VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2017

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER | | | | | | | | | Letter | |

MATHEMATICAL METHODS

Written examination 2

Thursday 9 November 2017

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 22 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f: R \to R, f(x) = 5\sin(2x) - 1$.

The period and range of this function are respectively

A. π and $[-1, 4]$

B. 2π and $[-1, 5]$

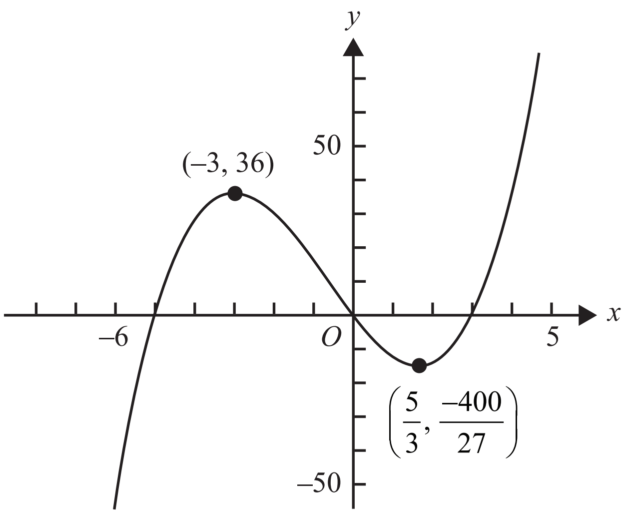
C. π and $[-6, 4]$

D. 2π and $[-6, 4]$

E. 4π and $[-6, 4]$

Question 2

Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown below.



$f'(x) < 0$ for the interval

A. $(0, 3)$

B. $(-\infty, -5) \cup (0, 3)$

C. $(-\infty, -3) \cup \left(\frac{5}{3}, \infty\right)$

D. $\left(-3, \frac{5}{3}\right)$

E. $\left(\frac{-400}{27}, 36\right)$

Question 3

A box contains five red marbles and three yellow marbles. Two marbles are drawn at random from the box without replacement.

The probability that the marbles are of **different** colours is

A. $\frac{5}{8}$

B. $\frac{3}{5}$

C. $\frac{15}{28}$

D. $\frac{15}{56}$

E. $\frac{30}{28}$

Question 4

Let f and g be functions such that $f(2) = 5$, $f(3) = 4$, $g(2) = 5$, $g(3) = 2$ and $g(4) = 1$.

The value of $f(g(3))$ is

A. 1

B. 2

C. 3

D. 4

E. 5

Question 5

The 95% confidence interval for the proportion of ferry tickets that are cancelled on the intended departure day is calculated from a large sample to be (0.039, 0.121).

The sample proportion from which this interval was constructed is

A. 0.080

B. 0.041

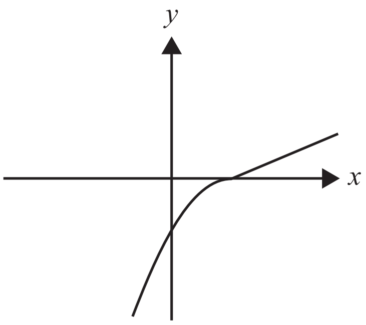
C. 0.100

D. 0.062

E. 0.059

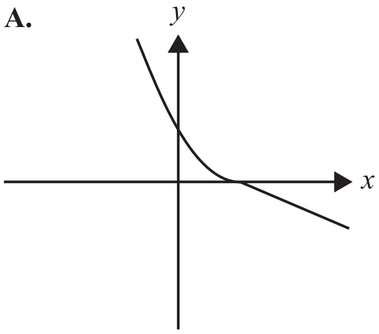
Question 6

Part of the graph of the function f is shown below. The same scale has been used on both axes.

