MATHS METHODS

PRACTICE EXAM 1
SOLUTIONS
2019

Question 1 a.

Answer:

$$f'(x) = \frac{\cos(x)\,(x+2) - \sin(x)}{(x+2)^2}$$

Notes:

We can use the quotient rule to get the answer here. We can use the product rule when we have one function divided by another function, like

$$f(x) = \frac{u}{v}$$

where u and v represent different functions here. Then the derivative will be given by

$$f'(x) = \frac{u'v - v'u}{v^2}$$

where u' is the derivative of u and v' is the derivative of v. Note that the order in the numerator is very important here: $u'v$ needs a positive sign and needs a positive sign and $v'u$ needs a negative sign.

Alternatively, if you're not a big fan of the quotient rule, you could turn the function into the form $f(x) = (x+2)^{-1}\sin(x)$ and then use the product rule on it to find the derivative.

Anything equivalent to the above equation gets you one mark. I would recommend not trying to fiddle and simplify this fraction as VCAA will not be expecting you to.

Make sure you have brackets around the $(x + 2)$: your answer is not correct if you don't have these.

Note that this is a very standard opening question on an Exam 1. If you can't already, I would recommend learning how to work out a derivative with the quotient rule in just one step, which is a little bit of a time saver to allow you more time for the trickier questions later on.

Question 1 b.

Answer:

$$g'(x) = -x\log_e(x^2) + (3-x)\frac{2x}{x^2}$$

$$g'(1) = -1\log_e(1^2) + (3-1)\frac{2(1)}{1^2} = 0 + 4 = 4$$

Notes:

You would get 1 mark for a correct equation for the derivative, and 1 mark for the answer (4).

We can use the product rule to find the derivative here. The product rule is used when we have the product of two functions. Then, to evaluate $g'(1)$, substitute in $x = 1$.

This is another very standard component of the first question of an Exam 1. Again, I would recommend being able to work out a product rule in one simple line of working to save you time.

Question 2 a.

Answer:

$$f'(x) = \frac{1}{2\left(\cos\left(x + \frac{\pi}{4}\right)\right)^2}$$

Notes:

There is 1 mark for a derivative equal to the above equation. Note that you can write the bottom of the fraction as $2\cos^2\left(x + \frac{\pi}{4}\right)$, which involves the mathematical shorthand for the square of a trigonometric function.

If you do Specialist Maths, you can also write the equivalent $f'(x) = 2\sec^2\left(x + \frac{\pi}{4}\right)$.

Work out this derivative with the formula for deriving $\tan(x)$ on the formula sheet. In essence, we're using the chain rule, but since the derivative of $\left(x + \frac{\pi}{4}\right)$ is 1, we just end up with a 1 on the top of our fraction.

Question 2 b.

Answer:

$$m = \tan(\theta) = \tan(45°) = 1$$

$$f'(x) = \frac{1}{2\left(\cos\left(x + \frac{\pi}{4}\right)\right)^2} = 1$$

$$\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$x + \frac{\pi}{4} = \cdots, -\frac{7\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \ldots$$

$$x = \cdots, -\frac{5\pi}{2}, -\frac{\pi}{2}, 0, \frac{3\pi}{2}, \ldots$$

So $x = \frac{3\pi}{2}$

Notes:

