

The value of b is

A. m
B. $\frac{m}{2}$
C. $2\pi m$
D. $\frac{\pi m}{2}$
E. $4\pi m$

Question 20

In a large flock of birds, it is known that 60% of the birds are magpies. A scientist takes a sample of 10 birds from the flock. For samples of 10 birds, $\hat{P}$ is the random variable of the distribution of sample proportions of magpies. (Do not use a normal approximation.)

$\Pr(\hat{P} > 0.8 \mid \hat{P} > 0.5)$ is closest to

A. 0.05
B. 0.07
C. 0.17
D. 0.20
E. 0.63

END OF SECTION A

Instructions

Answer **all** questions in the spaces provided.
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Question 1 (13 marks)

Let $f: [-\pi, 2\pi] \to R$, $f(x) = 2x\sin(x)$.

a. State the rule for the derivative function f'.

1 mark

b. Find the range of f. Give values correct to two decimal places.

1 mark

c. Find the total area of the regions bounded the $y = f(x)$ and the x-axis. Give your answer in exact form.

1 mark

d. The transformation $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ maps the graph of $y = f(x)$ onto the graph of $y = (2x + \pi)\cos(x) + 2$.
State the values of c and d.

2 marks

Let $g: [a, b] \to R$, $g(x) = 2x\sin(x)$, where $-\pi \leq a < b \leq 2\pi$.

e. State the largest difference between a and b for which g^{-1}, the inverse function of g, exists. Give the value correct to two decimal places.

1 mark

Let $h: [-\pi, 2\pi] \to R$, $h(x) = 2x\sin(x) + c$, where $c \in R^{-}$.

f. Find the value of c for which the average value of h is $-\frac{8}{3}$.

2 marks

g. Find the value of c for which the tangent to the graph of h at $x = -\frac{\pi}{2}$ passes through the point $(4, -21)$.

2 marks

h. Find the value of c for which the area bounded by the graph of h, the x-axis and the lines $x = -\pi$ and $x = 2\pi$ is 200π.

2 marks

i. Find the value of c for which $h(x)$ has only two solutions, correct to two decimal places.

1 mark

Question 2 (10 marks)

Dan runs a company which produces bubble bath mixture. If one measure of bubble bath mixture is added to a bathtub of water, the volume, V litres, of bubbles in the bath after t minutes of adding the mixture is given by rule

$$V(t) = k^2 t\, e^{-\frac{k}{100}t}$$

where k is the percentage (%) of sodium laureth sulphate in the mixture.

a. Find the time when the volume of bubbles is a maximum. Express your answer in terms of k.

1 mark

b. If Dan decides to use 5% sodium laureth sulphate in his bubble bath mixture, what is the maximum amount of bubbles that will form? State your answer in exact form.

1 mark

c. After 30 minutes, Dan would like there to be at least 100 litres of bubbles. Which values for k give this result? Give values correct to two decimal places.

1 mark

Dan's company also manufactures detergent. If one measure of detergent is added to a sink of water, the volume, D litres, of foam in the sink after t minutes of adding the detergent is given by

$$D(t) = ae^{-bt}$$

where a and b are positive real numbers and $t > 0$.

d. Find the time, in terms of b, when only half of the initial volume of foam will remain.

2 marks

e. Find the rule for D^{-1}, in terms of a and b.

2 marks

For the foam produced by the detergent from Dan's company, $b = 1$. For the foam produced by the detergent from a rival company, $b = 2$.

Dan runs an experiment. In Sink 1, he puts one measure of the detergent from his company. At the same time, he puts one measure of the detergent from the rival company in Sink 2. Initially, the volume of foam in each sink is the same.

f. At $t = 2$, find the exact ratio of the volume of foam in Sink 1 to the volume of the foam in Sink 2. Express your answer in the form $1 : c$, where c is a real number in exact form.

1 mark

g. At a certain time, the volume of foam in Sink 1 is W.
Find the volume of foam in Sink 2 at this time. Express your answer as a fraction, in terms of W and B.

1 mark

h. Find the value of time such that rate at which the volume of foam in Sink 2 is changing with respect to time is double the rate at which the volume of foam in Sink 1 is changing with respect to time. Express your answer in the form $\log_e(d)$, where d is a positive integer.

1 mark

Question 3 (16 marks)

Each morning she goes to school, Eliza catches the bus to school. The time that Eliza spends on the bus going to school each morning varies.

