YEAR 12 *Trial Exam Paper*

2018

MATHEMATICAL METHODS

Written examination 2

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips on how to approach the exam.

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Section A – Multiple-choice questions

Question 1

Answer: D

Explanatory notes

The gradient of the line that passes through (–3, 2) and (1, 5) is $\frac{5-2}{1-(-3)}=\frac{3}{4}$.

Therefore, the gradient of the line perpendicular to the line that passes through (–3, 2) and (1, 5) is $-\frac{4}{3}$.

Tips

- *Never leave a multiple-choice question unanswered.*
- *There are a variety of methods for answering a multiple-choice question:*
  - *Calculate an answer and match the answer with an option.*
  - *Eliminate wrong options (some questions are deliberately designed so that this is the only reasonable method).*
  - *Guess (sometimes used in combination with the method above).*

Question 2

Answer: E

Explanatory notes

Method 1:

First, check if there is a maximum turning point inside the domain $x \in (-1, \ 7]$.

Either complete the square to get the 'turning point' form $(f(x) = -(x-2)^2 + 4 - b)$ or use calculus.

There is a maximum turning point at $x = 2$.

$f(-1) = -5 - b, \quad f(7) = -21 - b, \ f(2) = 4 - b.$

Therefore, $f(7) < f(-1) < f(2)$.

So the range is $[f(7), \ f(2)]$; that is, $[-21 - b, \ 4 - b]$.

Note: The end point $x = 7$ of the domain is included; therefore, the end point $f(7) = -21 - b$ of the range is included.

Method 2:

Substitute a convenient value for *b* and draw the graph for $x \in (-1, \ 7]$ using CAS. The range can be seen by inspection. Test each option by substituting the chosen value of *b* and reject the options that give the wrong range.

For example, choose $b = 0$ so that $f(x) = -x^2 + 4x$.

There is a maximum turning point at (2, 4).

By inspection, the range associated with the domain $x \in (-1, \ 7]$ is $[-21, \ 4]$.

Substitute $b = 0$ into each option and reject the options that do not give a range of $[-21, \ 4]$.

Answer: $[-21 - b, \ 4 - b]$

Question 3

Answer: B

Explanatory notes

The function is in the standard form $a\tan(bx + c) + d$:

$$f(x) = -2\tan\left(-\frac{x}{3} + \frac{2\pi}{5}\right) + 4$$

$$\text{Period} = \frac{\pi}{|b|} = \frac{\pi}{\frac{1}{3}} = 3\pi$$

Question 4

Answer: A

Explanatory notes

Reflect in the y-axis: $y=\frac{1}{\sqrt{-x}}$

Now translate 4 units left and 3 units down: $y=\frac{1}{\sqrt{-x-4}}-3$

Question 5

Answer: E

Explanatory notes

Note that $\text{ran} f=(-\infty,1]=\text{dom} f^{-1}$. Now find the inverse function f^{-1}. $y=1-2\sqrt{x+1}$, and swap x, y: $x=1-2\sqrt{y+1}$.

So $y=\frac{1}{4}(x-1)^2-1$.

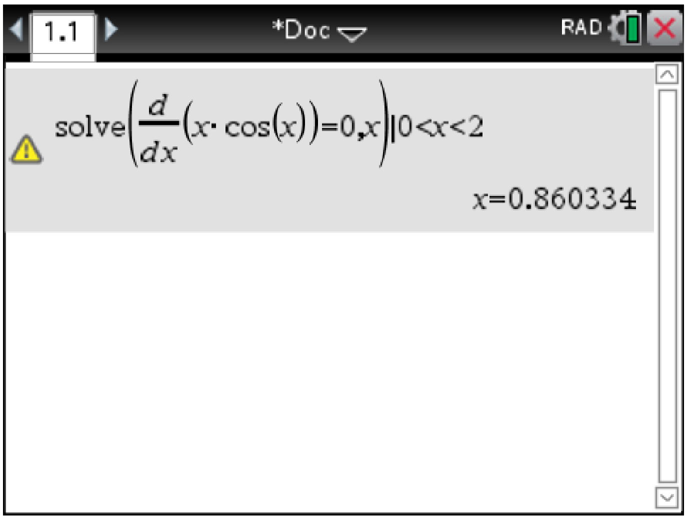
So $f^{-1}:(-\infty,1]\to R$, $f^{-1}(x)=\frac{1}{4}(x-1)^2-1$.

Question 6

Answer: D

Explanatory notes

Use CAS to draw a graph and to determine the turning point.



So $x=0.86$.

Question 7

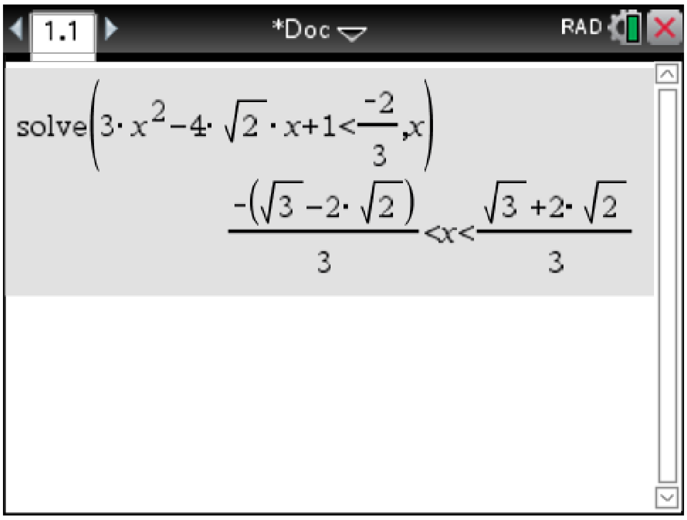
Answer: C

Explanatory notes

Require $\text{ran } g \subseteq \text{dom } f$.

Therefore, $g(x) = 3x^2 - 4\sqrt{2}x + 1 < -\frac{2}{3}$.

Solve $3x^2 - 4\sqrt{2}x + 1 < -\frac{2}{3}$ using CAS:



So $B = \left\{ x : \frac{2\sqrt{2} - \sqrt{3}}{3} < x < \frac{2\sqrt{2} + \sqrt{3}}{3} \right\}$

Warning:

The given domain of f is $\left(-\infty,\ -\frac{2}{3}\right)$ and this is NOT the maximal domain of f.

The maximal domain of f is $\left\{x:\ 2-3x>0 \Rightarrow x<\frac{2}{3}\right\}$.

Therefore, finding the rule for $f(g(x))$:

$$\begin{aligned} f(g(x)) &= \log_e\left(2-3g(x)\right) \\ &= \log_e(2-3(3x^2-4\sqrt{2}x+1)) \\ &= \log_e(-9x^2+12\sqrt{2}x-1) \end{aligned}$$

and using it:

$$-9x^2+12\sqrt{2}x-1>0 \qquad \Rightarrow \frac{2\sqrt{2}-\sqrt{7}}{3}<x<\frac{2\sqrt{2}+\sqrt{7}}{3} \text{ (option B)}$$

does NOT give the correct answer.

The method of finding the rule for $f(g(x))$ and then using this rule gives the correct maximal domain of $f(g(x))$ only when f is defined on its maximal domain.

Question 8

Answer: C

Explanatory notes

The coordinates of the *y*-intercept and the 'peak' define the scale of the *x*-axis to be such that the coordinates of the *x*-intercepts are $\left(\frac{a}{2}, 0\right)$ and $\left(\frac{3a}{2}, 0\right)$.

$$\overline{f} = \frac{\int_0^{2a} f(x)\,dx}{2a-0} = \frac{\int_0^{\frac{a}{2}} f(x)\,dx + \int_{\frac{a}{2}}^{\frac{3a}{2}} f(x)\,dx + \int_{\frac{3a}{2}}^{2a} f(x)\,dx}{2a}$$

$$\int_0^{\frac{a}{2}} f(x)\,dx = -\text{(area of triangle)} = -\frac{1}{2}\left(\frac{a}{2}\right)(a) = -\frac{a^2}{4}$$

$$\int_{\frac{a}{2}}^{\frac{3a}{2}} f(x)\,dx = \text{(area of triangle)} = \frac{1}{2}(a)(a) = \frac{a^2}{2}$$

$$\int_{\frac{3a}{2}}^{2a} f(x)\,dx = 0$$

Therefore:

$$\overline{f} = \frac{-\frac{a^2}{4} + \frac{a^2}{2} + 0}{2a} = \frac{a}{8}$$

Alternatively, this question can also be done using a geometric method by sectioning the area under the curve into known shapes and calculating their areas.

The area of the triangle from 0 to $\frac{a}{2}$ is $\frac{1}{2}\times\frac{a}{2}\times -a = -\frac{a^2}{4}$.

It can be seen that the area from $\frac{a}{2}$ to a is equal to, but has the opposite sign of, the previously calculated area, so is given by $\frac{a^2}{4}$.

This means that the area from 0 to a is $-\frac{a^2}{4}+\frac{a^2}{4}=0$.

The area of the triangle from a to $\frac{3a}{2}$ is $\frac{1}{2}\times\frac{a}{2}\times a=\frac{a^2}{4}$.

From $\frac{3a}{2}$ to $2a$ the area under the curve is 0.

Therefore, the total area under the curve is $\frac{a^2}{4}$.

Since the average value can be thought of as the height of a rectangle that has the same area as the area under the curve for the same domain, we can solve for this ‘height’.

$$\text{average value}\times 2a = \frac{a^2}{4}$$

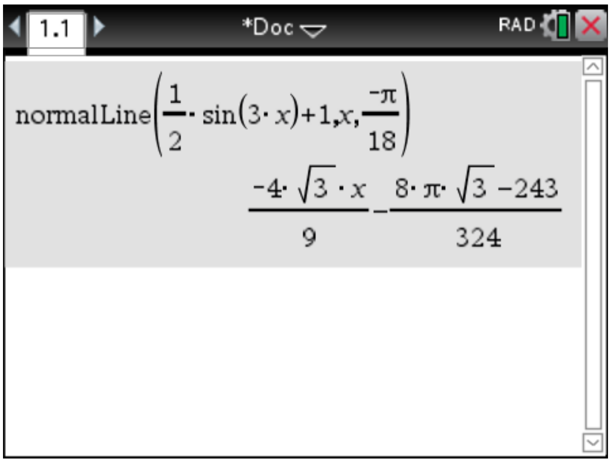
$$\Rightarrow \text{average value} = \frac{a}{8}$$

Question 9

Answer: E

Explanatory notes

Use CAS:



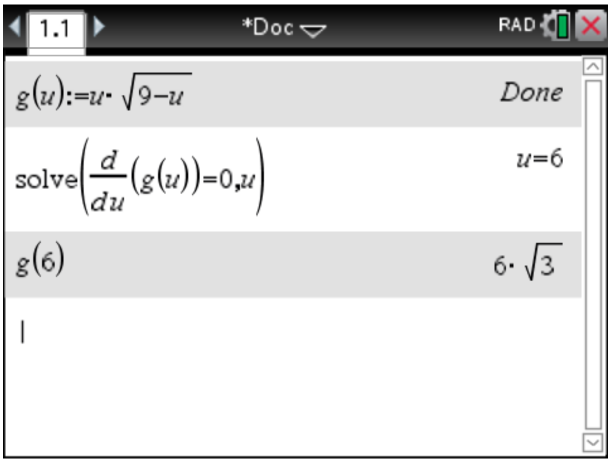
Question 10

Answer: B

Explanatory notes

The height of the rectangle $v = \sqrt{9 - u}$ and so the area $A = u\sqrt{9 - u}$.