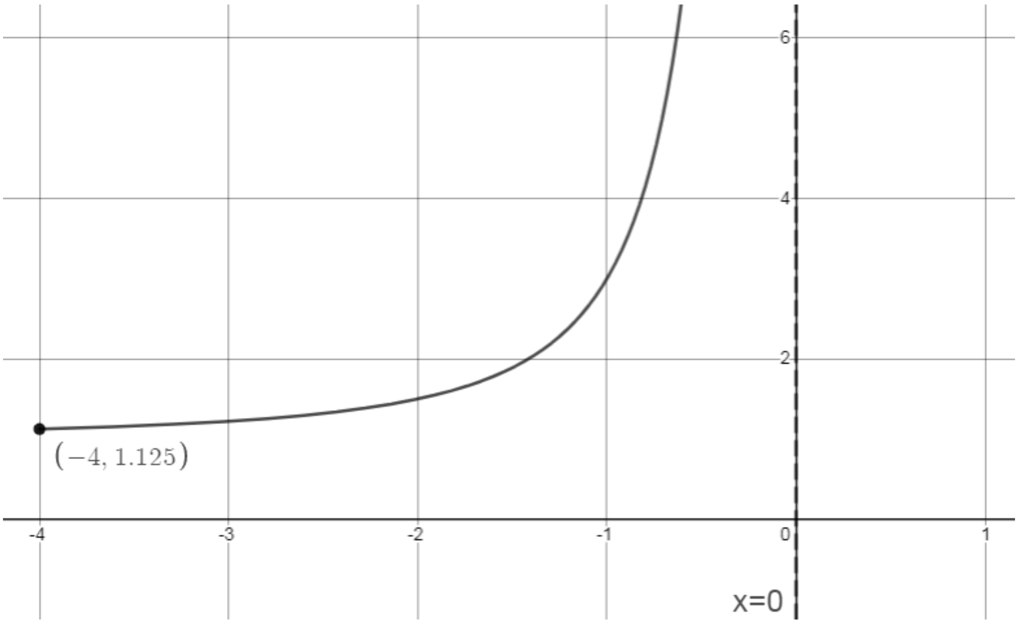
There is 1 mark getting the equation: $\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, and 1 mark for the answer $x = \frac{3\pi}{2}$.

The first step here is to note that we need to use the formula $m = \tan(\theta)$, where m is the gradient of f and θ is the angle asked for in the question. Substituting in $\theta = 45°$ and then using an exact value tells us that we want the gradient of f to be 1.

So we take the derivative $f'(x)$ and find when this is equal to 1. Solving for x yields a bunch of different answers, the question asks for the smallest positive one, so we'll take $\frac{3\pi}{2}$ as the answer.

Question 3 a.

Answer:



Notes:

You would get 1 mark for the correct asymptote at $x = 0$, labelled, dashed, and straight, 1 mark for the labelled endpoint at $\left(-4, \frac{5}{4}\right)$, and 1 mark for the correct shape, including the asymptotic behaviour.

If the function's domain was R, it would be necessary to label the asymptote at $y = 1$. However, due to the restricted domain, the graph does not approach this asymptote.

Make sure that you are careful plotting graphs that have asymptotes. VCAA is quite particular about a graph's asymptotic behaviour: they'll expect your graph to demonstrate this behaviour. So if your graph doesn't seem to approach the asymptote, or approaches but then curls a little away from it, they could take marks off there.

Question 3 b.

Answer:

$$\int_{-3}^{-1} \frac{2}{x^2} + 1\,dx = \left[-\frac{2}{x} + x\right]_{-3}^{-1} = \left(-\frac{2}{-1} + -1\right) - \left(-\frac{2}{-3} + -3\right) = 1 + \frac{7}{3} = \frac{10}{3}$$

Notes:

There is 1 mark for successfully antidifferentiating the function, and 1 mark for the answer $\frac{10}{3}$.

The area asked for will be found by integrating the function f between the bounds -3 and -1. The trickiest step here is the antidifferentiation. We need to bring the x^2 to the top of the fraction, changing it into the form x^{-2}.

Question 4 a.

Answer:

$$E(\hat{P}) = \frac{1}{3}$$

Notes:

For a sample, $E(\hat{P}) = p$. Here, the population proportion p is one third.

Question 4 b.

Answer:

$$\hat{P} = \frac{X}{n} = \frac{1}{3} = \frac{X}{3}$$

$$X = 1$$

$$\Pr\left(\hat{P} \geq \frac{1}{3}\right) = \Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - {}^{3}C_{0}\left(\frac{1}{3}\right)^{0}\left(1 - \frac{1}{3}\right)^{3-0} = 1 - \frac{3!}{0!\,(3-0)!}\left(\frac{2}{3}\right)^{3}$$

$$= 1 - \frac{8}{27} = \frac{19}{27}$$

Notes:

You would get 1 mark for $\Pr\left(\hat{P} \geq \frac{1}{3}\right) = 1 - \Pr(X = 0)$, or something equivalent, and 1 mark for the answer $\frac{19}{27}$.