The continuous random variable T, which models the time, t, in minutes, that Eliza spends each morning on the bus, has a probability density function f, where

$$f(t) = \begin{cases} \frac{4}{8625}(t^3 - 63t^2 + 1275t - 8125) & 15 \le t \le 30 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the **mean** time, in minutes, that Eliza spends on the bus each morning she goes to school, correct to two decimal places.

1 mark

b. Find the **median** time, in minutes, that Eliza spends on the bus each morning she goes to school, correct to two decimal places.

2 marks

Eliza will be late for school if she spends over 25 minutes on the bus. Assume that the amount of time spent on the bus on any day is independent of the time spent on the bus on any other day.

c. Find the probability that Eliza will be late for school.

2 marks

d. On a particular morning, Eliza was not late for school.
Find the probability that she spent less than 20 minutes on the bus.

2 marks

e. Eliza catches the bus to school on five consecutive days. Find the probability that Eliza was late to school on more than two of these days. Give your answer correct to three decimal places.

2 marks

The time it takes Eliza to walk from the bus stop to school is known to be normally distributed with a mean of 8 minutes and a standard deviation of σ minutes. 80% of the time Eliza's walk from the bus stop to the school will take more than 7 minutes.

f. Find the value of σ, correct to two decimal places.

2 marks

As she travels to school, Eliza will listen to music from her large collection of songs. 40% of these songs were released before the year 2000.

g. Find the probability that a random sample of 10 songs from Eliza's collection will contain exactly three songs released before the year 2000. Give your answer correct to 3 decimal places.

2 marks

For samples of size n from Eliza's collection, $\hat{P}$ is the random variable of the distribution of sample proportions of songs released before the year 2000.

h. For $n = 10$, find the probability that $\Pr(\hat{P} \geq 0.5 \mid \hat{P} \geq 0.3)$, correct to three decimal places. Do not use the normal approximation.

2 marks

i. Eliza takes a sample of n songs. For this sample, the standard deviation of $\hat{P}$ is $\frac{\sqrt{6}}{30}$. Find the value of n.

1 mark

Question 4 (21 marks)

Kaylee is an urban planner who is planning the construction of a new road. Kaylee is located at (0,0). An existing curved road nearby follows the curve with the equation $y = -(x-4)^3 + 2$, where $x \geq 0$. All measurements are in kilometres.

Kaylee is planning to build a straight road which will intersect the existing road at $x = k$. The straight road will be modelled by $y = mx$, where $0 \leq x \leq k$ and m is a positive real number.

a. Find the length, L kilometres, of the straight road, in terms of k.

1 mark

b. Find the values of m such that the straight road has a length of 5 kilometres. Give your answers correct to two decimal places.

3 marks

c. Find the value of m such that the length of the straight road is a minimum. Give your answer correct to two decimal places.

2 marks

d. Find the value of m such that the region bounded by the straight road, the existing road and the x-axis has an area of 18 square kilometres.

3 marks

Kaylee decides that the straight road should be built with $m = 2$. The straight road is constructed with this value of m.

e. Find the acute angle, θ degrees, between the straight road and the line $x = 0$. Give your answer to the nearest degree.

2 marks

Once the road is constructed, Kaylee returns to the point $(0,0)$. She then wishes to travel to the point $(3,3)$. She plans to travel along the straight road until she reaches $x = a$. When she reaches this point, Kaylee will leave the straight road and she will travel in a straight line towards the point $(3,3)$.

f. Find the distance from the point where Kaylee leaves the straight road to the point $(3,3)$, in terms of a.

1 mark

Kaylee will travel along the straight road at 6 km per hour. When she is not travelling along the straight road, she will travel at 3 km per hour.

g. Show that the total time, T hours, taken to get to the point $(3,3)$ is given by

$$T = \frac{2\sqrt{5a^2 - 18a + 18} + \sqrt{5}a}{6}$$

2 marks

h. Find the value for a for which the time taken to get to the point $(3,3)$ is a minimum. Give your answer in the form $\frac{1}{5}\left(m - \sqrt{n}\right)$, where m and n are positive integers.

2 marks

Instead, Kaylee decided to travel to the point $(3,3)$ by a different path. She will travel along the straight road until it intersects with the road that follows the curve with the equation $y = -(x-4)^3 + 2$. Then, she will travel along this curved road until she reaches the point $(3,3)$. The amount of time she will spend on the straight road will be the same as the length of time she will spend on the curved road.

i. Find Kaylee's average distance from the origin across the duration of her journey from the origin to the point $(3,3)$. Give your answer correct to two decimal places.

5 marks

END OF QUESTION AND ANSWER BOOK