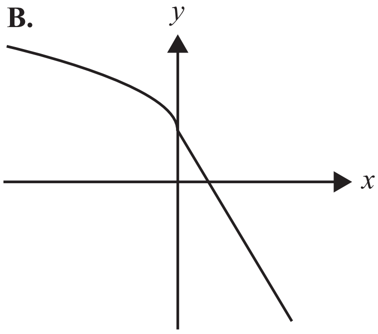


The corresponding part of the graph of the inverse function f^{-1} is best represented by

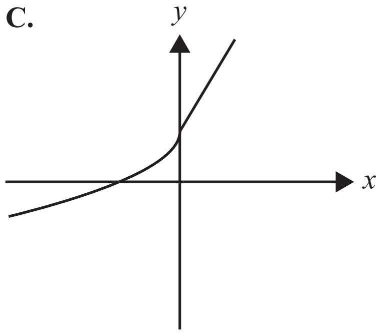
A.



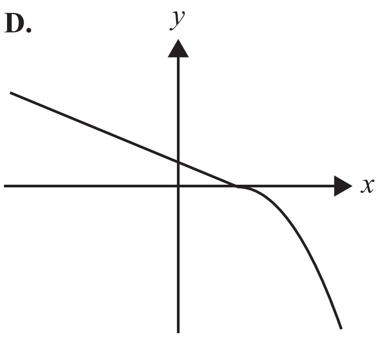
B.



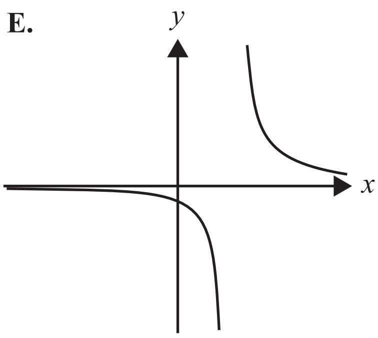
C.



D.



E.



Question 7

The equation $(p-1)x^2 + 4x = 5 - p$ has no real roots when

A. $p^2 - 6p + 6 < 0$

B. $p^2 - 6p + 1 > 0$

C. $p^2 - 6p - 6 < 0$

D. $p^2 - 6p + 1 < 0$

E. $p^2 - 6p + 6 > 0$

Question 8

If $y = a^{b-4x} + 2$, where $a > 0$, then x is equal to

A. $\frac{1}{4}\left(b - \log_a(y-2)\right)$

B. $\frac{1}{4}\left(b - \log_a(y+2)\right)$

C. $b - \log_a\left(\frac{1}{4}(y+2)\right)$

D. $\frac{b}{4} - \log_a(y-2)$

E. $\frac{1}{4}\left(b + 2 - \log_a(y)\right)$

Question 9

The average rate of change of the function with the rule $f(x) = x^2 - 2x$ over the interval $[1, a]$, where $a > 1$, is 8.

The value of a is

A. 9

B. 8

C. 7

D. 4

E. $1 + \sqrt{2}$

Question 10

A transformation $T: R^2 \to R^2$ with rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps the graph of $y = 3\sin\left(2\left(x + \frac{\pi}{4}\right)\right)$ onto the graph of

A. $y = \sin(x + \pi)$

B. $y = \sin\left(x - \frac{\pi}{2}\right)$

C. $y = \cos(x + \pi)$

D. $y = \cos(x)$

E. $y = \cos\left(x - \frac{\pi}{2}\right)$

Question 11

The function $f: R \to R, f(x) = x^3 + ax^2 + bx$ has a local maximum at $x = -1$ and a local minimum at $x = 3$.

The values of a and b are respectively

A. -2 and -3

B. 2 and 1

C. 3 and -9

D. -3 and -9

E. -6 and -15

Question 12

The sum of the solutions of $\sin(2x) = \frac{\sqrt{3}}{2}$ over the interval $[-\pi, d]$ is $-\pi$.

The value of d could be

A. 0

B. $\frac{\pi}{6}$

C. $\frac{3\pi}{4}$

D. $\frac{7\pi}{6}$

E. $\frac{3\pi}{2}$

Question 13

Let $h: (-1, 1) \to R, h(x) = \frac{1}{x-1}$.

Which one of the following statements about h is **not** true?

A. $h(x)h(-x) = -h(x^2)$

B. $h(x) + h(-x) = 2h(x^2)$

C. $h(x) - h(0) = xh(x)$

D. $h(x) - h(-x) = 2xh(x^2)$

E. $(h(x))^2 = h(x^2)$

Question 14

The random variable X has the following probability distribution, where $0 < p < \frac{1}{3}$.

x	-1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

The variance of X is

A. $2p(1 - 3p)$

B. $1 - 4p$

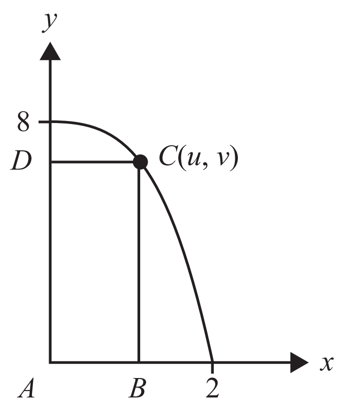
C. $(1 - 3p)^2$

D. $6p - 16p^2$

E. $p(5 - 9p)$

Question 15

A rectangle $ABCD$ has vertices $A(0, 0)$, $B(u, 0)$, $C(u, v)$ and $D(0, v)$, where (u, v) lies on the graph of $y = -x^3 + 8$, as shown below.



The maximum area of the rectangle is

A. $\sqrt[3]{2}$

B. $6\sqrt[3]{2}$

C. 16

D. 8

E. $3\sqrt[3]{2}$

Question 16

For random samples of five Australians, $\hat{P}$ is the random variable that represents the proportion who live in a capital city.

Given that $\Pr\left(\hat{P} = 0\right) = \frac{1}{243}$, then $\Pr\left(\hat{P} > 0.6\right)$, correct to four decimal places, is

A. 0.0453

B. 0.3209

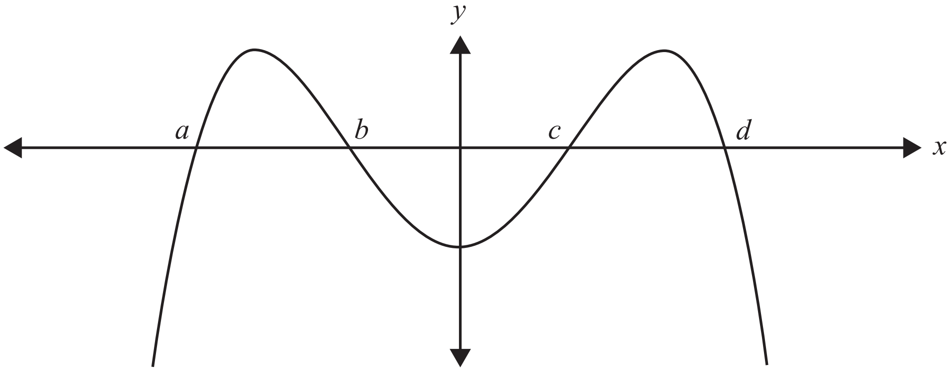
C. 0.4609

D. 0.5390

E. 0.7901

Question 17

The graph of a function f, where $f(-x) = f(x)$, is shown below.



The graph has x-intercepts at $(a, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$ only.

The area bound by the curve and the x-axis on the interval $[a, d]$ is

A. $\int_a^d f(x)\,dx$

B. $\int_a^b f(x)\,dx - \int_c^b f(x)\,dx + \int_c^d f(x)\,dx$

C. $2\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

D. $2\int_a^b f(x)\,dx - 2\int_b^{b+c} f(x)\,dx$

E. $\int_a^b f(x)\,dx + \int_c^b f(x)\,dx + \int_d^c f(x)\,dx$

Question 18

Let X be a discrete random variable with binomial distribution $X \sim \text{Bi}(n, p)$. The mean and the standard deviation of this distribution are equal.