We want to find the probability that the population proportion is equal to or greater than a third. To work this one out, we need to see what values of X will satisfy the required condition on the population proportion. A nice way to do that is by using the formula $\hat{P} = \frac{X}{n}$ where n is the sample size.

Using this formula gives us $\Pr\left(\hat{P} \geq \frac{1}{3}\right) = \Pr(X \geq 1)$. So that means that $X = 1$, 2 and 3 will all satisfy $\hat{P} \geq \frac{1}{3}$. We could work out each of those individually, or shortcut all that working out by noting that $\Pr(X \geq 1)$ is a complementary event to $\Pr(X = 0)$.

To work out $\Pr(X = 0)$, we can use a binomial formula: we have 3 trials and the probability of success is $\frac{1}{3}$. Evaluating and simplifying yields the answer of $\frac{19}{27}$.

Question 4c.

Answer:

$$\mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{\frac{1}{3}\left(1 - \frac{1}{3}\right)}{72}} = \sqrt{\frac{\frac{2}{9}}{72}} = \sqrt{\frac{2}{9} \times \frac{1}{72}} = \sqrt{\frac{1}{9} \times \frac{1}{36}} = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$$

Notes:

Answer must be in the form $\frac{1}{18}$ to receive the mark. The formula for standard deviation of a sample proportion, when assuming normal distribution, is on the formula sheet.

Question 4 d.

Answer:

$$E(\hat{P}) = p = \frac{1}{3}$$

$$\text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{\frac{1}{3}\left(1-\frac{1}{3}\right)}{72}} = \sqrt{\frac{\frac{2}{9}}{72}} = \sqrt{\frac{2}{9}\times\frac{1}{72}} = \sqrt{\frac{1}{9}\times\frac{1}{36}} = \frac{1}{3}\times\frac{1}{6} = \frac{1}{18}$$

$$\hat{P} \sim N\left(\frac{1}{3}, \frac{1}{18}\right)$$

$$\Pr\left(\hat{P} \geq \frac{1}{3}\right) = \Pr\left(Z \geq \frac{\frac{1}{3}-\frac{1}{3}}{\frac{1}{18}}\right) = \Pr(Z \geq 0) = 0.5$$

Notes:

There is 1 mark for some explanation of the answer $\Pr\left(\hat{P} > \frac{1}{3}\right) = 0.5$, and 1 mark for your answer. Acceptable explanations include comparing $\hat{P}$ to Z as done in the solution above, or giving reasoning about $\Pr\left(\hat{P} \geq \frac{1}{3}\right) = \Pr\left(\hat{P} \geq E(\hat{P})\right) = 0.5$ since $E(\hat{P})$ is also the median of $\hat{P}$.

We are told to assume that $\hat{P}$ is normally distributed. This means that we can compare the value $\frac{1}{3}$ on the $\hat{P}$ curve to the matching value on the Z curve using the formula

$$Z = \frac{X-\mu}{\sigma}$$

When we substitute in $\mu = \frac{1}{3}$ and $\sigma = \frac{1}{18}$, we find the Z value is zero. So $\Pr\left(\hat{P} \geq \frac{1}{3}\right) = \Pr(Z \geq 0)$. Since the median of the standard normal distribution is zero, $\Pr(Z \geq 0) = 0.5$.

Question 5a.

Answer:

$h(x) = (2\cos(x))^2$

Notes:

You could have also written this in an alternative way, such as $4\cos^2(x)$.

Question 5 b. i.

Answer:

$a = \infty$

Notes:

All real numbers can go into the function g, come out of g and then go into f. So R is a possible domain for h. However, g has a restricted domain (namely, $[0, a)$), so $[0, \infty)$ is the largest possible domain for h.

Question 5 b. ii.

Answer:

Range: $[0,4]$

Notes:

At the moment we know that $h(x) = (2\cos(x))^2$ and its domain is $[0, \infty)$. So, let's think about what will happen to the domain $[0, \infty)$ as it passes through h. As we feed those numbers into the $2\cos(x)$, the numbers coming out will be $[-2,2]$, the range of a cos graph with an amplitude of 2. Then all these numbers are squared, resulting in the range $[0,4]$.

