Year 2017

VCE

Mathematical Methods

Trial Examination 1

Solutions

KILBAHA MULTIMEDIA PUBLISHING PO BOX 2227 KEW VIC 3101 AUSTRALIA	TEL: (03) 9018 5376 FAX: (03) 9817 4334 kilbaha@gmail.com http://kilbaha.com.au

Question 1

a. $f(x) = e^{\tan(3x)}$

$y = e^{\tan(3x)} = e^u \quad , \; u = \tan(3x)$

$\dfrac{dy}{du} = e^u \qquad \dfrac{du}{dx} = \dfrac{3}{\cos^2(3x)}$ Chain Rule M1

$f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{3}{\cos^2(3x)} e^{\tan(3x)}$

$f'\left(\dfrac{\pi}{9}\right) = \dfrac{3}{\cos^2\left(\dfrac{\pi}{3}\right)} e^{\tan\left(\frac{\pi}{3}\right)} = \dfrac{3}{\left(\dfrac{1}{2}\right)^2} e^{\sqrt{3}}$

$f'\left(\dfrac{\pi}{9}\right) = 12e^{\sqrt{3}}$ A1

b. $\tan(120^0) = -\sqrt{3}$

$g(x) = \cos(2x) \quad , \; g'(x) = -2\sin(2x)$

$g'(p) = -2\sin(2p) = -\sqrt{3} \quad , \; 0 < p < \pi$

$\sin(2p) = \dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad 2p = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3} \; , \; \pi - \dfrac{\pi}{3}$ M1

$2p = \dfrac{\pi}{3}, \dfrac{2\pi}{3} \quad , \quad 0 < 2p < 2\pi$

$p = \dfrac{\pi}{6}, \dfrac{\pi}{3}$ A1

Question 2

i. $\hat{p} = \dfrac{1}{10} \qquad \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} \le \dfrac{1}{50}$

$\sqrt{\dfrac{0.1 \times 0.9}{n}} = \sqrt{\dfrac{0.09}{n}} = \dfrac{0.3}{\sqrt{n}} \le \dfrac{1}{50} \quad \Rightarrow \quad \sqrt{n} \ge 50 \times 0.3 = 15$ M1

$n = 15^2 = 225$ A1

ii. $S \overset{d}{=} \text{Bi}(n = 5, p = 0.1)$

$\Pr(S = 3) = \dbinom{5}{3} \times 0.1^3 \times 0.9^2 = \dfrac{5 \times 4}{2} \times 0.1^3 \times 0.9^2$

$= 0.0081 = \dfrac{81}{10000}$ A1

Question 3

a. $h: R \to R$, $h(x) = 3 - 6\cos\left(\frac{\pi x}{3}\right)$

amplitude 6, period $\frac{2\pi}{\frac{\pi}{3}} = 6$, range $[-3, 9]$ A1

b. $3 - 6\cos\left(\frac{\pi x}{3}\right) = 0$

$6\cos\left(\frac{\pi x}{3}\right) = 3 \quad \Rightarrow \quad \cos\left(\frac{\pi x}{3}\right) = \frac{1}{2}$

$\frac{\pi x}{3} = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right) = 2n\pi \pm \frac{\pi}{3}$

$x = 6n \pm 1$, $n \in J$ A1

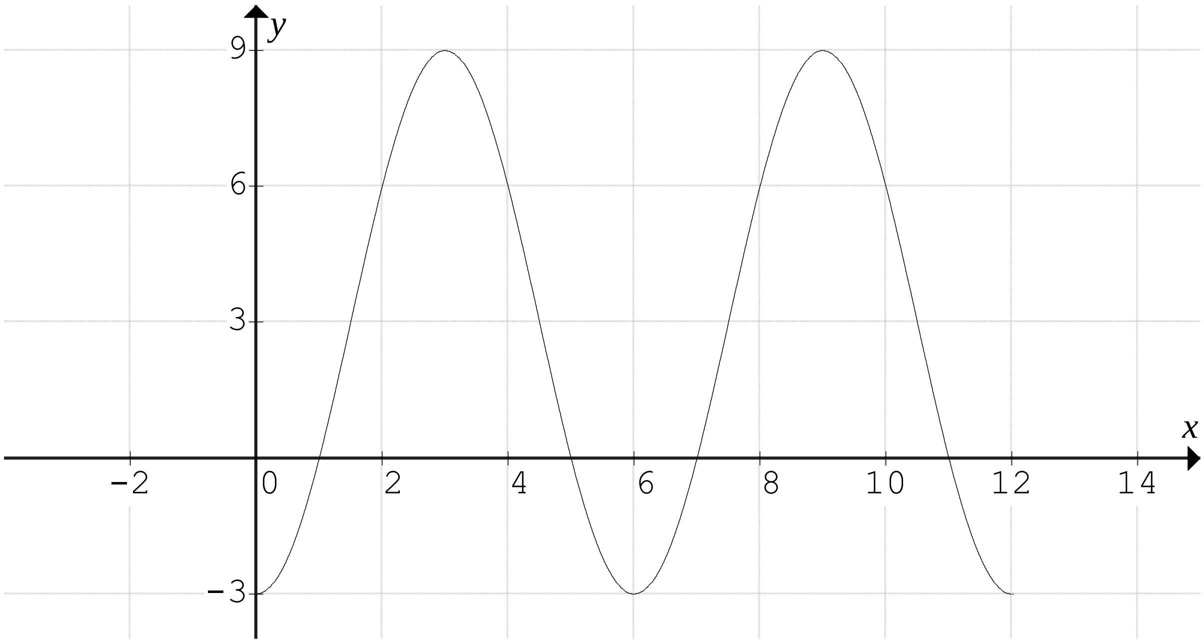
c. $f : [0,12] \to R$, $f(x) = 3 - 6\cos\left(\frac{\pi x}{3}\right)$

crosses the *x*-axis $x = 6n \pm 1$, $x = 1, 5, 7, 11$, $(1,0)$, $(5,0)$, $(7,0)$, $(11,0)$

endpoint $x = 0$ $y = 3 - 6\cos(0) = -3$ $(0, -3)$

endpoint $x = 12$ $y = 3 - 6\cos(4\pi) = -3$ $(12, -3)$ A1

maximum $(3,9)$, $(9,9)$ minimum $(6,-3)$ G1

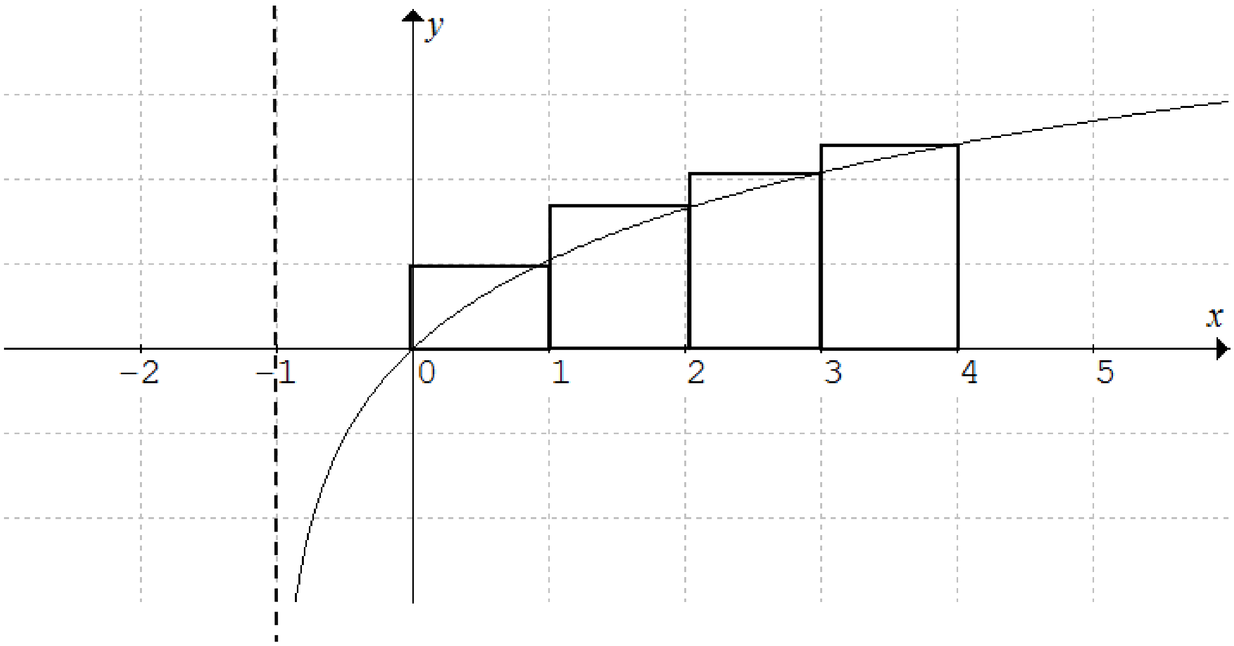


Question 4

$f: D \to R$, $f(x) = 3\log_e(x+1)$

a. $x = -1$ vertical asymptote, domain $(-1, \infty)$, range R A1

graph, shape, passes through the origin $(0, 0)$ G1



$x = -1$ *VA*

b.i Drawing the rectangles above G1

ii.

x	0	1	2	3	4
$y = 3\log_e(x+1)$	0	$3\log_e(2)$	$3\log_e(3)$	$3\log_e(4)$	$3\log_e(5)$

$A_u = 1\left[3\log_e(2) + 3\log_e(3) + 3\log_e(4) + 3\log_e(5)\right]$

$= 3\log_e(2 \times 3 \times 4 \times 5)$

$= 3\log_e(120)$ $b = 3$ $c = 120$ A1

c. $f:\ y = 3\log_e(x+1)$ swap x and y

$f^{-1}:\ x = 3\log_e(y+1)$

$\frac{x}{3} = \log_e(y+1) \Rightarrow y + 1 = e^{\frac{x}{3}}$ M1

$y = f^{-1}(x) = e^{\frac{x}{3}} - 1$ A1

d.i. $\int_0^{\log_e(125)} \left(e^{\frac{x}{3}} - 1\right) dx$

$= \left[3e^{\frac{x}{3}} - x\right]_0^{\log_e(125)}$ A1

$= \left(3e^{\frac{1}{3}\log_e 5^3} - 3\log_e(5)\right) - \left(3e^0 - 0\right)$

$= 15 - 3\log_e(5) - 3$

$= 12 - 3\log_e(5)$ A1

ii. Now the area of the inverse function $A_1 = \int_0^{\log_e(125)} \left(e^{\frac{x}{3}} - 1\right) dx = 12 - 3\log_e(5)$ units^2

and the area of original function $A_2 = \int_0^4 3\log_e(x+1)\,dx$ equals the area of the rectangle $A_1 + A_2 = 4 \times 3\log_e(5)$, so that

$A_2 = \int_0^4 3\log_e(x+1)\,dx = 12\log_e(5) - \left(12 - 3\log_e(5)\right)$

$= 15\log_e(5) - 12$ units^2 A1

Question 5 $f : [0, \infty) \to R$, $f(x) = k\sqrt{3x+4}$

a. $f(4) = k\sqrt{16} = 4k$ $f(0) = 2k$

average rate of change $\dfrac{f(4) - f(0)}{4 - 0} = \dfrac{4k - 2k}{4} = \dfrac{k}{2} = 9$ A1

$k = 18$ A1

b. average value $\dfrac{1}{4-0}\int_0^4 18\sqrt{3x+4}\,dx$

$= \dfrac{9}{2} \times \dfrac{2}{3} \times \dfrac{1}{3}\left[(3x+4)^{\frac{3}{2}}\right]_0^4$ A1

$= 16^{\frac{3}{2}} - 4^{\frac{3}{2}} = 64 - 8$

$= 56$ A1

Question 6

$y = 5 - 2\sqrt{55 - 6x - x^2} = 5 - 2\sqrt{55 - \left(x^2 + 6x + 9\right) + 9}$

$y - 5 = -2\sqrt{64 - (x+3)^2}$

$y - 5 = -4\sqrt{\dfrac{64 - (x+3)^2}{4}} = -4\sqrt{16 - \left(\dfrac{x+3}{2}\right)^2}$ M1

image curve $\dfrac{y' - 5}{-4} = \sqrt{16 - \left(\dfrac{x'+3}{2}\right)^2}$ original curve $y = \sqrt{16 - x^2}$ M1

$x = \dfrac{x'+3}{2}$, $y = \dfrac{y'-5}{-4}$

$x' = 2x - 3$, $y' = -4y + 5$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ $a = 2$, $b = -4$, $h = -3$, $k = 5$ A1

Question 7 $f(x) = 4 - x^2$

$s = d(OP) = \sqrt{p^2 + \left(f(p)\right)^2}$ M1

$= \sqrt{p^2 + \left(4 - p^2\right)^2}$

$= \sqrt{p^2 + 16 - 8p^2 + p^4}$

$= \sqrt{16 - 7p^2 + p^4} = \left(16 - 7p^2 + p^4\right)^{\frac{1}{2}}$ A1

$\dfrac{ds}{dp} = \dfrac{1}{2} \times \left(4p^3 - 14p\right)\left(16 - 7p^2 + p^4\right)^{-\frac{1}{2}}$

$\dfrac{ds}{dp} = \dfrac{p\left(2p^2 - 7\right)}{\sqrt{16 - 7p^2 + p^4}} = 0$ solving, since $0 \le p \le 2$ gives $p = 0$, $\sqrt{\dfrac{7}{2}}$ M1

when $p = 0$, $s = 4$ this is a maximum distance

the minimum distance occurs when $p = \sqrt{\dfrac{7}{2}} = \dfrac{\sqrt{14}}{2}$ A1

Question 8

a. Using the product rule

$$\frac{d}{dx}\left[x\cos(2x)\right]=\cos(2x)-2x\sin(2x)$$ A1

b.i. $\int\left(\cos(2x)-2x\sin(2x)\right)dx=x\cos(2x)$

$$\int\cos(2x)\,dx-2\int x\sin(2x)\,dx=x\cos(2x)$$

$$2\int x\sin(2x)\,dx=\int\cos(2x)\,dx\ -x\cos(2x)$$

$$2\int x\sin(2x)\,dx=\frac{1}{2}\sin(2x)-x\cos(2x)+C$$ M1

$$\int x\sin(2x)\,dx=\frac{1}{4}\sin(2x)-\frac{x}{2}\cos(2x)+c$$

$$k\int_0^{\frac{\pi}{2}}x\sin(2x)\,dx=k\left[\frac{1}{4}\sin(2x)-\frac{x}{2}\cos(2x)\right]_0^{\frac{\pi}{2}}=1 \quad \text{since it is a pdf}$$ M1

$$=k\left[\left(\frac{1}{4}\sin(\pi)-\frac{\pi}{4}\cos(\pi)\right)-\left(\frac{1}{4}\sin(0)-0\right)\right]=1$$

$$\frac{k\pi}{4}=1 \quad \Rightarrow \quad k=\frac{4}{\pi}$$

ii. the median m, satisfies $\frac{4}{\pi}\int_0^m x\sin(2x)\,dx=\frac{4}{\pi}\left[\frac{1}{4}\sin(2x)-\frac{x}{2}\cos(2x)\right]_0^m=\frac{1}{2}$

$$=\frac{4}{\pi}\left[\left(\frac{1}{4}\sin(2m)-\frac{m}{2}\cos(2m)\right)-\left(\frac{1}{4}\sin(0)-0\right)\right]=\frac{1}{2}$$ M1

$$=\frac{1}{\pi}\left(\sin(2m)-2m\cos(2m)\right)=\frac{1}{2}$$

$$\sin(2m)-2m\cos(2m)=\frac{\pi}{2} \quad , \quad p=\frac{\pi}{2}$$ A1

iii. the mode *M*, satisfies $f'(M)=0$

$f(x)=x\sin(2x)$

$f'(x)=\sin(2x)+2x\cos(2x)$

$f'(M)=\sin(2M)+2M\cos(2M)=0$ M1

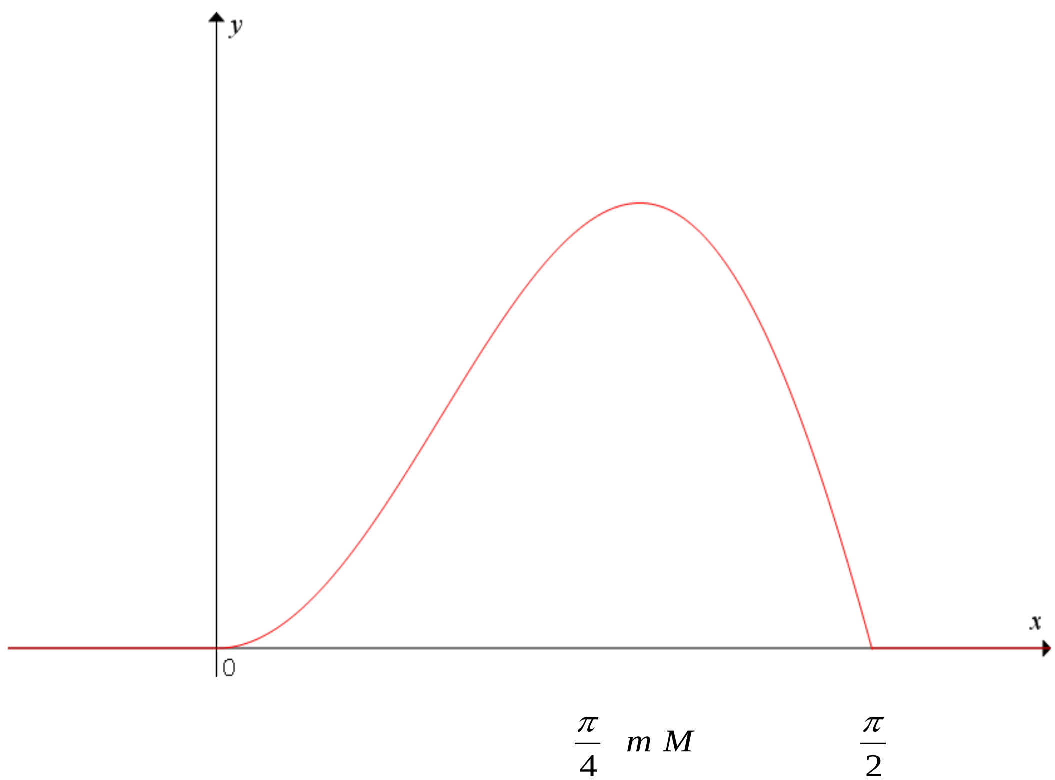
$\sin(2M)=-2M\cos(2M)$

$\frac{\sin(2M)}{2M\cos(2M)}=-1$

$\frac{\tan(2M)}{2M}=-1 \quad , \quad q=-1$ A1

iv. graph, correct shape, zero elsewhere G1

correct ordering $0<\frac{\pi}{4}<m<M<\frac{\pi}{2}$ A1



END OF SUGGESTED SOLUTIONS