The area is maximised when $\frac{dA}{du} = 0$. This can be done by hand or using CAS:



The area is $6\sqrt{3}$, which occurs when $u = 6$.

Question 11

Answer: B

Explanatory notes

From the quotient rule:

$$h'(x)=\frac{g'(x)f(x)-f'(x)g(x)}{\left(f(x)\right)^2}\qquad\Rightarrow h'(0)=\frac{g'(0)f(0)-f'(0)g(0)}{\left(f(0)\right)^2}$$

Substitute the given values:

$$h'(0)=\frac{(2)(3)-(-6)(4)}{(3)^2}=\frac{10}{3}$$

Question 12

Answer: E

Explanatory notes

Let $j(x)=\sqrt{g(x)}=\left(g(x)\right)^{\frac{1}{2}}$ so that $k(x)=f(x)\cdot j(x)$.

From the product rule:

$$k'(x)=f'(x)j(x)+f(x)j'(x)\qquad\Rightarrow k'(0)=f'(0)j(0)+f(0)j'(0)$$

From the chain rule:

$$j'(x)=\frac{g'(x)}{2\sqrt{g(x)}}\qquad\Rightarrow j'(0)=\frac{g'(0)}{2\sqrt{g(0)}}$$

Therefore:

$$\begin{aligned}k'(0)&=f'(0)j(0)+f(0)\frac{g'(0)}{2\sqrt{g(0)}}\\&=f'(0)\sqrt{g(0)}+f(0)\frac{g'(0)}{2\sqrt{g(0)}}\end{aligned}$$

Substitute the given values:

$$k'(0)=(-6)\sqrt{4}+(3)\frac{2}{2\sqrt{4}}=-\frac{21}{2}$$

Question 13

Answer: C

Explanatory notes

$h(x) = x(x+2) - (3x-2)$

$\text{dom}(h) = \left[-6, -\frac{3}{2}\right]$

From a simple graph, it is seen that $\text{ran}(h) = \left[\frac{23}{4},\ 44\right]$ (since there is no turning point in the interval $\left[-6, -\frac{3}{2}\right]$).

$\text{dom}\left(h^{-1}\right) = \text{ran}(h) = \left[\frac{23}{4},\ 44\right]$

Question 14

Answer: A

Explanatory notes

Definition: f is a strictly increasing function for an interval I when $f(x_1) > f(x_2)$ for all $x_2 > x_1$, where $x_1, x_2 \in I$.

Answer: [–2, 2]

Note: If $f'(x) > 0$ for an interval, then $f(x)$ is strictly increasing for that interval. But the converse is **not** always true:

If a function is strictly increasing for an interval, it is NOT always true that $f'(x) > 0$ for that **entire** interval. $f'(x)$ may equal zero or may not even exist at some points in the interval.

Solving $f'(x) > 0$ will still usually be an important step when trying to find the set of points for which $f(x)$ is strictly increasing but it does not always give the maximal set.

Question 15

Answer: A

Explanatory notes

The graph of $y = f(x)$ has turning points at $x = 2$ and $x \in (-2, \ -1)$.

Therefore, the graph of $y = f'(x)$ has x-intercepts at $x = 2$ and $x \in (-2, \ -1)$.

This eliminates options B, D and E.

From the graph of $y = f(x)$ it can be seen that $f'(x) < 0$ for $x \le -2$. This eliminates option C.

Option A is the only remaining option and therefore must be the answer.

Question 16

Answer: E

Explanatory notes

$\mathrm{E}(X) = \dfrac{40}{7}$:

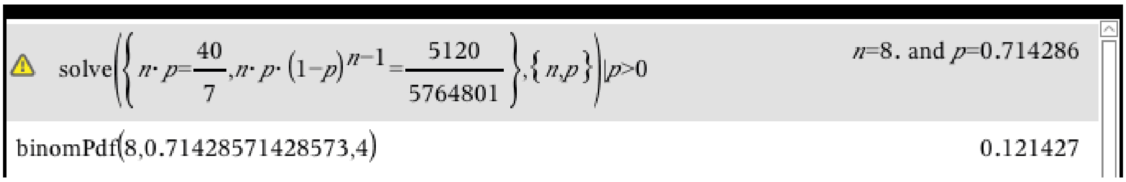
$np = \dfrac{40}{7}$ equation (1)

$\Pr(X = 1) = \dfrac{5120}{5\ 764\ 801}$:

$^{n}C_1 p^1 (1-p)^{n-1} = \dfrac{5120}{5\ 764\ 801}$

$\Rightarrow np(1-p)^{n-1} = \dfrac{5120}{5\ 764\ 801}$. equation (2)

Use the CAS calculator to solve equations (1) and (2) simultaneously for n and p:



$n = 8$ and $p = \dfrac{5}{7}$.

Use the CAS calculator to find $\Pr(X = 4)$.

Answer: 0.1214

Question 17

Answer: C

Explanatory notes

$\hat{p}$ is the midpoint of the interval: $\hat{p} = \dfrac{0.3023 + 0.4377}{2} = 0.37$

The value of $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ is found using one of the end points of the interval (the upper end point is used here):

$$0.37 + 1.96\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}} = 0.4377$$

$$\Rightarrow \sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}} = \frac{0.4377 - 0.37}{1.96} = 0.03454$$

The 90% confidence interval is then:

$(0.37 - 1.645 \times 0.03454, 0.37 + 1.645 \times 0.03454) = (0.3132, 0.4268)$.

Option C is the answer.

It should be noted that options D and E can be rejected immediately as they are wider than the original interval. The 90% confidence interval must be narrower than the corresponding 95% confidence interval.

Question 18

Answer: B

Explanatory notes

Requirement 1: $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$\int_0^1 4kx^3 + 3x^2 - k^2 dx = 1$

$\Rightarrow \left[kx^4 + x^3 - k^2x\right]_0^1 = 1 \qquad \Rightarrow k + 1 - k^2 = 1 \qquad \Rightarrow k(1-k) = 0$

$\Rightarrow k = 0, 1$

Requirement 2: $f(x) \geq 0$ for $x \in R$. In particular, it is required that $f(x) \geq 0$ for $0 < x \leq 1$.

$\underline{k = 0}$: $f(x) = \begin{cases} 3x^2 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Since $3x^2 > 0$ for $0 < x \leq 1$, $k = 0$ is a valid solution.

$\underline{k = 1}$: $f(x) = \begin{cases} 4x^3 + 3x^2 - 1 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

But $-1 < 4x^3 + 3x^2 - 1 < 6$ for $0 < x \leq 1$.

Therefore, $f(x)$ is not always greater than or equal to zero.

Therefore, $k = 1$ is **not** a valid solution.

Answer: $k = 0$

Question 19

Answer: A

Explanatory notes

The sum of the probabilities equals 1: $8a = 1 \Rightarrow a = \frac{1}{8}$

Then $E(X^2) = (-2)^2 \times 1/8 + (-1)^2 \times 2/8 + 0^2 \times 3/8 + 1^2 \times 2/8 = 1$.

$E(X) = -2 \times \frac{1}{8} - 1 \times \frac{2}{8} + 0 \times \frac{3}{8} + 1 \times \frac{2}{8} = -\frac{1}{4}$

$\text{Var}(X) = E(X^2) - (E(X))^2 = 1 - \frac{1}{16} = \frac{15}{16}$

Question 20

Answer: D

Explanatory notes

Method 1:

$$X = a \Rightarrow Z = \frac{a-1.5}{0.4}$$

therefore:

$$\Pr(X < a) = \Pr\left(Z > \frac{a}{3}\right) \quad \Rightarrow \Pr\left(Z < \frac{a-1.5}{0.4}\right) = \Pr\left(Z > \frac{a}{3}\right)$$

To compare the *Z*-values, both inequalities must be in the same direction.

By symmetry: $\Pr\left(Z > \frac{a}{3}\right) = \Pr\left(Z < -\frac{a}{3}\right)$

Therefore:

$$\Pr\left(Z < \frac{a-1.5}{0.4}\right) = \Pr\left(Z < -\frac{a}{3}\right)$$

$$\Rightarrow \frac{a-1.5}{0.4} = -\frac{a}{3}$$

Solve for *a* using the CAS calculator: $a = \frac{45}{34}$

Method 2:

Use the CAS calculator to test the equality of $\Pr(X < a) = \Pr\left(Z > \frac{a}{3}\right)$ for each option.

This method is time-inefficient.

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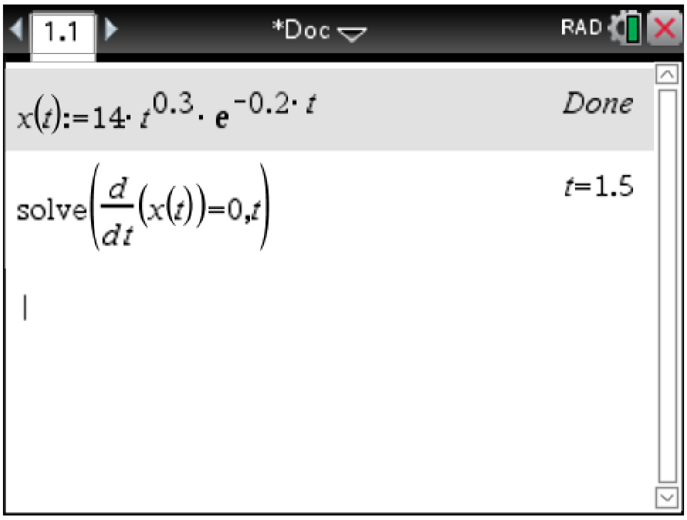
SECTION B

Question 1a.

Worked solution

Use a CAS calculator to find the maximum turning point by solving $x'(t) = 0$:



Answer: $t = \frac{3}{2}$ or 1.5.

Mark allocation: 1 mark

- 1 mark for correct answer

Tips

- *For **parts a.**, **b.**, **c.**, **d.** and **e.** it is advantageous to define and use the function $x(t) = 14t^{0.3}e^{-0.2\,t}$ on a CAS calculator. This should be identified during reading time.*
- *Be guided by the number of marks allotted for the question. In this case there is 1 mark, so there is no need to show any working, just use a CAS to get the answer.*

Question 1b.

Worked solution

From **part a.** $t = \frac{3}{2}$

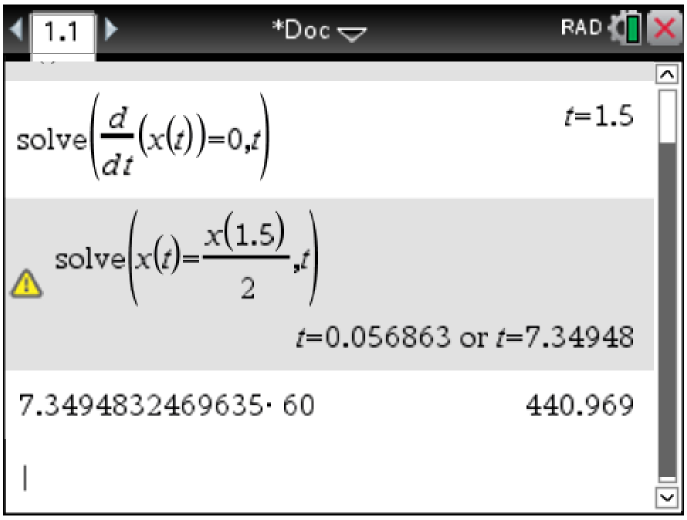
Therefore:

$$x_{\max} = 14\left(\frac{3}{2e}\right)^{0.3} \qquad \Rightarrow \frac{1}{2}x_{\max} = 7\left(\frac{3}{2e}\right)^{0.3}$$

Use a CAS calculator to solve $x(t) = 7\left(\frac{3}{2e}\right)^{0.3}$.

$t = 0.056863$ or $t = 7.349483$.

But the concentration is falling, therefore the larger value of t is required: $t = 7.349483$.



Answer: 441 minutes

Mark allocation: 2 marks

- 1 mark for the equation $x(t) = 7\left(\frac{3}{2e}\right)^{0.3}$ (where $x(t) = 14t^{0.3}e^{-0.2\,t}$ has been defined in **part a.**)
- 1 mark for correct answer

Tips

- *The answer to **part a.** is used to answer **part b.** It is often the case that an answer to an earlier part of a question will be either useful or required in order to answer a later part of the question.*
- *Graphing the function allows you to visualise how the blood concentration changes over time and is a way to check/validate your answer.*

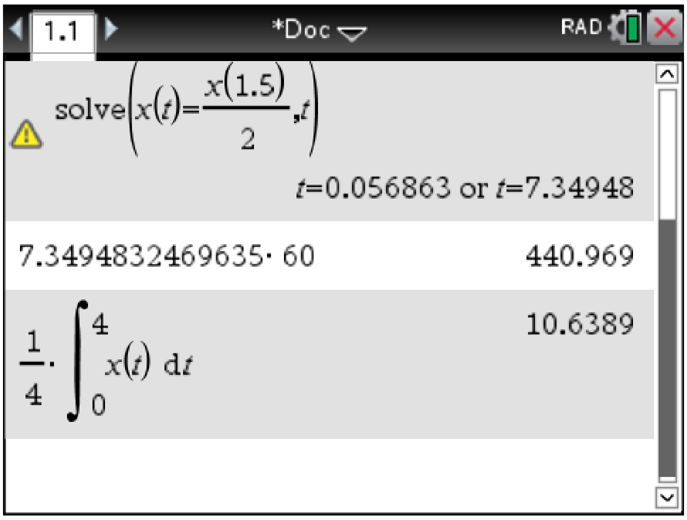
Question 1c.

Worked solution

9 a.m.: $t = 0$

1 p.m.: $t = 4$

$$\bar{x} = \frac{\int_0^4 x(t)\,dt}{4-0}, \text{ where } x(t) = 14t^{0.3}e^{-0.2t}.$$



Answer: 10.6389

Mark allocation: 2 marks

- 1 mark for an appropriate integral: $\bar{x} = \dfrac{\int_0^4 x(t)\,dt}{4-0}$
- 1 mark for correct answer

Tip

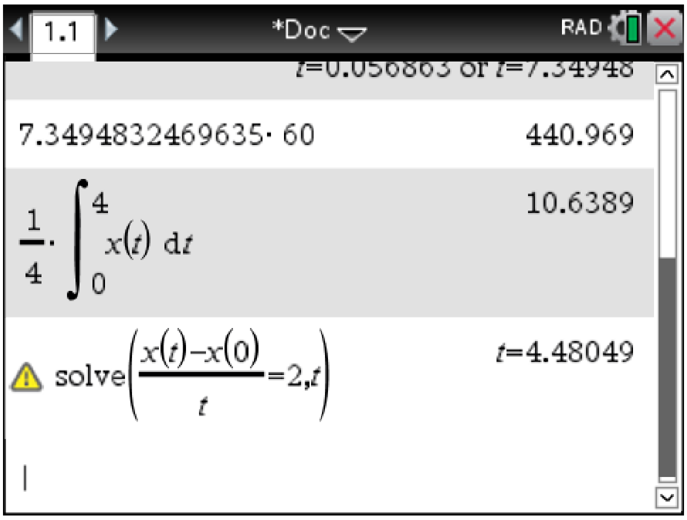
- *For* ***parts c., d., e.*** *and* ***f.*** *it is advantageous to define and use the function* $x(t) = 14t^{0.3}e^{-0.2t}$ *on CAS.*

Question 1d.

Worked solution

$$\frac{\Delta x}{\Delta t} = 2 = \frac{x(t_1) - x(0)}{t_1 - 0} = \frac{14t_1^{0.3}e^{-0.2\,t_1} - 0}{t_1}$$

Use a CAS calculator to solve $\dfrac{14t^{0.3}e^{-0.2\,t}}{t} = 2,\ t > 0.$



Answer: $t_1 = 4.48$

Mark allocation: 2 marks

- 1 mark for solving $\dfrac{14t^{0.3}e^{-0.2\,t}}{t} = 2$
- 1 mark for correct answer

Question 1e.

Worked solution

First dose at 9 a.m.: $t = 0$

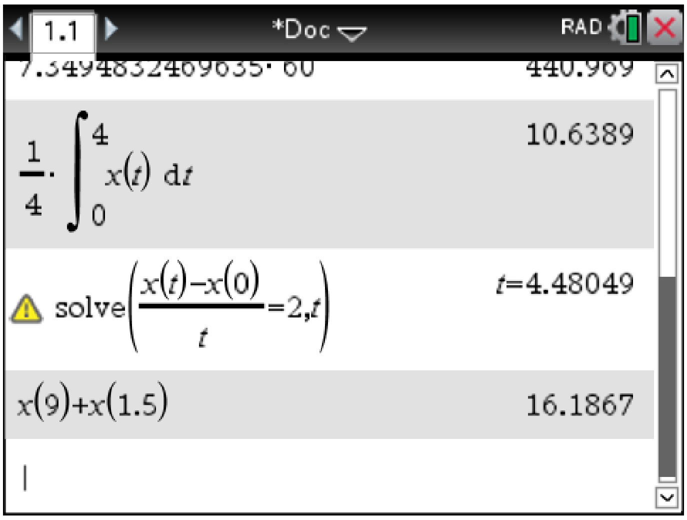
Second dose at 4.30 p.m.: $t = 7.5$

Define $C(t)$ to be the concentration of drug in the bloodstream t hours after the first dose:

$C(t) = x(t) + x(t - 7.5)$, where t is the number of hours after the first dose.

6 p.m.: $t = 9$

$C(9) = x(9) + x(9 - 7.5) = x(9) + x(1.5)$



Answer: 16.1867

Mark allocation: 2 marks

- 1 mark for the equation $C(t) = x(t) + x(t - 7.5)$ and definition of $C(t)$
- 1 mark for correct answer

Question 1f.

Worked solution

From a CAS calculator: $\frac{dx}{dt} = a\left(\frac{b}{t} - c\right)e^{-ct}$

Solve $\frac{dx}{dt} = 0$: $t = \frac{b}{c}$.

Substitute $t = \frac{b}{c}$ into $x = at^b e^{-ct}$:

Answer: $x_{\max} = a\left(\frac{b}{c}\right)^b e^{-b} = a\left(\frac{b}{ec}\right)^b$.

Mark allocation: 2 marks

- 1 mark for finding $t = \frac{b}{c}$
- 1 mark for correct answer

Question 1g.i.

Worked solution

x has a maximum value when $t = 1.5$ (i.e. 90 minutes).

From **parts a.** and **f.**: $\frac{b}{c} = 1.5$ equation (1)

Substitute $x = 12$ when $t = 1.5$ into $x = at^b e^{-ct}$:

$12 = a(1.5)^b e^{-1.5c}$

Substitute equation (1): $12 = a(1.5)^b e^{-b}$ equation (2)

x has half of its maximum value when $t = 7.5$.

Substitute $x = 6$ when $t = 7.5$ into $x = at^b e^{-ct}$:

$6 = a(7.5)^b e^{-7.5c}$ equation (3)

Substitute equation (1): $6 = a(7.5)^b e^{-5b}$

$$\frac{(2)}{(3)}: \quad \frac{12}{6} = \frac{a(1.5)^b e^{-b}}{a(7.5)^b e^{-5b}} = \frac{(1.5)^b e^{-b}}{(7.5)^b e^{-5b}}$$

$$\Rightarrow 2 = \frac{(1.5)^b e^{-b}}{(7.5)^b e^{-5b}} = \left(\frac{1.5}{7.5}\right)^b e^{4b} = \left(\frac{1}{5}\right)^b e^{4b} = \left(\frac{e^4}{5}\right)^b$$

Use a CAS calculator to solve for b:

Answer: $b = \frac{\log_e(2)}{2^2 - \log_e(5)}$

Mark allocation: 3 marks

- 1 mark for equation (1)
- 1 mark for equation (2)
- 1 mark for correct answer

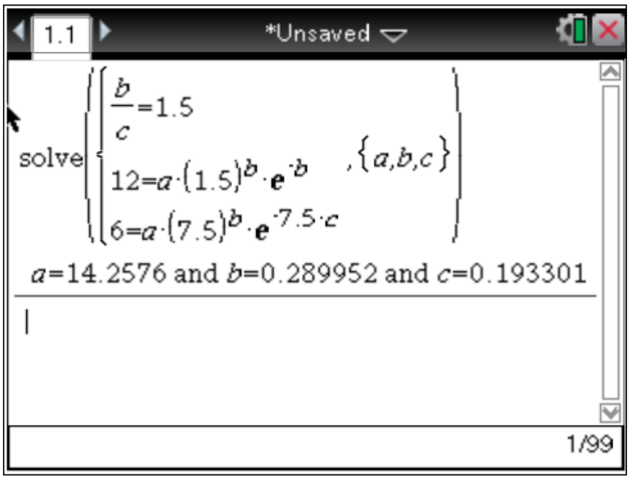
Tip

- *Make sure your final answer is written in the required form* $\frac{\log_e(p)}{p^2 - \log_e(q)}$, *where* p *and* q *are positive integers.*

Question 1g.ii.

Worked solution

From **part g.i.**, solve equations (1), (2) and (3) simultaneously, using a CAS calculator:



Answers: $a = 14.2576$ and $c = 0.1933$.

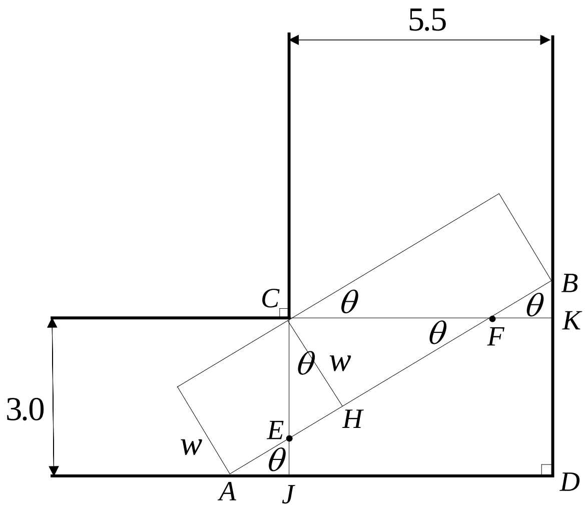
Mark allocation: 1 mark

- 1 mark for correct answers

Question 2a.i.

Worked solution

The following annotated diagram will be useful for defining notation that facilitates writing out the solutions and helps identify useful right-angled triangles for answering this question.



From the right-angled triangle *ECH*:

$$\cos(\theta) = \frac{w}{EC}$$

$$\Rightarrow EC = \frac{w}{\cos(\theta)}$$

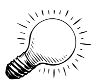
From the right-angled triangle *CFH*:

$$\sin(\theta) = \frac{w}{CF}$$

$$\Rightarrow CF = \frac{w}{\sin(\theta)}$$

Mark allocation: 2 marks

- 1 mark for recognising and using an appropriate right-angled triangle
- 1 mark for correct answers



Tip

- *It is often useful to annotate a given diagram in order to define notation that facilitates writing out a solution. In this case, it also helps identify useful right-angled triangles for answering the question.*

Question 2a.ii.

Worked solution

From the right-angled triangle *ECF*:

$$(EF)^2 = (EC)^2 + (CF)^2 \quad = \left(\frac{w}{\cos(\theta)}\right)^2 + \left(\frac{w}{\sin(\theta)}\right)^2$$

Simplify for *EF* using a CAS calculator: $EF = \dfrac{w}{\sin(\theta)\cos(\theta)}$, which is to be shown.

<u>Alternative solution:</u>

For the right-angled triangle *ECF*, where the angle at vertex *F* is θ we can say that:

$$\cos(\theta) = \frac{\frac{w}{\sin(\theta)}}{EF} \quad \text{or} \quad \sin(\theta) = \frac{\frac{w}{\cos(\theta)}}{EF}$$

Rearranging either ratio to make *EF* the subject will give the required result:

$$EF = \frac{w}{\sin(\theta)\cos(\theta)}$$

Mark allocation: 1 mark

- 1 mark for recognising an appropriate right-angled triangle, applying Pythagoras' theorem and substituting the result from **part a.i.**

OR

- 1 mark for using appropriate right-angled triangle trigonometric ratios with results from **part a. i.** to show the length

Tips

- *This is a 'show' question; therefore, extra care must be taken to give appropriate working. One way of answering a 'show' question such as this one is to follow the steps below:*

  1. Ignore the given result and instead treat the question as

  'Find, in terms of w and θ, an expression for length EC.'

  2. Write out a solution.

  3. Compare your answer with the given expression.

- *The result given in a 'show question' will usually be needed in the solution to a later question. In this case, it is needed in* ***part c.***
- *The word 'Hence …' means that the answer from the previous part must be used as part of the solution.*

Question 2b.i.

Worked solution

From the right-angled triangle *AEJ*:

$$\sin(\theta) = \frac{JE}{AE} = \frac{JC - EC}{AE}$$

$$\Rightarrow AE = \frac{JC - EC}{\sin(\theta)}$$

Substitute $EC = \dfrac{w}{\cos(\theta)}$ (from **part a.i.**) and $JC = 3.0$: $AE = \dfrac{3.0 - \dfrac{w}{\cos(\theta)}}{\sin(\theta)}$

Simplify using a CAS calculator: $AE = \dfrac{3\cos(\theta) - w}{\sin(\theta)\cos(\theta)}$, which is to be shown.

Alternatively, from the diagram it can be seen that $AF = \dfrac{3}{\sin(\theta)}$.

So $AE = AF - EF = \dfrac{3}{\sin(\theta)} - \dfrac{w}{\sin(\theta)\cos(\theta)}$.

Simplifying the above then gives the result $AE = \dfrac{3\cos(\theta) - w}{\sin(\theta)\cos(\theta)}$, as required.

Mark allocation: 2 marks

- 1 mark for recognising and using an appropriate right-angled triangle
- 1 mark for substituting the answer to **part a.i.**

OR

- 1 mark for recognising $AF = \dfrac{3}{\sin(\theta)}$
- 1 mark for using the answer from **part a.ii.** in the equation $AE = AF - EF$ and simplifying to get the required result

Question 2b.ii.

Worked solution

From the right-angled triangle *FKB*:

$$\cos(\theta) = \frac{FK}{FB} = \frac{CK - CF}{FB}$$

$$\Rightarrow FB = \frac{CK - CF}{\cos(\theta)}.$$

Substitute $CF = \dfrac{w}{\sin(\theta)}$ and $CK = 5.5$:

$$FB = \frac{5.5 - \dfrac{w}{\sin(\theta)}}{\cos(\theta)}$$

Simplify using CAS.

Answer: $FB = \dfrac{5.5\sin(\theta) - w}{\sin(\theta)\cos(\theta)}$

Mark allocation: 3 marks

- 1 mark for recognising and using an appropriate right-angled triangle
- 1 mark for substituting $CF = \dfrac{w}{\sin(\theta)}$ and $CK = 5.5$
- 1 mark for correct answer

Question 2c.

Worked solution

$$L = AE + EF + FB$$

Substitute $AE = \dfrac{3\cos(\theta) - w}{\sin(\theta)\cos(\theta)}$, $EF = \dfrac{w}{\sin(\theta)\cos(\theta)}$, $FB = \dfrac{5.5\sin(\theta) - w}{\sin(\theta)\cos(\theta)}$:

$$L = \frac{3\cos(\theta) + 5.5\sin(\theta) - w}{\cos(\theta)\sin(\theta)}$$

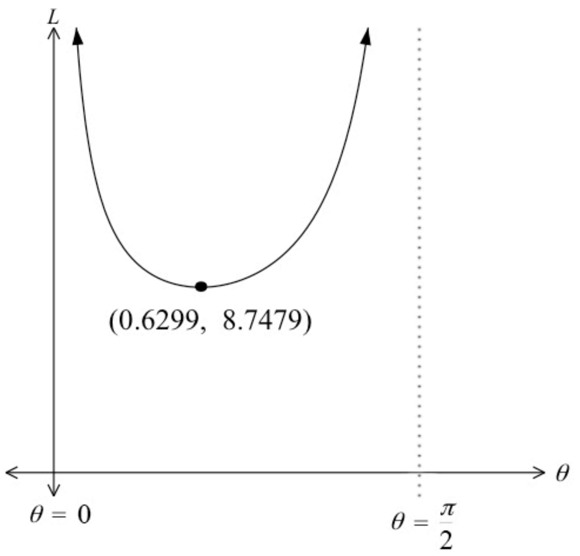
Mark allocation: 1 mark

- 1 mark for recognising *L* as a sum of three lengths and substituting correct expressions for each of those lengths

Question 2d.i.

Worked solution

$$L = \frac{3\cos(\theta) + 5.5\sin(\theta) - 1.5}{\cos(\theta)\sin(\theta)}$$



Minimum turning point: (0.6299, 8.7479)

Vertical asymptotes: $\theta = 0$ and $\theta = \frac{\pi}{2}$

Vertical asymptotes will occur when either sin (θ) = 0 or cos (θ) = 0.

Mark allocation: 3 marks

- 1 mark for correctly labelled minimum turning point (0.6299, 8.7479)
- 1 mark for correctly labelled vertical asymptotes $\theta = 0$ and $\theta = \frac{\pi}{2}$
- 1 mark for correct shape

Tips

- *Make sure that you label all of the features that the question asks you to label.*
- *Make sure that you use the accuracy specified in the question (four decimal places) when labelling the coordinates of the stationary points.*
- *Make sure that the correct symbols are used when labelling equations of asymptotes. The correct equations of the vertical asymptote are $\theta = 0$ and $\theta = \frac{\pi}{2}$* ***not*** *$x = 0$ and $x = \frac{\pi}{2}$.*

Question 2d.ii.

Worked solution

Width of widest possible rectangle is equal to width of narrowest part of polygon:

Answer: $w = 3.0$

Mark allocation: 1 mark

- 1 mark for correct answer

Question 2d.iii.

Worked solution

For a given width, the longest rectangle that can be rotated within the polygon will be the shortest one that remains in contact with the inside corner and walls.

Therefore, a rectangle of width 1.5 units and length 4.4 units can be rotated within the polygon because $L_{\text{rectangle}} = 4.4 < L_{\text{min}} = 8.7479$.

Mark allocation: 1 mark

- 1 mark for correct conclusion and justification ($L_{\text{rectangle}} = 4.4 < L_{\text{min}} = 8.7479$)

Question 3a.i.

Worked solution

Define the random variable:

Let X be the random variable: amount of homework completed on a given weeknight by a Year 9 student.

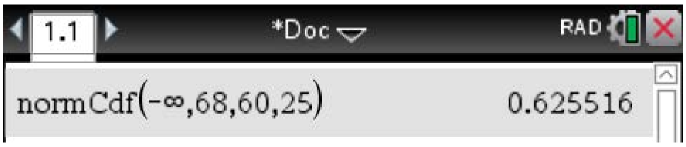
Define the distribution that the random variable follows:

$X \sim \text{Normal}(\mu = 60,\ \sigma = 25)$

Define the problem in terms of a probability statement:

$\Pr(X < 68) = ?$

Use the Normal Cdf command on a CAS calculator:



Answer: 0.6255

Mark allocation: 1 mark

- 1 mark for correct answer

Tips

- *Even though the question is worth only 1 mark and therefore only an answer is required, it is important that you do the following:*
  - *clearly define the random variable relevant to the question (use a different symbol for each new random variable)*
  - *define the distribution that the random variable follows*
  - *define the problem in terms of a probability statement (this is very important because it helps clarify in your mind how to get the correct answer, and it is often helpful for showing appropriate working in questions that follow, which are worth more than 1 mark)*
  - *correctly round and give your answer correct to the required accuracy.*

Question 3a.ii.

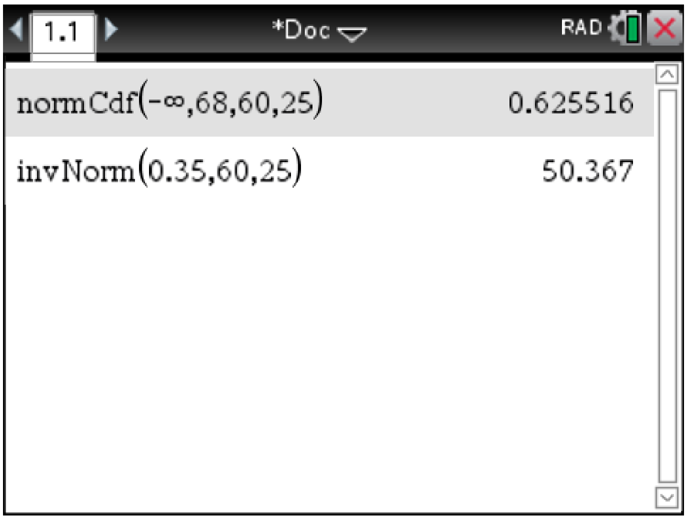
Worked solution

Define the problem in terms of a probability statement:

$\Pr(X > a) = 0.65$

$\Rightarrow \Pr(X < a) = 0.35$

Use the Inverse Normal command on CAS:



Answer: $a = 50.4$

Check for reasonableness:

Since $\Pr(X > a) > 0.5$, the answer must be less than the mean of 60. ✓

Mark allocation: 1 mark

- 1 mark for correct answer

Tips

- *Where possible and when time permits, it is useful to check an answer for its reasonableness.*

  Since $\Pr(X > a) > 0.5$ *the answer must be less than the mean of 60.* ✓

- *Even though the question is worth only 1 mark and therefore working is not required, you should nevertheless define the problem in terms of a probability statement using a clearly defined random variable (this variable may have been defined in a previous part of the question, as is the case here) because it helps clarify in your mind how to get the correct answer.*

Question 3b.i.

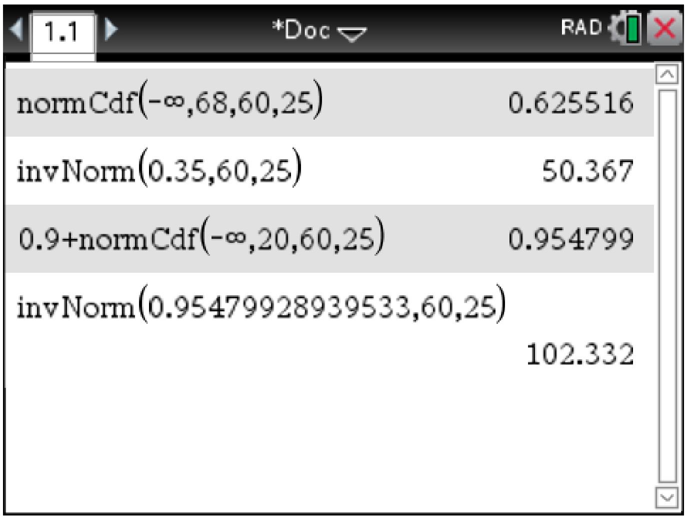
Worked solution

Define the problem in terms of a probability statement:

$$\Pr(20 < X < b) = 0.9$$
$$\Rightarrow \quad \Pr(X < b) - \Pr(X < 20) = 0.9$$
$$\Rightarrow \quad \Pr(X < b) = 0.9 + \Pr(X < 20)$$
$$\Rightarrow \quad \Pr(X < b) = 0.954799$$

Note: To avoid a rounding error in the final answer, more accuracy than is required in the final answer should always be used during a calculation. Since the final answer must be correct to one decimal place, a value of $\Pr(X < 20)$ that is correct to six decimal places is used.

Use the Inverse Normal command on CAS:



Answer: $b = 102.3$

Check for reasonableness:

Use the Normal Cdf command on a CAS calculator to calculate

$\Pr(20 < X < 102.3)$: $\Pr(20 < X < 102.3) = 0.9$ ✓

Mark allocation: 2 marks

- 1 mark for the equation $\Pr(X < b) - \Pr(X < 20) = 0.9$
- 1 mark for correct answer

Tip

- *Make sure that you define the problem in terms of a probability statement using a clearly defined random variable.*

Question 3b.ii.

Worked solution

Define the problem in terms of a probability statement:

The idea of conditional probability is required.

The given condition (i.e. the event that is known to have occurred) is that the student is typical:

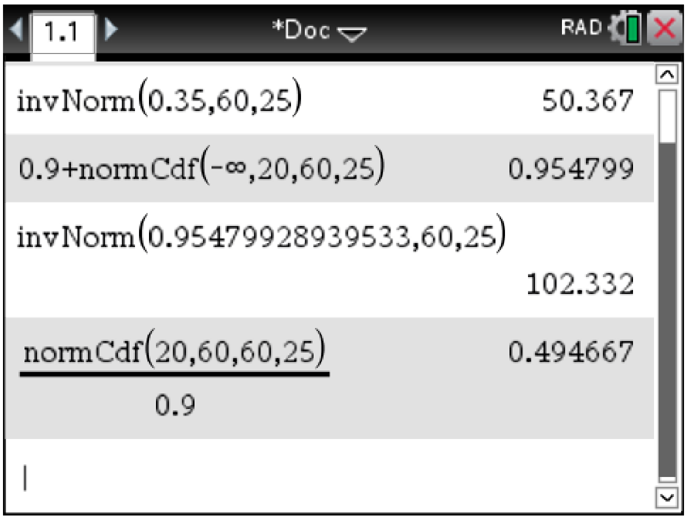
$\Pr(20 < X < 60 \mid \text{typical student}) = ?$

Note: Typical student means $20 < X < b$ therefore the required event is $20 < X < 60$ **not** $X < 60$.

$$\Pr(20 < X < 60 \mid \text{typical student}) = \frac{\Pr(20 < X < 60)}{\Pr(20 < X < b)}$$

Substitute $\Pr(20 < X < b) = 0.9$:

$$\frac{\Pr(20 < X < 60)}{0.9} = 0.4947$$



Answer: 0.4947

Mark allocation: 2 marks

- 1 mark for recognition of conditional probability and 0.9 in the denominator
- 1 mark for correct answer

Question 3c.i.

Worked solution

Define the random variable:

Let Y be the random variable: number of Year 9 students that will complete between 40 and 60 minutes of homework on a given weeknight.

Define the distribution that the random variable follows:

$Y \sim \text{Binomial}(n = 9,\ p = \Pr(40 < X < 60))$

Use the Normal Cdf command on a CAS calculator to calculate

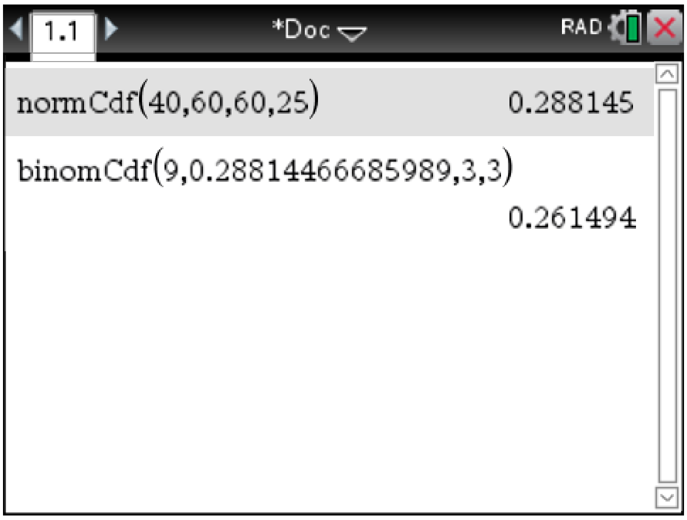
$p = \Pr(40 < X < 60)$

and so

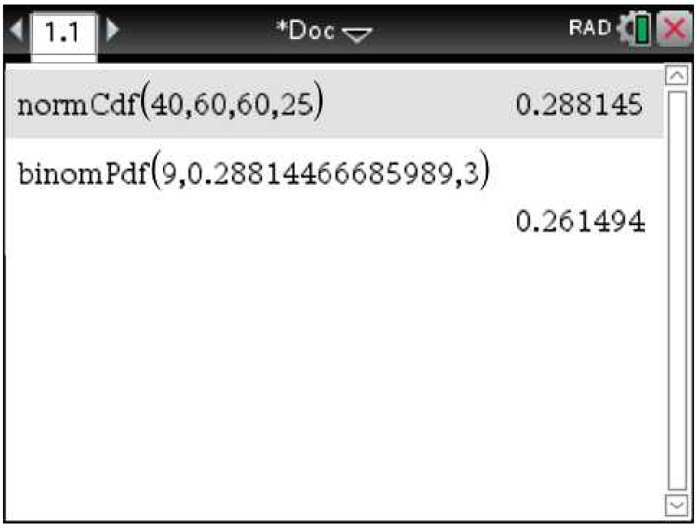
$Y \sim \text{Binomial}(n = 9, p = 0.288145)$

Define the problem in terms of a probability statement: $\Pr(Y = 3) = ?$

Use the Binomial Cdf command on CAS:



Alternatively, use the Binomial Pdf command on CAS:



Answer: 0.2615

Mark allocation: 2 marks

- 1 mark for recognition of the binomial distribution with correct values $n = 9$ and $p = \Pr(40 < X < 60)$ for its parameters
- 1 mark for correct answer

Tips

- *Be ready for a 'switch' of random variable (and a possible link to the old random variable). Switching from a continuous random variable (such as normal) to a discrete random variable (such as binomial) is a common question type.*
- *Make sure that you clearly define a new random variable. A different symbol (such as Y) from the ones used in* **parts a.** *and* **b.** *must be used.*
- *Make sure that you define the distribution that the random variable follows.*

Question 3c.ii.

Worked solution

Define the random variable:

Let W be the random variable: number of Year 9 students who complete an insufficient amount of homework.

Define the distribution that the random variable follows:

$W \sim \text{Binomial}(n = \ ?,\ p = \Pr(X < 20))$

Substitute $p = \Pr(X < 20) = 0.054799$ (found during the calculation in **part b.i.**):

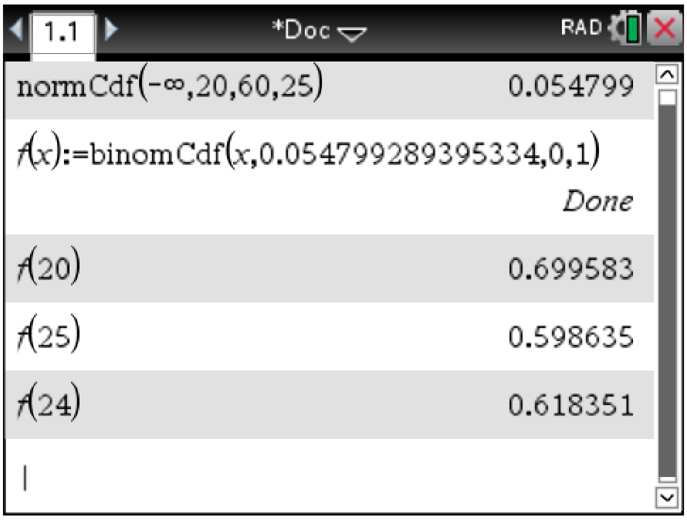
$W \sim \text{Binomial}\left(n = \ ?,\ p = 0.054799\right)$

Define the problem in terms of a probability statement:

The smallest value of n such that $\Pr(W > 1) > 0.4$ is required.

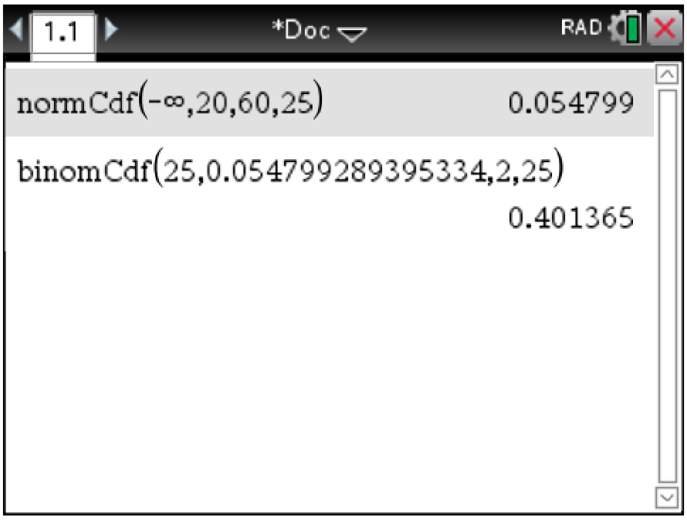
$\Pr(W > 1) > 0.4 \Rightarrow \Pr\left(W \le 1\right) < 0.6$

Define the function f(x) = binomcdf (x, 0.054799, 0, 1) on a CAS calculator and use a table (or trial and error, as shown here) to obtain the answer:



Answer: $n = 25$

Note that we can check this result easily:



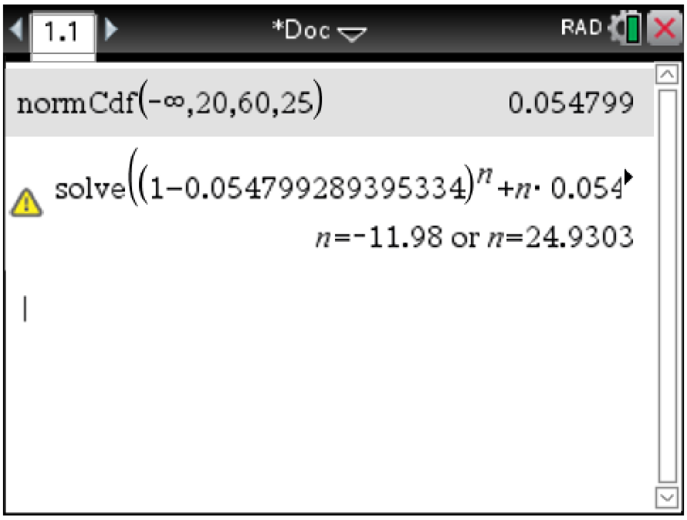
Alternatively, we could solve $\Pr(W=0)+\Pr(W=1)=0.6$,

Where $W \sim \text{Binomial}(n= \ ?,\ p=0.054799)$.

Note that:

$$\begin{aligned}\Pr(W=0)+\Pr(W=1) &= \binom{n}{0}p^0(1-p)^n+\binom{n}{1}p^1(1-p)^{n-1}\\ &=(1-p)^n+np(1-p)^{n-1}\end{aligned}$$

Use the CAS to solve:



So the smallest value of n is 25.

Mark allocation: 3 marks

- 1 mark for recognition of the binomial distribution with correct value of the parameter $p = 0.054799$
- 1 mark for $\Pr(W>1)>0.4 \Rightarrow \Pr(W \le 1)<0.6$ or equivalent
- 1 mark for correct answer

Tips

- *Finding a minimum sample size is a common question type. Make sure you can use your CAS calculator in the way described in the solution to answer this type of question.*
- *Make sure that you clearly define a new random variable. A different symbol (such as W) from the ones used in* ***parts a.***, ***b.*** *and* ***c.i.*** *must be used.*

Question 3d.

Worked solution

Let V be the random variable: number of Year 9 students who complete more than 90 minutes of homework on a given night.

Then $\hat{P} = \frac{V}{60}$, where $V \sim \text{Binomial}(n = 60,\ p = \Pr(X > 90))$.

Use the Normal Cdf command on a CAS calculator to calculate $p = \Pr(X > 90)$: $p = 0.11507$

Therefore,

$V \sim \text{Binomial}(n = 60, p = 0.11507)$.

$\hat{P} = 0.10 \Rightarrow V = 6$.

$\hat{P} = 0.15 \Rightarrow V = 9$.

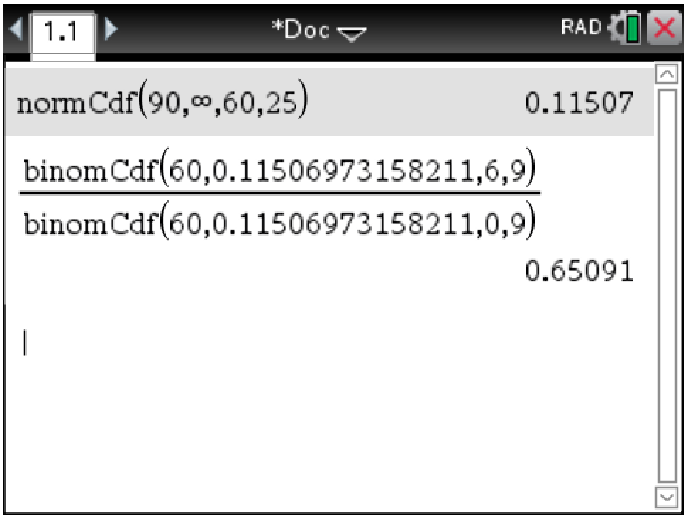
Therefore:

$\Pr(\hat{P} \geq 0.10 \mid \hat{P} \leq 0.15)$

$= \Pr(V \geq 6 \mid V \leq 9)$

$= \dfrac{\Pr(6 \leq V \leq 9)}{\Pr(V \leq 9)}$.

Use the Binomial Cdf command on a CAS calculator to calculate the answer:



Answer: 0.6509

Mark allocation: 4 marks

- 1 mark for definition of $\hat{P} = \frac{V}{60}$
- 1 mark for correct values $n = 60$ and $p = 0.11507$ for its parameters
- 1 mark for correct statement of the conditional probability, with the random variable defined
- 1 mark for correct answer

Tips

- *It is often convenient to define the sample proportion of a population in terms of a binomial random variable.*
- *Again, make sure that you clearly define a new random variable. A different symbol (such as V) from the ones used in previous parts must be used.*
- *Make sure that you define the distribution that the random variable follows.*
- *Defining the random variable and its distribution and defining the problem in terms of a probability statement will often be recognised as appropriate working. Use a CAS to get the final answer.*

Question 3e.

Worked solution

Define the random variable.

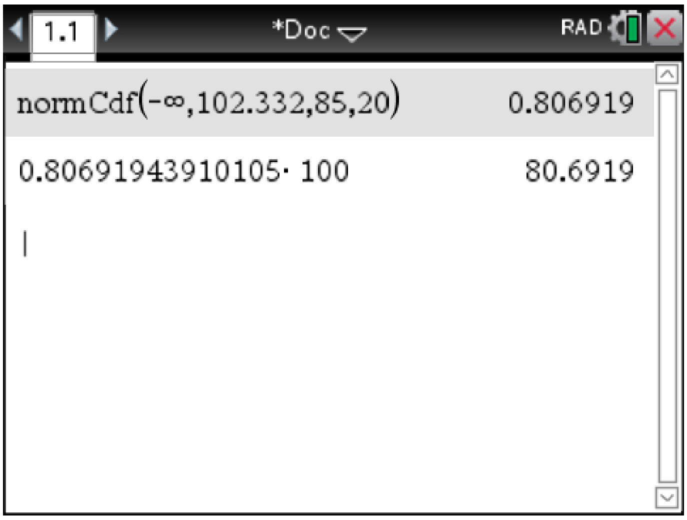
Let L be the random variable: amount of time required for a Year 9 student to complete their homework.

Define the distribution of the random variable as follows:

$L \sim \text{Normal}(\mu = 85,\ \sigma = 20)$

Define the problem in terms of a probability statement:

$\Pr(L < b) = ?$ where $b = 102.3$ (from **part b.i.**)



Answer: 80.69%

Mark allocation: 2 marks

- 1 mark for defining the problem in terms of a probability statement, using clearly defined random variables
- 1 mark for correct answer

Question 3f.

Worked solution

Define the random variable.

Let C be the random variable: amount of homework required for a Year 7 student to complete their homework.

Define the distribution the random variable follows:

$$C \sim \text{Normal}\left(\mu = 30,\ \sigma = \ ?\right)$$

Define the problem in terms of a probability statement:

The value of σ such that $\Pr(C > 25) = 0.83$ is required.

$\Pr(Z > z^*) = 0.83 \Rightarrow \Pr(Z < z^*) = 0.17$

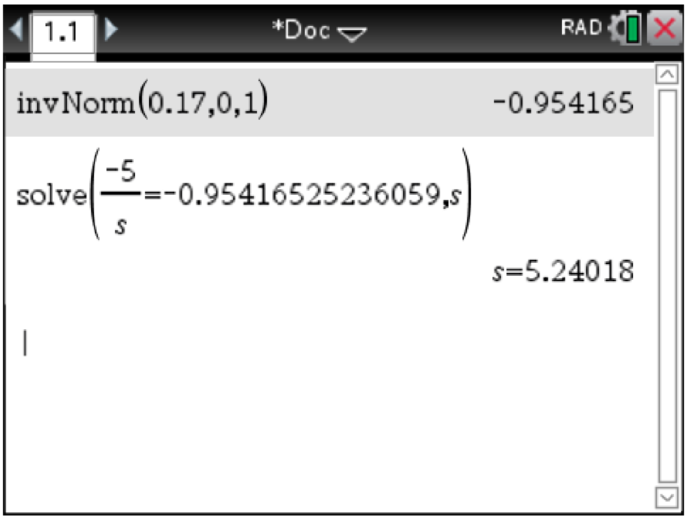
Use the Inverse Normal command on CAS:

$z^* = -0.9541652$

$$Z = \frac{C - \mu}{\sigma} \quad \Rightarrow z^* = \frac{25 - 30}{\sigma} = -\frac{5}{\sigma}$$

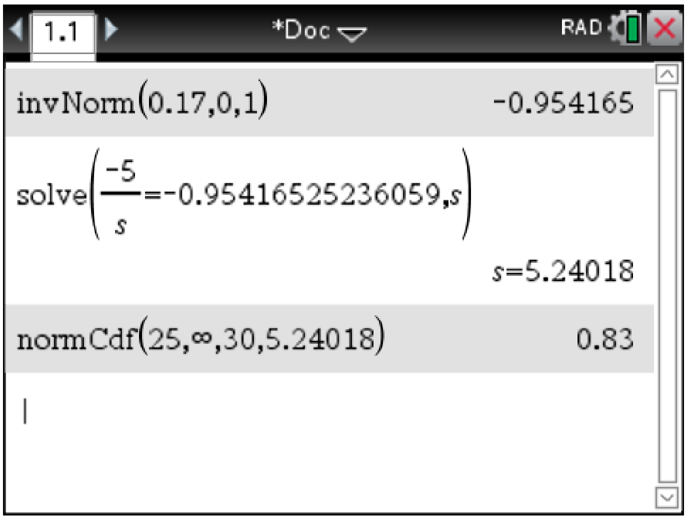
$$\Rightarrow -0.9541652 = -\frac{5}{\sigma}$$

Solve for σ using CAS:



Answer: $\sigma = 5.24$

We can check this result quickly:



Mark allocation: 2 marks

- 1 mark for getting a correct equation for σ using $z^* = -0.9541652$
- 1 mark for correct answer

Question 3g.

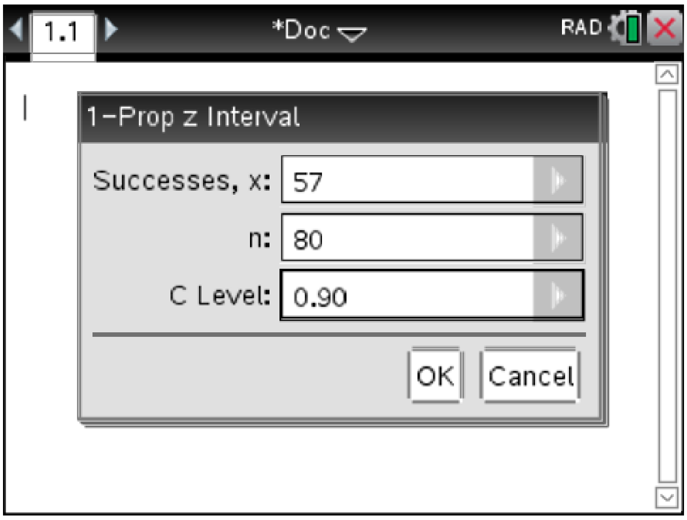
Worked solution

Use the appropriate commands on a CAS calculator.

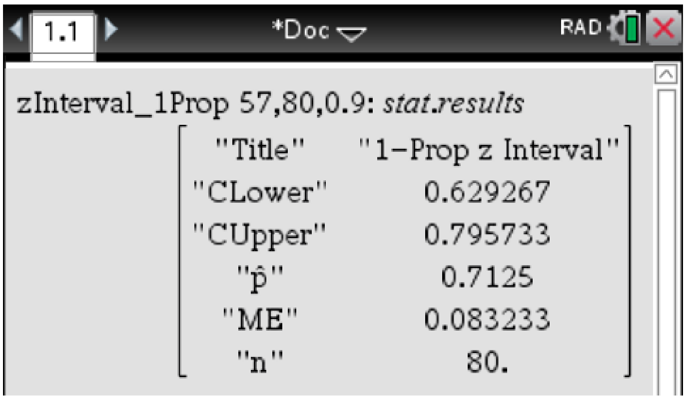
For example, using the TI-Nspire:

Menu>Statistics>Confidence Intervals>1-Prop z Interval

Complete the information as follows:



This produces the result:



Answer: $(0.63, 0.80)$

Mark allocation: 1 mark

- 1 mark for correct answer

Tip

- *Make sure that you can use your CAS calculator to directly get a confidence interval. It is highly recommended that you include the required command and its syntax in your bound reference.*

Question 4a.

Worked solution

Method 1:

Identify the transformations and their order that map $y=\sqrt{5-4x}=\sqrt{-4x+5}$ to $y=\sqrt{x}$.

Reflect in *y*-axis: $f(x)\to f(-x)=\sqrt{4x+5}=g(x)$

Dilate by a factor of 4 from the *y*-axis: $g(x)\to g\left(\frac{x}{4}\right)=\sqrt{x+5}=h(x)$

Translate 5 units along the *x*-axis: $h(x)\to h(x-5)=\sqrt{x}$

Note: The translation must be applied *last* in the sequence of transformations because the 'translation' part $\begin{bmatrix} c \\ d \end{bmatrix}$ of the transformation *T* is applied to $\begin{bmatrix} x \\ y \end{bmatrix}$ *after* the 'dilations and reflections' part $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$.

Therefore, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Compare with $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} c \\ d \end{bmatrix}$:

Answer: $a=-4, c=5, d=0$

Mark allocation: 2 marks

- 1 mark for correct identification of the three transformations and that translation comes last in the sequence
- 1 mark for correct answers

<u>Method 2:</u>

Let $y=\sqrt{5-4x}$.

Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x' \\ y' \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} c \\ d \end{bmatrix}=\begin{bmatrix} ax+c \\ y+d \end{bmatrix}$

Therefore:

$x'=ax+c \quad \Rightarrow x=\frac{x'-c}{a}$ equation (1)

$y'=y+d \quad \Rightarrow y=y'-d$ equation (2)

Substitute equations (1) and (2) into $y=\sqrt{5-4x}$:

$$y'-d=\sqrt{5-4\left(\frac{x'-c}{a}\right)}$$

$$\Rightarrow y'=\sqrt{5-\frac{4}{a}x'+\frac{4c}{a}}+d=\sqrt{-\frac{4}{a}x'+5+\frac{4c}{a}}+d$$

Therefore, the equation of the transformed function is

$$y=b\sqrt{-\frac{4}{a}x+5+\frac{4c}{a}}+d .$$

Compare with $y=\sqrt{x}$:

$d=0$

$-\frac{4}{a}=1\Rightarrow a=-4$

$5+\frac{4c}{a}=0$

Substitute $a=-4$: $5+\frac{4c}{(-4)}=0\Rightarrow 5-c=0\Rightarrow c=5$

Answer: $a=-4, c=5, d=0$

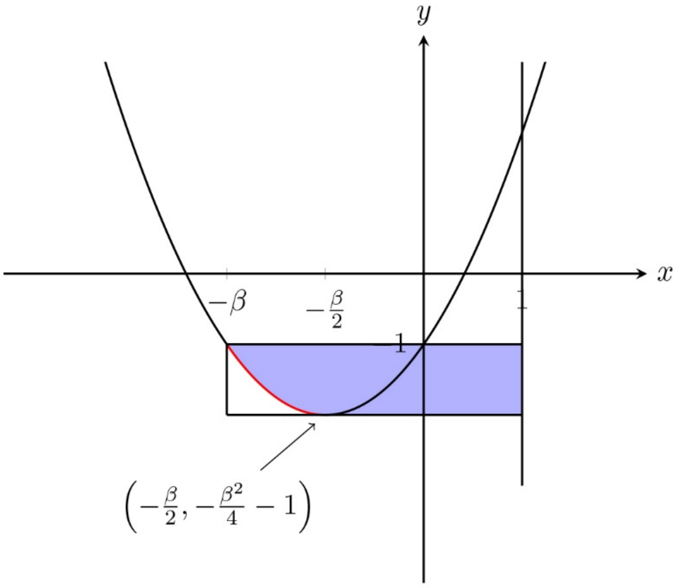
Mark allocation: 2 marks

- 1 mark for correct equation of transformed function
- 1 mark for correct answers

Question 4b.

Worked solution

Draw a simple graph of $y = x^2 + \beta x - 1$ over $-\beta \le x \le -\dfrac{\beta}{2}$ to see the required area:



Use either calculus, turning point form (complete the square) or formula to get the *x*-coordinate of the turning point: $x = -\dfrac{\beta}{2}$

y-coordinate of turning point: $y = \left(-\dfrac{\beta}{2}\right)^2 + \beta\left(-\dfrac{\beta}{2}\right) - 1 = -\dfrac{\beta^2}{4} - 1$

$x = -\beta \Rightarrow y = (-\beta)^2 + \beta(-\beta) - 1 = -1$

The required area, *A*, is the area between the graph of $y = x^2 + \beta x - 1$ and the line $x = 1$ from $y = -\dfrac{\beta^2}{4} - 1$ to $y = -1$.

Therefore, $A = \text{(area of rectangle)} - \displaystyle\int_{-\beta}^{-\frac{\beta}{2}} (x^2 + \beta x - 1) - \left(-\frac{\beta^2}{4} - 1\right) dx,$

Upper curve – Lower curve

where the area of the rectangle is width × height $= (1+\beta)\left(\dfrac{\beta^2}{4}\right) = \dfrac{(1+\beta)\beta^2}{4}$.

From CAS: $\displaystyle\int_{-\beta}^{-\frac{\beta}{2}} (x^2 + \beta x - 1) - \left(-\frac{\beta^2}{4} - 1\right) dx = \frac{\beta^3}{24}$

From CAS: $A = \dfrac{(1+\beta)\beta^2}{4} - \dfrac{\beta^3}{24} = \dfrac{\beta^2(5\beta + 6)}{24}$

Answer: $\dfrac{\beta^2(5\beta+6)}{24}$

Mark allocation: 2 marks

- 1 mark for any integral that gives the correct required area
- 1 mark for correct answer

Question 4c.

Worked solution

$f(x) = g(x)$

$\Rightarrow \sqrt{5-4x} = x^2 + \beta x - 1$ equation (1)

$\Rightarrow \sqrt{5-4x} + 1 = x^2 + \beta x$ equation (2)

The solutions to equation (2) are the *x*-coordinates of the intersection points of the graphs of $y = \sqrt{5-4x} + 1$ and $y = x^2 + \beta x$.

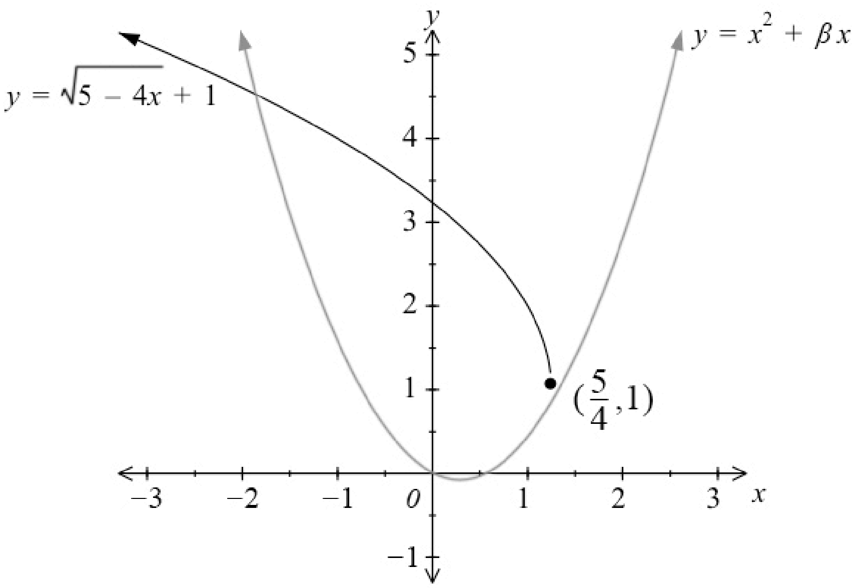
Note: Rearranging equation (1) into equation (2) is done because the graph of $y = x^2 + \beta x$ is easier to work with than the graph of $y = x^2 + \beta x - 1$.

Therefore, equation (1) has exactly one solution if the graphs of $y = \sqrt{5-4x} + 1$ and $y = x^2 + \beta x$ have exactly one intersection point.

<u>Case 1:</u> $\beta < 0$

Since $y = \sqrt{5-4x} + 1$ is strictly decreasing, it is sufficient that

$x^2 + \beta x < 1$ for $x = \frac{5}{4}$:



$$\left(\frac{5}{4}\right)^2 + \beta\left(\frac{5}{4}\right) < 1$$

$$\Rightarrow \beta < -\frac{9}{20}$$

<u>Case 2:</u> $\beta > 0$

Since $y = \sqrt{5-4x}+1$ is strictly decreasing, it is sufficient that $x^2 + \beta x < 1$ for $x = \frac{5}{4}$.

No solution for $\left(\frac{5}{4}\right)^2 + \beta\left(\frac{5}{4}\right) < 1$ and $\beta > 0$.

Answer: $\beta < -\frac{9}{20}$

Mark allocation: 2 marks

- 1 mark for consideration of the case $\beta < 0$ and a correct inequality
- 1 mark for correct answer

Alternatively, as the parabola is upright and passes through the y-axis at $y = -1$, it will always have one positive root and one negative root. If the positive root is less than $\frac{5}{4}$, then there will be two solutions to the equation $f(x) = g(x)$. Suppose the positive root occurs at $x = \frac{5}{4}$. Then:

$$g\left(\frac{5}{4}\right) = \left(\frac{5}{4}\right)^2 + \beta\left(\frac{5}{4}\right) - 1 = 0$$

and so $\beta = -\frac{9}{20}$. In this case, the turning point of $g(x)$ occurs when $x = \frac{9}{40}$. Setting $\beta < -\frac{9}{20}$ moves the turning point to the right and so the graphs of $f(x)$ and $g(x)$ meet only once.

Mark allocation (alternative): 2 marks

- 1 mark for consideration of the equation $g\left(\frac{5}{4}\right) = \left(\frac{5}{4}\right)^2 + \beta\left(\frac{5}{4}\right) - 1 = 0$
- 1 mark for correct answer

Question 4d.i.

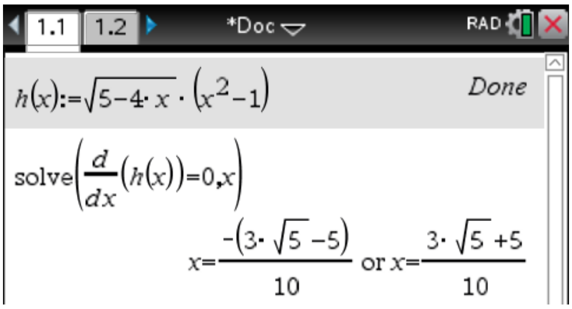
Worked solution

Substitute $\beta = 0$:

$h(x) = (x^2 - 1)\sqrt{5 - 4x}$

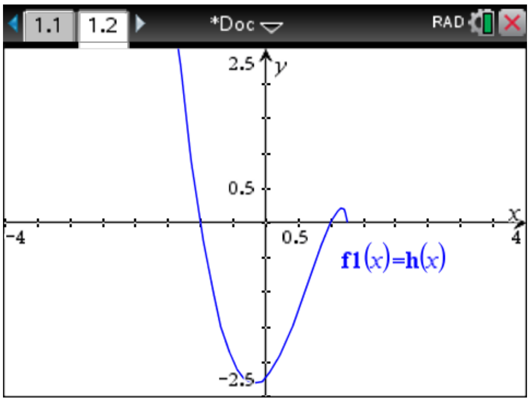
and the maximal domain of h is $5 - 4x \geq 0$; that is, $x \in \left(-\infty, \frac{5}{4}\right]$.

Solve $h'(x) = 0$ using CAS, where $x = \frac{5 \pm 3\sqrt{5}}{10}$.



Both of these solutions lie in the maximal domain $x \in \left(-\infty, \frac{5}{4}\right]$.

From a graph of $y = (x^2 - 1)\sqrt{5 - 4x}$ (plotted using a CAS calculator) it is clear that the local maximum turning point occurs when $x = \frac{5 + 3\sqrt{5}}{10}$.



Answer: $x = \frac{5 + 3\sqrt{5}}{10}$

Mark allocation: 1 mark

- 1 mark for correct answer

Tip

- *Always consider the domain when finding solutions to an equation.*

Question 4d.ii.

Worked solution

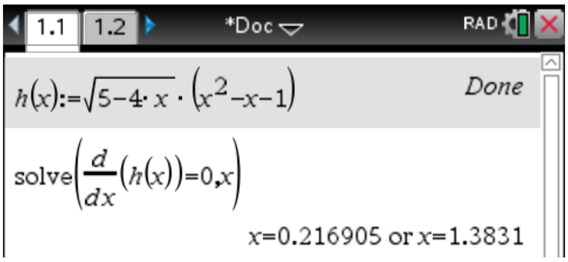
Substitute $\beta = -1$:

$h(x) = (x^2 - x - 1)\sqrt{5 - 4x}$

and the maximal domain of h is $5 - 4x \geq 0$; that is, $x \in \left(-\infty, \frac{5}{4}\right]$.

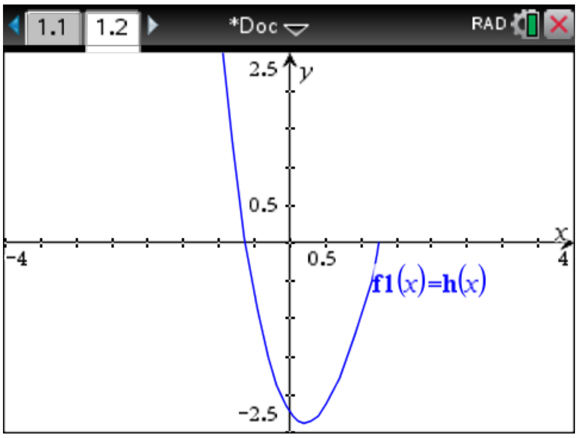
Solve $h'(x) = 0$ using a CAS calculator:

$x = \dfrac{8 \pm \sqrt{34}}{10}$



Only $x = \dfrac{8 - \sqrt{34}}{10}$ lies in the maximal domain $x \in \left(-\infty, \frac{5}{4}\right]$.

From a graph of $y = (x^2 - 1)\sqrt{5 - 4x}$ (plotted using a CAS calculator) it is clear that $x = \dfrac{8 - \sqrt{34}}{10}$ corresponds to a local minimum turning point.



Therefore, no maximum turning point exists.

Answer: A maximum turning point does not exist. The only solution of $h'(x) = 0$ that lies in the maximal domain of h is $x = \dfrac{8 - \sqrt{34}}{10}$ and this corresponds to a minimum turning point.

Mark allocation: 1 mark

- 1 mark for correct answer

Question 4e.

Worked solution

Part d. suggests how to approach this question.

The maximal domain of $h(x)=(x^2+\beta x-1)\sqrt{5-4x}$ is $5-4x\geq 0$;

that is, $x\in\left(-\infty,\ \frac{5}{4}\right]$.

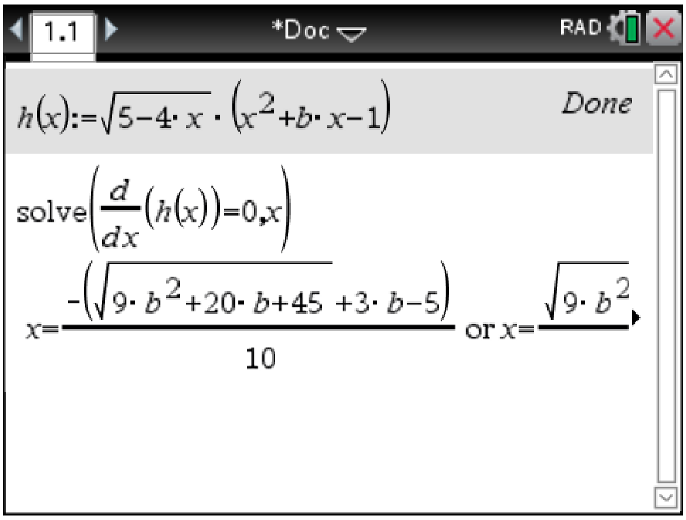
Therefore, values of β such that $h'(x)=0$ has only one solution over

$x\in\left(-\infty,\ \frac{5}{4}\right]$ are required.

From a CAS calculator: $h'(x)=\dfrac{-10x^2+(10-\beta)x+5\beta+2}{\sqrt{5-4x}}$

Therefore, the values of x such that $-10x^2+(10-\beta)x+5\beta+2=0$ has only one solution over $x\in\left(-\infty,\ \frac{5}{4}\right]$ are required.

Solve using a CAS calculator: $x=\dfrac{5-3\beta\pm\sqrt{9\beta^2+20\beta+45}}{10}$



Therefore, there are two cases for which $h'(x)=0$ has only one solution.

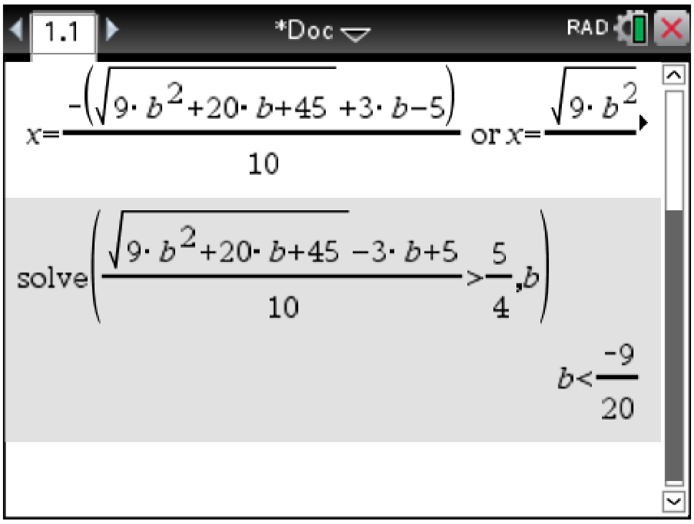
<u>Case 1:</u> $9\beta^2+20\beta+4=0$ and the resulting solution $x=\dfrac{5-3\beta}{10}$ lies inside the maximal domain $x\in\left(-\infty,\ \frac{5}{4}\right]$; that is, $\dfrac{5-3\beta}{10}<\dfrac{5}{4}$.

$9\beta^2+20\beta+4=0$ has no real solution.

Therefore, case 1 contributes no real values for β.

Case 2: The larger solution $x = \dfrac{5 - 3\beta + \sqrt{9\beta^2 + 20\beta + 45}}{10}$ lies outside the maximal domain $x \in \left(-\infty, \dfrac{5}{4}\right]$; that is, $\dfrac{5 - 3\beta + \sqrt{9\beta^2 + 20\beta + 45}}{10} > \dfrac{5}{4}$.

Solve this inequality using CAS: $\beta < -\dfrac{9}{20}$



Answer: $\beta < -\dfrac{9}{20}$

Mark allocation: 3 marks

- 1 mark for recognising that there is only one solution to $h'(x) = 0$; over $x \in \left(-\infty, \dfrac{5}{4}\right]$ via an appropriate equation with restricted domain; for example, $-10x^2 + (10 - \beta)x + 5\beta + 2 = 0$ has only one solution over $x \in \left(-\infty, \dfrac{5}{4}\right]$
- 1 mark for a correct inequality involving the larger solution to $h'(x) = 0$ –; for example, $x = \dfrac{5 - 3\beta + \sqrt{9\beta^2 + 20\beta + 45}}{10}$ lies outside $x \in \left(-\infty, \dfrac{5}{4}\right]$
- 1 mark for correct answer $\beta < -\dfrac{9}{20}$

END OF WORKED SOLUTIONS