Question 5 c.

Answer:

g^{-1} will exist when g is a one-to-one function, which is when g contains no turning points.

$$g'(x) = -2\sin(x) = 0$$

$$x = \cdots, 0, \pi, 2\pi, \ldots$$

Since the domain of g is $[0, a)$, $a = \pi$.

Notes:

You would get 1 mark for some explanation of how g will have an inverse function (it is not required to differentiate g), and 1 mark for the answer $a = \pi$.

Question 6 a.

Answer:

$$\Pr(A \cap B) = \Pr(A)\Pr(B) = \frac{2}{3}p$$

$$\Pr(B|A') = \frac{\Pr(B \cap A')}{\Pr(A')} = \frac{p - \frac{2}{3}p}{1 - \frac{2}{3}} = \frac{\frac{1}{3}p}{\frac{1}{3}} = p$$

Notes:

We can use the formula for independent events to find $\Pr(A \cap B)$. Note that this formula isn't on the formula sheet, so you do need to commit it to memory. Then, we can evaluate $\Pr(A|B')$ using the conditional probability formula.

Question 6 b.

Answer:

$$\Pr(B|A') = \frac{\Pr(B \cap A')}{\Pr(A')} = \frac{p}{1 - \frac{2}{3}} = \frac{p}{\frac{1}{3}} = 3p$$

Notes:

Drawing a Venn diagram is a great way to visualise this question. Since A and B are mutually exclusive, their Venn diagram circles will have no overlap.

Question 7 a.

Answer:

$$\Pr(\text{rejected}) = \frac{1}{2} + \left(1 - \frac{1}{2}\right)\left(\frac{2}{5}\right) + \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{5}\right)\frac{1}{6} = \frac{1}{2} + \frac{1}{5} + \frac{1}{30} = \frac{11}{15}$$

Notes:

A tree diagram is a great way to represent the information given in this question. Note that when a toaster is rejected, it will not be subjected to later tests. This means that branches of the tree diagram will end whenever a toaster is rejected.

Question 7 b.

Answer:

$$\Pr(\text{not rejected by first test} \mid \text{not rejected}) = \frac{\Pr(\text{not rejected by first test} \cap \text{not rejected})}{\Pr(\text{not rejected})}$$

$$= \frac{\frac{2}{5} + \left(1 - \frac{2}{5}\right)\frac{1}{6}}{1 - \frac{11}{15}} = \frac{\frac{2}{5} + \frac{1}{15}}{\frac{12}{15}} = \frac{\frac{7}{15}}{\frac{12}{15}} = \frac{7}{12}$$

Notes:

You would get 1 mark for the successful use of the conditional probability formula, and 1 mark for the answer $\frac{7}{12}$.

Use a tree diagram and the conditional probability formula. To work out the numerator of the fraction, we are only interested in the part of the tree diagram after the 'passed the first test' branch.

Question 8 a.

Answer:

Let $A = \sin(x)$

$$4A^2 + 1 = 4A$$

$$A^2 - A + \frac{1}{4} = 0$$

$$\left(A - \frac{1}{2}\right)^2 = 0$$

$$A = \frac{1}{2}$$

$$\sin(x) = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Notes:

There is 1 mark for the line $\left(A-\frac{1}{2}\right)^2=0$ or $\left(\sin(x)-\frac{1}{2}\right)^2=0$, or an equivalent statement, and 1 mark for the solutions $x=\frac{\pi}{6},\frac{5\pi}{6}$.

The equation we're asked to solve is a 'quadratic in disguise': there's a squared term, a linear term and a constant coefficient. This can be seen a lot more clearly when we make a substitution like $A=\sin(x)$. Making this substitution isn't necessary, but does make the solving a little 'cleaner'.

The solution of the quadratic is $\frac{1}{2}$, so we then need to solve $\sin(x)=\frac{1}{2}$.

Question 8 b.