Given that $0 < p < 1$, the smallest number of trials, n, such that $p \leq 0.01$ is

A. 37

B. 49

C. 98

D. 99

E. 101

Question 19

A probability density function f is given by

$$f(x) = \begin{cases} \cos(x) + 1 & k < x < (k+1) \\ 0 & \text{elsewhere} \end{cases}$$

where $0 < k < 2$.

The value of k is

A. 1

B. $\dfrac{3\pi - 1}{2}$

C. $\pi - 1$

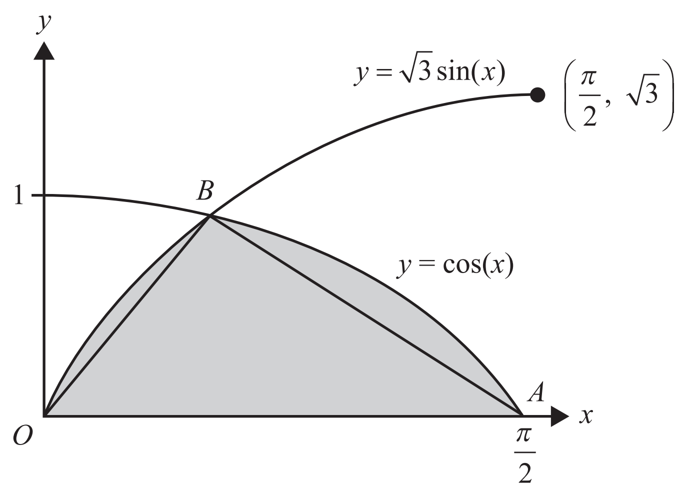
D. $\dfrac{\pi - 1}{2}$

E. $\dfrac{\pi}{2}$

Question 20

The graphs of $f : \left[0, \frac{\pi}{2}\right] \to R,\ f(x) = \cos(x)$ and $g : \left[0, \frac{\pi}{2}\right] \to R,\ g(x) = \sqrt{3}\sin(x)$ are shown below.

The graphs intersect at B.



The ratio of the area of the shaded region to the area of triangle OAB is

A. 9 : 8

B. $\sqrt{3} - 1 : \dfrac{\sqrt{3}\pi}{8}$

C. $8\sqrt{3} - 3 : 3\pi$

D. $\sqrt{3} - 1 : \dfrac{\sqrt{3}\pi}{4}$

E. $1 : \dfrac{\sqrt{3}\pi}{8}$

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

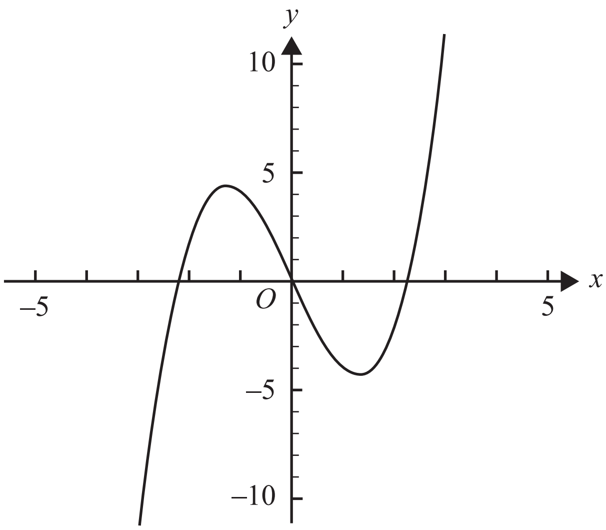
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (11 marks)

Let $f : R \to R,\ f(x) = x^3 - 5x$. Part of the graph of f is shown below.



a. Find the coordinates of the turning points. 2 marks

b. $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

i. Find the equation of the straight line through A and B. 2 marks

ii. Find the distance AB. 1 mark

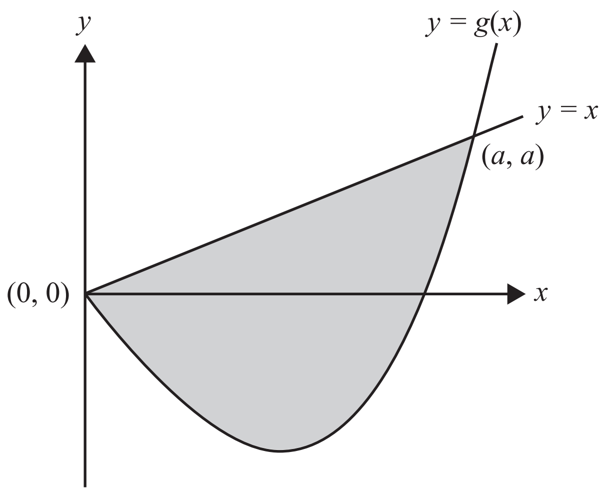
Let $g : R \to R, g(x) = x^3 - kx, k \in R^+$.

c. Let $C(-1, g(-1))$ and $D(1, g(1))$ be two points on the graph of g.

i. Find the distance CD in terms of k. 2 marks

ii. Find the values of k such that the distance CD is equal to $k + 1$. 1 mark

d. The diagram below shows part of the graphs of g and $y = x$. These graphs intersect at the points with the coordinates (0, 0) and (a, a).



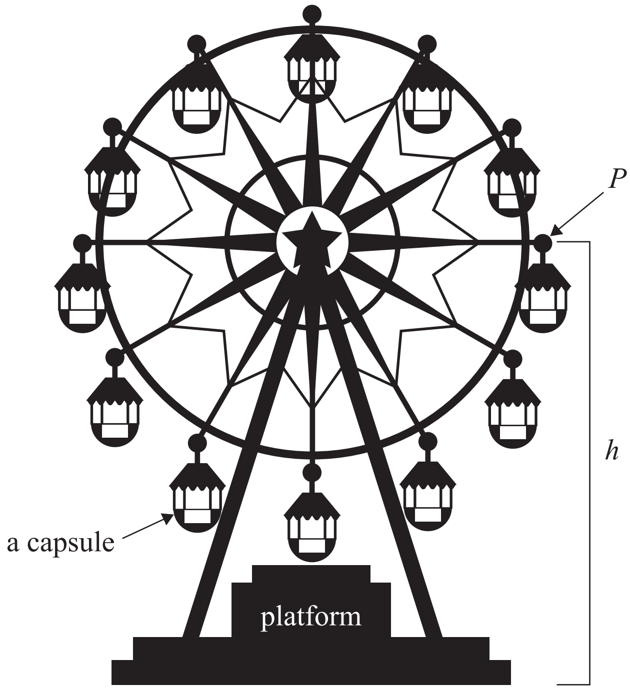
i. Find the value of a in terms of k. 1 mark

ii. Find the area of the shaded region in terms of k. 2 marks

Question 2 (12 marks)

Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise. The capsule is attached to the Ferris wheel at point *P*. The height of *P* above the ground, *h*, is modelled by $h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$, where *t* is the time in minutes after Sammy enters the capsule and *h* is measured in metres.

Sammy exits the capsule after one complete rotation of the Ferris wheel.

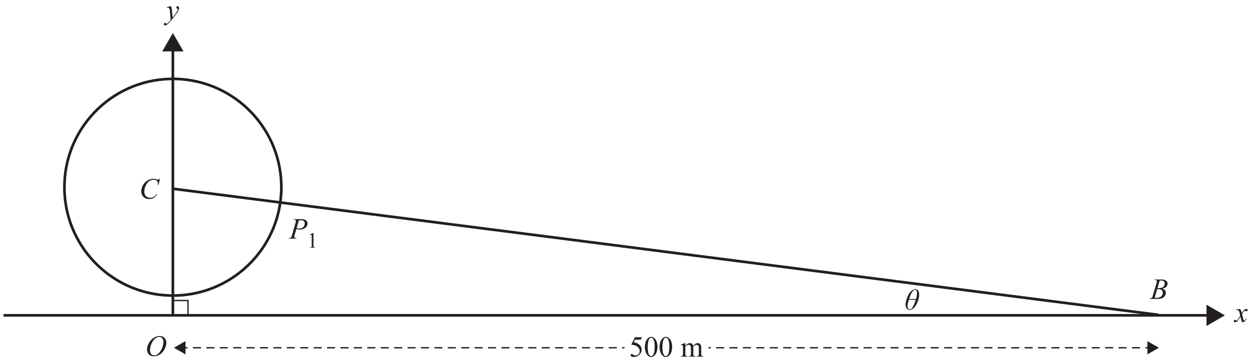


a. State the minimum and maximum heights of *P* above the ground. 1 mark

b. For how much time is Sammy in the capsule? 1 mark

c. Find the rate of change of *h* with respect to *t* and, hence, state the value of *t* at which the rate of change of *h* is at its maximum. 2 marks

As the Ferris wheel rotates, a stationary boat at B, on a nearby river, first becomes visible at point P_1. B is 500 m horizontally from the vertical axis through the centre C of the Ferris wheel and angle $CBO = \theta$, as shown below.

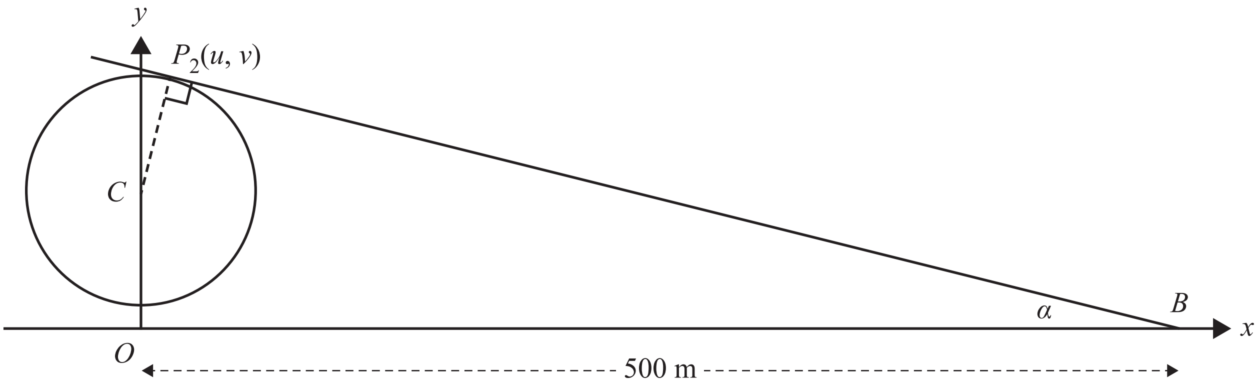


d. Find θ in degrees, correct to two decimal places. 1 mark

Part of the path of P is given by $y = \sqrt{3025 - x^2} + 65$, $x \in [-55, 55]$, where x and y are in metres.

e. Find $\frac{dy}{dx}$. 1 mark

As the Ferris wheel continues to rotate, the boat at B is no longer visible from the point $P_2(u, v)$ onwards. The line through B and P_2 is tangent to the path of P, where angle $OBP_2 = \alpha$.



f. Find the gradient of the line segment P_2B in terms of u and, hence, find the coordinates of P_2, correct to two decimal places. 3 marks

g. Find α in degrees, correct to two decimal places. 1 mark

h. Hence or otherwise, find the length of time, to the nearest minute, during which the boat at B is visible. 2 marks

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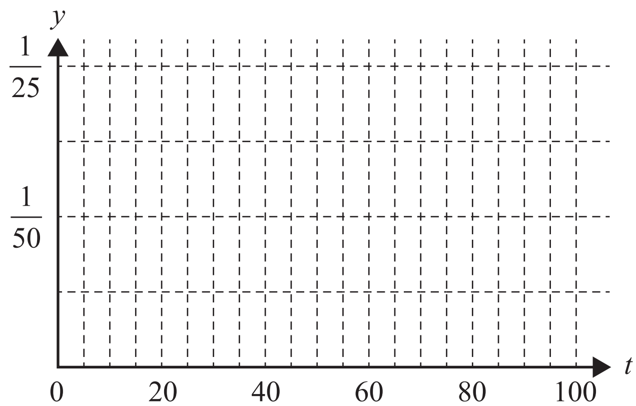
Question 3 (19 marks)

The time Jennifer spends on her homework each day varies, but she does some homework every day.

The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f, where

$$f(t)=\begin{cases}\frac{1}{625}(t-20) & 20\le t<45\\ \frac{1}{625}(70-t) & 45\le t\le 70\\ 0 & \text{elsewhere}\end{cases}$$

a. Sketch the graph of f on the axes provided below. 3 marks



b. Find $\Pr(25 \le T \le 55)$. 2 marks

c. Find $\Pr(T \le 25 \mid T \le 55)$. 2 marks

d. Find a such that $\Pr(T \geq a) = 0.7$, correct to four decimal places. 2 marks

e. The probability that Jennifer spends more than 50 minutes on her homework on any given day is $\frac{8}{25}$. Assume that the amount of time spent on her homework on any day is independent of the time spent on her homework on any other day.

i. Find the probability that Jennifer spends more than 50 minutes on her homework on more than three of seven randomly chosen days, correct to four decimal places. 2 marks

ii. Find the probability that Jennifer spends more than 50 minutes on her homework on at least two of seven randomly chosen days, given that she spends more than 50 minutes on her homework on at least one of those days, correct to four decimal places. 2 marks

Let p be the probability that on any given day Jennifer spends more than d minutes on her homework.

Let q be the probability that on two or three days out of seven randomly chosen days she spends more than d minutes on her homework.

f. Express q as a polynomial in terms of p. 2 marks

g. **i.** Find the maximum value of q, correct to four decimal places, and the value of p for which this maximum occurs, correct to four decimal places. 2 marks

ii. Find the value of d for which the maximum found in **part g.i.** occurs, correct to the nearest minute. 2 marks

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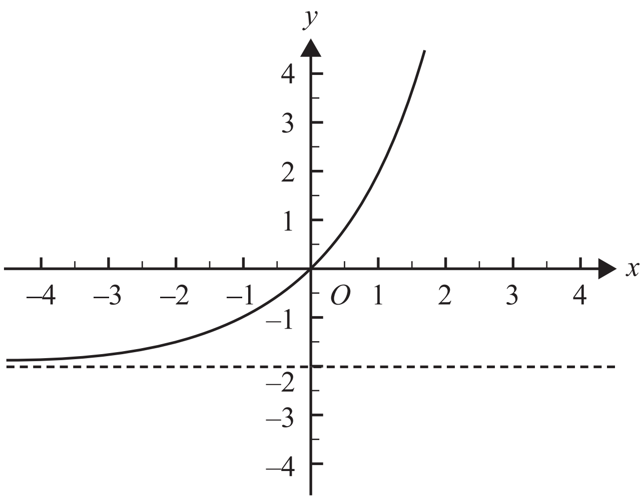
Question 4 (18 marks)

Let $f: R \to R: f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below.



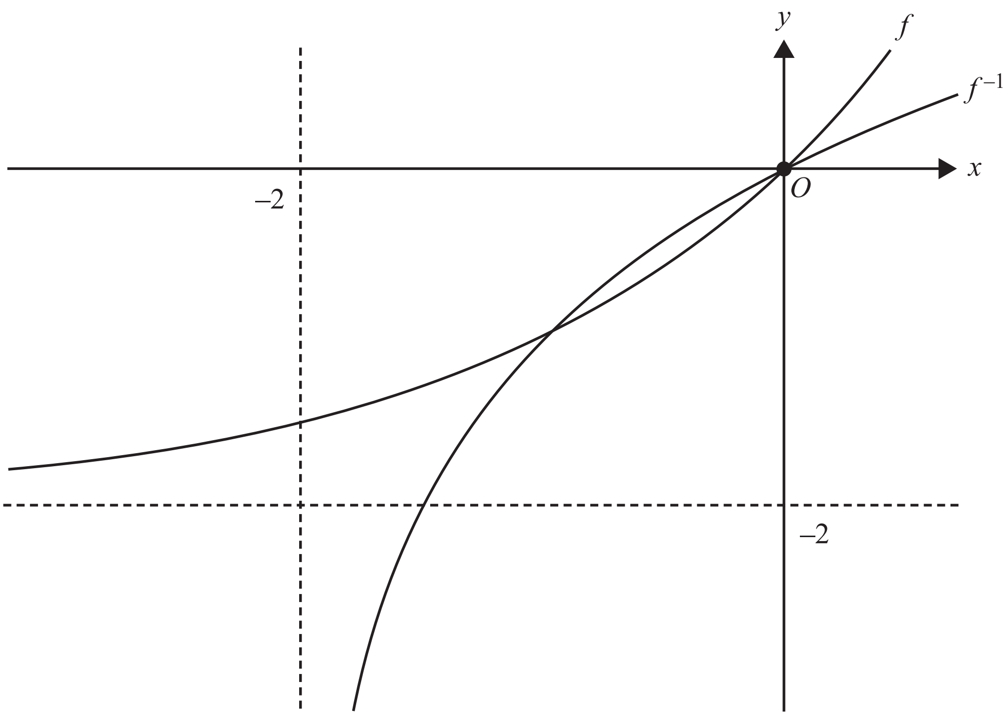
a. The transformation $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ maps the graph of $y = 2^x$ onto the graph of f.

State the values of c and d. 2 marks

b. Find the rule and domain for f^{-1}, the inverse function of f. 2 marks

c. Find the area bounded by the graphs of f and f^{-1}. 3 marks

d. Part of the graphs of f and f^{-1} are shown below.



Find the gradient of f and the gradient of f^{-1} at $x = 0$. 2 marks

The functions of g_k, where $k \in R^+$, are defined with domain R such that $g_k(x) = 2e^{kx} - 2$.

e. Find the value of k such that $g_k(x) = f(x)$. 1 mark

f. Find the rule for the inverse functions g_k^{-1} of g_k, where $k \in R^+$. 1 mark

g. **i.** Describe the transformation that maps the graph of g_1 onto the graph of g_k. 1 mark

ii. Describe the transformation that maps the graph of g_1^{-1} onto the graph of g_k^{-1}. 1 mark

h. The lines L_1 and L_2 are the tangents at the origin to the graphs of g_k and g_k^{-1} respectively.

Find the value(s) of k for which the angle between L_1 and L_2 is 30°. 2 marks

i. Let p be the value of k for which $g_k(x) = g_k^{-1}(x)$ has only one solution.

i. Find p. 2 marks

ii. Let $A(k)$ be the area bounded by the graphs of g_k and g_k^{-1} for all $k > p$.

State the smallest value of b such that $A(k) < b$. 1 mark

END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM
AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2017

MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = E(X)$	variance	$\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

Probability distribution		**Mean**	**Variance**
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$E(\hat{P}) = p$
standard deviation	$\text{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET