VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2017

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER | | | | | | | | | Letter

MATHEMATICAL METHODS

Written examination 2

Wednesday 7 June 2017

Reading time: 2.00 pm to 2.15 pm (15 minutes)
Writing time: 2.15 pm to 4.15 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 21 pages.
- Formula sheet.
- Answer sheet for multiple-choice questions.

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The gradient of a line perpendicular to the line that passes through (3, 0) and (0, –6) is

A. $-\frac{1}{2}$

B. –2

C. $\frac{1}{2}$

D. 4

E. 2

Question 2

The function with rule $f(x) = 2\sin\left(\frac{x}{4}\right) + 1$ has period

A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. π

D. 4π

E. 8π

Question 3

The simultaneous linear equations $mx + 7y = 12$ and $7x + my = m$ have a unique solution only for

A. $m = 7$ or $m = -7$

B. $m = 12$ or $m = 3$

C. $m \in R \backslash \{-7, 7\}$

D. $m = 4$ or $m = 3$

E. $m \in R \backslash \{12, 1\}$

Question 4

The graph of the function $f:[0,\infty)\to R$, where $f(x)=4x^{\frac{1}{3}}$, is reflected in the x-axis and then translated five units to the right and six units vertically down.
Which one of the following is the rule of the transformed graph?

A. $y=4(x-5)^{\frac{1}{3}}+6$

B. $y=-4(x+5)^{\frac{1}{3}}-6$

C. $y=-4(x+5)^{\frac{1}{3}}+6$

D. $y=-4(x-5)^{\frac{1}{3}}-6$

E. $y=4(x-5)^{\frac{1}{3}}+1$

Question 5

Which one of the following is the inverse function of the function $f:(-\infty,3)\to R,\ f(x)=\dfrac{2}{\sqrt{3-x}}+1$?

A. $f^{-1}:(-\infty,3)\to R,\ f^{-1}(x)=-\dfrac{4}{(x-1)^2}+3$

B. $f^{-1}:(1,\infty)\to R,\ f^{-1}(x)=-\dfrac{4}{(x-3)^2}+1$

C. $f^{-1}:(1,\infty)\to R,\ f^{-1}(x)=-\dfrac{4}{(x-1)^2}+3$

D. $f^{-1}:(1,\infty)\to R,\ f^{-1}(x)=-\dfrac{4}{x^2}+3$

E. $f^{-1}:R^+\to R,\ f^{-1}(x)=-\dfrac{4}{(x-1)^2}+3$

Question 6

Let $f:D\to R,\ f(x)=\dfrac{3x-5}{2-x}$, where D is the maximal domain of f.
Which of the following are the equations of the asymptotes of the graph of f?

A. $x=2$ and $y=\dfrac{5}{3}$

B. $x=2$ and $y=-3$

C. $x=-2$ and $y=3$

D. $x=-3$ and $y=2$

E. $x=2$ and $y=3$

Question 7

The graph of $y = kx - 2$ will not intersect or touch the graph of $y = x^2 + 3x$ when

A. $3 - 2\sqrt{2} < k < 3 + 2\sqrt{2}$

B. $\{k : k < 3 - 2\sqrt{2}\} \cup \{k : k > 3 + 2\sqrt{2}\}$

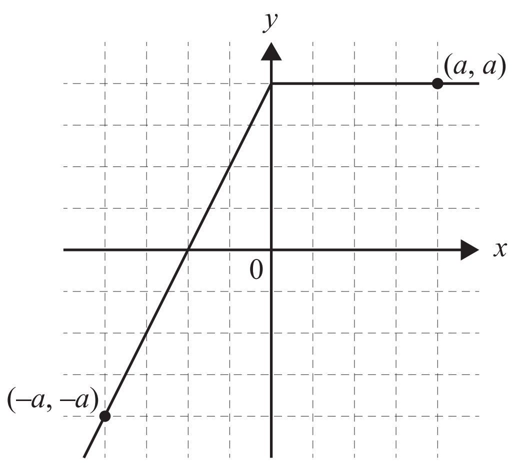
C. $-5 < k < 11$

D. $3 - 2\sqrt{2} \le k \le 3 + 2\sqrt{2}$

E. $k \in R^+$

Question 8

Part of the graph of a function f is shown below.