Answer:

$$2\log_3(3-t)-\log_3(t+7)=2$$

$$\log_3\left(\frac{(3-t)^2}{t+7}\right)=2$$

$$\frac{(3-t)^2}{t+7}=3^2=9$$

$$(3-t)^2=9(t+7)$$

$$t^2-15t+54=0$$

$$(t+3)(t-18)=0$$

$$\text{So } t=-3 \text{ or } t=18.$$

$$-7<t<3, \text{ so } t=-3$$

Notes:

There is 1 mark for the line $t^2-15t+54=0$, and 1 mark for the solution $t=-3$. No answer mark is awarded if $t=18$ is listed as a valid solution.

Question 9 a.

Answer:

$$\frac{dy}{dx}=\cos(2\pi x+k)-2\pi x\sin(2\pi x+k)$$

Notes:

There is 1 mark for the correct derivative. Note that the product rule is needed to differentiate y.

Question 9 b.

Answer:

$$\frac{d}{dx}(x\cos(2\pi x+k)) = \cos(2\pi x+k) - 2\pi x\sin(2\pi x+k)$$

$$\int_0^1 \left(\frac{d}{dx}(x\cos(2\pi x+k))\right)dx = \int_0^1 (\cos(2\pi x+k))\,dx + \int_0^1 (-2\pi x\sin(2\pi x+k))\,dx$$

$$[x\cos(2\pi x+k)]_0^1 = \left[\frac{1}{2\pi}\sin(2\pi x+k)\right]_0^1 - 2\pi\int_0^1 (x\sin(2\pi x+k))\,dx$$

$$\cos(2\pi+k) - 0 = \frac{1}{2\pi}\sin(2\pi+k) - \frac{1}{2\pi}\sin(k) - 2\pi\int_0^1 (x\sin(2\pi x+k))\,dx$$

$$\cos(k) = \frac{1}{2\pi}\sin(k) - \frac{1}{2\pi}\sin(k) - 2\pi\int_0^1 (x\sin(2\pi x+k))\,dx$$

$$\cos(k) = -2\pi\int_0^1 (x\sin(2\pi x+k))\,dx$$

$$-\frac{1}{2\pi}\cos(k) = \int_0^1 (x\sin(2\pi x+k))\,dx$$

$$\mu = \int_0^1 (x\ \sin(2\pi x+k)+1)\,dx = \int_0^1 (x\ \sin(2\pi x+k))\,dx + \int_0^1 1\,dx = -\frac{1}{2\pi}\cos(k) + [x]_0^1$$
$$= -\frac{1}{2\pi}\cos(k) + 1 - 0 = \frac{1}{2\pi}(\pi - \cos(k))$$

Notes:

Using the formula for expected value gives us

$$\mu = \int_0^1 (x\ \sin(2\pi x+k)+1)\,dx$$

This is a function which we are not able to differentiate directly. However, we can recognise that it is similar to a part from **part a.**, so we can do differentiation by recognition.

It is important that in a 'Show that ...' question, all your working is clear and that you make no logical leaps in your working. Since you are given the answer to aim for, assessors are often harsh if they think you're just trying to bluff your way to the answer.

Question 9 c.

Answer:

$$\mu = \frac{1}{2\pi}(\pi - \cos(k))$$

$$\frac{d\mu}{dk} = \frac{1}{2\pi}\sin(k) = 0$$

$$\sin(k) = 0$$

$$k = \cdots, -\pi, \pi, 3\pi, \ldots$$

Since $0 \leq k < 2\pi$, $k = \pi$.

$$\mu = \frac{1}{2\pi}(\pi - \cos(\pi)) = \frac{1}{2\pi}\big(\pi - (-1)\big) = \frac{\pi + 1}{2\pi}$$

Notes:

You would be given 1 mark for correctly deriving μ with respect to k and setting this derivative equal to 0, 1 mark for $k = \pi$, and 1 mark for the solution $\mu = \frac{\pi+1}{2\pi}$, or an equivalent form of this answer.

In **part b.**, we are given an expression for μ in terms of k. So to find the maximum value of μ, we can differentiate this expression, set it equal to 0 and solve for k.