Which one of the following is the average value of the function f over the interval $[-a, a]$?

A. 0

B. $\dfrac{3a}{4}$

C. $\dfrac{3a}{8}$

D. $\dfrac{a}{2}$

E. $\dfrac{a}{4}$

Question 9

The range of the function $f:\left(-\frac{1}{2},0\right)\cup(0,2]\rightarrow R,\ f(x)=\log_e(x^2)$ is

A. $\left(-2\log_e(2),2\log_e(2)\right]$

B. $\left(-\log_e\left(\frac{1}{4}\right),\log_e(4)\right]$

C. $\left(-\infty,2\log_e(2)\right]$

D. $R\backslash\left[-2\log_e(2),2\log_e(2)\right)$

E. R

Question 10

The tangent to the graph of $y = 3\sin(2x) - 1$ is parallel to the line with equation $y = 3x + 1$ at the points where x is equal to

A. $n\pi \pm \frac{\pi}{6},\ n\in Z$

B. $-\frac{\pi}{3},\ \frac{\pi}{3}$ only

C. $\frac{\pi}{6},\ \frac{5\pi}{6}$ only

D. $n\pi \pm \frac{\pi}{3},\ n\in Z$

E. $n\pi,\ n \in Z$

Question 11

A bag contains five blue marbles and four red marbles. A sample of four marbles is taken from the bag, without replacement.

The probability that the proportion of blue marbles in the sample is greater than $\frac{1}{2}$ is

A. $\frac{1}{2}$

B. $\frac{2}{9}$

C. $\frac{5}{14}$

D. $\frac{5}{9}$

E. $\frac{25}{63}$

Question 12

The maximum temperature reached by the water heated in a kettle each time it is used is normally distributed with a mean of 95 °C and a standard deviation of 2 °C.

When the kettle is used, the proportion of times that the maximum temperature reached by the water is greater than 98 °C is closest to

A. 0.0671

B. 0.0668

C. 0.8669

D. 0.9332

E. 0.9342

Question 13

The probability distribution for the discrete random variable X is defined by the following.

x	-1	0	1	2
$\Pr(X = x)$	a	$3b$	$2a$	b

The mean and variance of the distribution, respectively, are

A. $a + 2b$ and $3a + 4b$

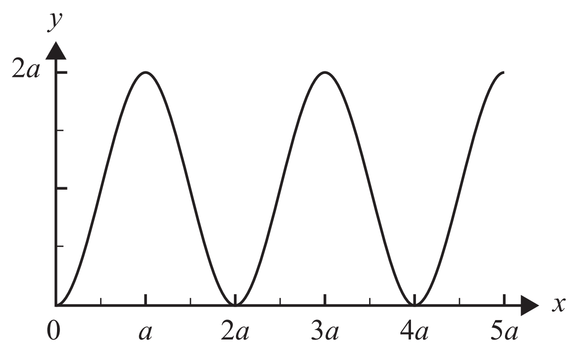
B. $a + 2b$ and $3a + 4b - a^2 - 4ab - 4b^2$

C. $3a + 2b$ and $3a + 4b - 9a^2 - 12ab - 4b^2$

D. $3a + 2b$ and $3a + 4b$

E. $3a + 5b$ and $a + b$

Question 14



The rule of the function with the graph shown above could be

A. $y = a\cos\left(\dfrac{\pi x}{2a}\right) + a$

B. $y = 2a\sin\left(\dfrac{\pi x}{a}\right) + a$

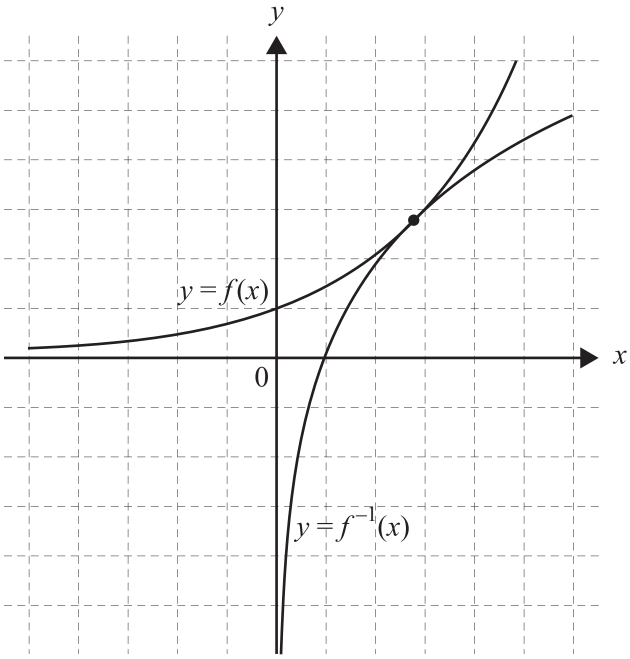
C. $y = -a\cos\left(\dfrac{\pi x}{2a}\right) + a$

D. $y = a\sin\left(\dfrac{\pi}{a}(x - a)\right) + a$

E. $y = -a\cos\left(\dfrac{\pi x}{a}\right) + a$

Question 15

The graph of f where $f(x) = e^{kx}$, $k > 0$, touches the graph of its inverse function f^{-1} at exactly one point, as shown below.



The value of k must be

A. e

B. e^2

C. 1

D. $\frac{1}{e^2}$

E. $\frac{1}{e}$

Question 16

The area under the graph of $y = f(x)$, between $x = 1$ and $x = 4$, where $f(x) > \frac{1}{2}$ over this interval, is eight units.

The area under the graph of $y = 2f(x) - 1$ over the same interval is

A. 15 units.

B. 13 units.

C. 10 units.

D. 7 units.

E. 3 units.

Question 17

A function f satisfies the relation $f(x^2) = f(x) + f(x+2)$.

A possible rule for f is

A. $f(x) = \sqrt{x+2}$

B. $f(x) = x + 2$

C. $f(x) = \log_{10}(x-1)$

D. $f(x) = \frac{1}{2}\left(x^2 - 1\right)$

E. $f(x) = \frac{1}{x-1}$

Question 18

Let $f(x) = x^m e^{ax}$, where a and m are non-zero real constants.

If $(x + 2)$ is a factor of $f'(x)$, then which one of the following must be true?

A. $m = 2$

B. $m = -2$

C. $m = 2 - a$

D. $m = 2a$

E. $m = -2a$

Question 19

The first digit of each house number in a large address database is a random variable D. The possible values of D are 1, 2, …, 9 and $\Pr(D = d) = \log_{10}\left(1 + \frac{1}{d}\right)$.

The probability that D is greater than 8, given that D is greater than 7, is

A. $1 - \log_{10}(9)$

B. $\frac{1 - \log_{10}(9)}{1 - \log_{10}(8)}$

C. $\frac{1 - \log_{10}(9)}{2 - \log_{10}(9) - \log_{10}(8)}$

D. $\log_{10}\left(\frac{8}{9}\right)$

E. $\log_{10}\left(\frac{9}{8}\right)$

Question 20

Consider the function $f : [2, \infty) \to R$, $f(x) = x^4 + 2(a-4)x^2 - 8ax + 1$, where $a \in R$.

The maximal set of values of a for which the inverse function f^{-1} exists is

A. $(-9, \infty)$

B. $(-\infty, 1)$

C. $[-9, 1]$

D. $[-8, \infty)$

E. $(-\infty, -8]$

END OF SECTION A

CONTINUES OVER PAGE

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (11 marks)

The temperature, T °C, in an office is controlled. For a particular weekday, the temperature at time t, where t is the number of hours after midnight, is given by the function

$T(t) = 19 + 6\sin\left(\frac{\pi}{12}(t-8)\right), 0 \le t \le 24.$

a. What are the maximum and minimum temperatures in the office? 2 marks

b. What is the temperature in the office at 6.00 am? 1 mark

c. Most of the people working in the office arrive at 8.00 am.

What is the temperature in the office when they arrive? 1 mark

d. For how many hours of the day is the temperature greater than or equal to 19 °C? 2 marks

e. What is the average rate of change of the temperature in the office between 8.00 am and noon? 2 marks

f. **i.** Find $T'(t)$. 1 mark

ii. At what time of the day is the temperature in the office decreasing most rapidly? 2 marks

Question 2 (13 marks)

Let $f : R \to R$, where $f(x) = (x - 2)^2 (x - 5)$.

a. Find $f'(x)$. 1 mark

b. For what values of x is $f'(x) < 0$? 1 mark

c. **i.** Find the gradient of the line segment joining the points on the graph of $y = f(x)$ where $x = 1$ and $x = 5$. 1 mark

ii. Show that the midpoint of the line segment in **part c.i.** also lies on the graph of $y = f(x)$. 2 marks

iii. Find the values of x for which the gradient of the tangent to the graph of $y = f(x)$ is equal to the gradient of the line segment joining the points on the graph where $x = 1$ and $x = 5$. 2 marks

Let $g: R \to R$, where $g(x) = (x - 2)^2 (x - a)$, where $a \in R$.

d. The coordinates of the stationary points of g are $P(2, 0)$ and $Q(p(a + 1), q(a - 2)^3)$, where p and q are rational numbers.

Find the values of p and q. 2 marks

e. Show that the gradient of the tangent to the graph of $y = g(x)$ at the point $(a, 0)$ is positive for $a \in R\backslash\{2\}$. 1 mark

f. **i.** Find the coordinates of another point where the tangent to the graph of $y = g(x)$ is parallel to the tangent at $x = a$. 2 marks

ii. Hence, find the distance between this point and point Q when $a > 2$. 1 mark

Question 3 (18 marks)

A company supplies schools with whiteboard pens.

The total length of time for which a whiteboard pen can be used for writing before it stops working is called its use-time.

There are two types of whiteboard pens: Grade A and Grade B.

The use-time of Grade A whiteboard pens is normally distributed with a mean of 11 hours and a standard deviation of 15 minutes.

a. Find the probability that a Grade A whiteboard pen will have a use-time that is greater than 10.5 hours, correct to three decimal places. 1 mark

The use-time of Grade B whiteboard pens is described by the probability density function

$$f(x) = \begin{cases} \dfrac{x}{576}(12 - x)(e^{\frac{x}{6}} - 1) & 0 \le x \le 12 \\ 0 & \text{otherwise} \end{cases}$$

where x is the use-time in hours.

b. Determine the expected use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places. 2 marks

c. Determine the standard deviation of the use-time of a Grade B whiteboard pen. Give your answer in hours, correct to two decimal places. 2 marks

d. Find the probability that a randomly chosen Grade B whiteboard pen will have a use-time that is greater than 10.5 hours, correct to four decimal places. 2 marks

A worker at the company finds two boxes of whiteboard pens that are not labelled, but knows that one box contains only Grade A whiteboard pens and the other box contains only Grade B whiteboard pens.

The worker decides to randomly select a whiteboard pen from one of the boxes. If the selected whiteboard pen has a use-time that is greater than 10.5 hours, then the box that it came from will be labelled Grade A and the other box will be labelled Grade B. Otherwise, the box that it came from will be labelled Grade B and the other box will be labelled Grade A.

e. Find the probability, correct to three decimal places, that the worker labels the boxes incorrectly. 2 marks

f. Find the probability, correct to three decimal places, that the whiteboard pen selected was Grade B, given that the boxes have been labelled incorrectly. 2 marks

As a whiteboard pen ages, its tip may dry to the point where the whiteboard pen becomes defective (unusable). The company has stock that is two years old and, at that age, it is known that 5% of Grade A whiteboard pens will be defective.

g. A school purchases a box of Grade A whiteboard pens that is two years old and a class of 26 students is the first to use them.

If every student receives a whiteboard pen from this box, find the probability, correct to four decimal places, that at least one student will receive a defective whiteboard pen. 2 marks

h. Let $\hat{P}_A$ be the random variable of the distribution of sample proportions of defective Grade A whiteboard pens in boxes of 100. The boxes come from the stock that is two years old.

Find $(\hat{P}_A > 0.04 \mid \hat{P}_A < 0.08)$. Give your answer correct to four decimal places. Do not use a normal approximation. 3 marks

i. A box of 100 Grade A whiteboard pens that is two years old is selected and it is found that six of the whiteboard pens are defective.

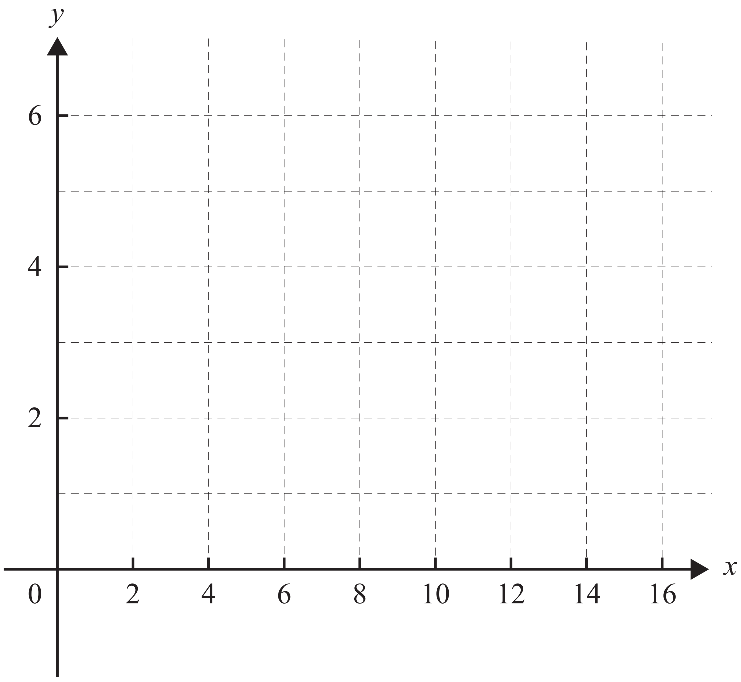
Determine a 90% confidence interval for the population proportion from this sample, correct to two decimal places. 2 marks

Question 4 (18 marks)

Let $f:[0,16]\to R,\ f(x)=4\sqrt{x}-x.$

a. Find the value of x at which f has a maximum and state the maximum value. 2 marks

b. Sketch the graph of $y=f(x)$ on the axes below. 1 mark



c. **i.** Find the area A of the region enclosed by the graph of f and the x-axis. 1 mark

ii. Let X be the point (16, 0). Triangles can be formed with vertices O, X and $C(c, f(c))$, where C is a point on the graph of $y = f(x)$.

Find the value of c for which triangle OCX has maximum area and find this maximum area. 2 marks

Let a and b be two values in the domain of f such that $f(b) = f(a)$, and assume $b > a$.

d. Show that $b = \left(\sqrt{a} - 4\right)^2$. 1 mark

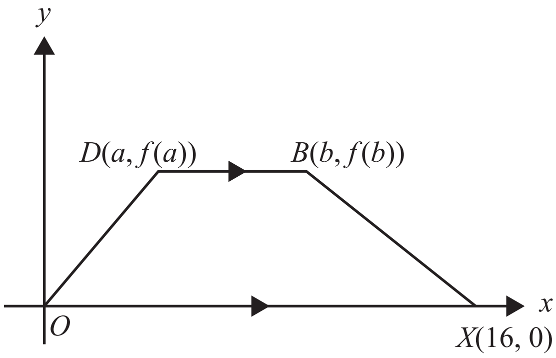
e. A rectangle is formed with vertices $(a, 0)$, $(b, 0)$, $(b, f(b))$ and $(a, f(a))$.

i. Find the area of this rectangle in terms of a. 2 marks

ii. Find the values of a and b for which the rectangle has maximum area. 3 marks

iii. Find the value of the maximum area in the form $\frac{m\sqrt{n}}{p}$, where m, n and p are positive integers. 2 marks

f. A trapezium $ODBX$ is formed with vertices at O, $X(16, 0)$, $B(b, f(b))$ and $D(a, f(a))$, as shown in the diagram below, where DB is parallel to OX.



i. Find the value of a for which the trapezium has maximum area A_T and find this maximum area. 3 marks

ii. Find $\frac{A_T}{A}$, where A is the area of the region enclosed by the graph of f and the x-axis. 1 mark

END OF QUESTION AND ANSWER BOOK

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MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

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Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET