

Year 2017

VCE

Mathematical Methods

Trial Examination 1

Solutions



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Question 1

a. $f(x) = e^{\tan(3x)}$
 $y = e^{\tan(3x)} = e^u$, $u = \tan(3x)$
 $\frac{dy}{du} = e^u$ $\frac{du}{dx} = \frac{3}{\cos^2(3x)}$ Chain Rule M1

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{3}{\cos^2(3x)} e^{\tan(3x)}$$

$$f'\left(\frac{\pi}{9}\right) = \frac{3}{\cos^2\left(\frac{\pi}{3}\right)} e^{\tan\left(\frac{\pi}{3}\right)} = \frac{3}{\left(\frac{1}{2}\right)^2} e^{\sqrt{3}}$$

$$f'\left(\frac{\pi}{9}\right) = 12e^{\sqrt{3}} \quad \text{A1}$$

b. $\tan(120^\circ) = -\sqrt{3}$
 $g(x) = \cos(2x)$, $g'(x) = -2\sin(2x)$
 $g'(p) = -2\sin(2p) = -\sqrt{3}$, $0 < p < \pi$
 $\sin(2p) = \frac{\sqrt{3}}{2} \Rightarrow 2p = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}, \pi - \frac{\pi}{3}$ M1

$$2p = \frac{\pi}{3}, \frac{2\pi}{3}, \quad 0 < 2p < 2\pi$$

$$p = \frac{\pi}{6}, \frac{\pi}{3} \quad \text{A1}$$

Question 2

i. $\hat{p} = \frac{1}{10}$ $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq \frac{1}{50}$
 $\sqrt{\frac{0.1 \times 0.9}{n}} = \sqrt{\frac{0.09}{n}} = \frac{0.3}{\sqrt{n}} \leq \frac{1}{50} \Rightarrow \sqrt{n} \geq 50 \times 0.3 = 15$ M1
 $n = 15^2 = 225$ A1

ii. $S \stackrel{d}{=} \text{Bi}(n=5, p=0.1)$
 $\Pr(S=3) = \binom{5}{3} \times 0.1^3 \times 0.9^2 = \frac{5 \times 4}{2} \times 0.1^3 \times 0.9^2$
 $= 0.0081 = \frac{81}{10000}$ A1

Question 3

a. $h: R \rightarrow R, h(x) = 3 - 6\cos\left(\frac{\pi x}{3}\right)$

amplitude 6, period $\frac{2\pi}{\frac{\pi}{3}} = 6$, range $[-3, 9]$ A1

b. $3 - 6\cos\left(\frac{\pi x}{3}\right) = 0$

$$6\cos\left(\frac{\pi x}{3}\right) = 3 \Rightarrow \cos\left(\frac{\pi x}{3}\right) = \frac{1}{2}$$

$$\frac{\pi x}{3} = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right) = 2n\pi \pm \frac{\pi}{3}$$

$$x = 6n \pm 1, n \in J \quad \text{A1}$$

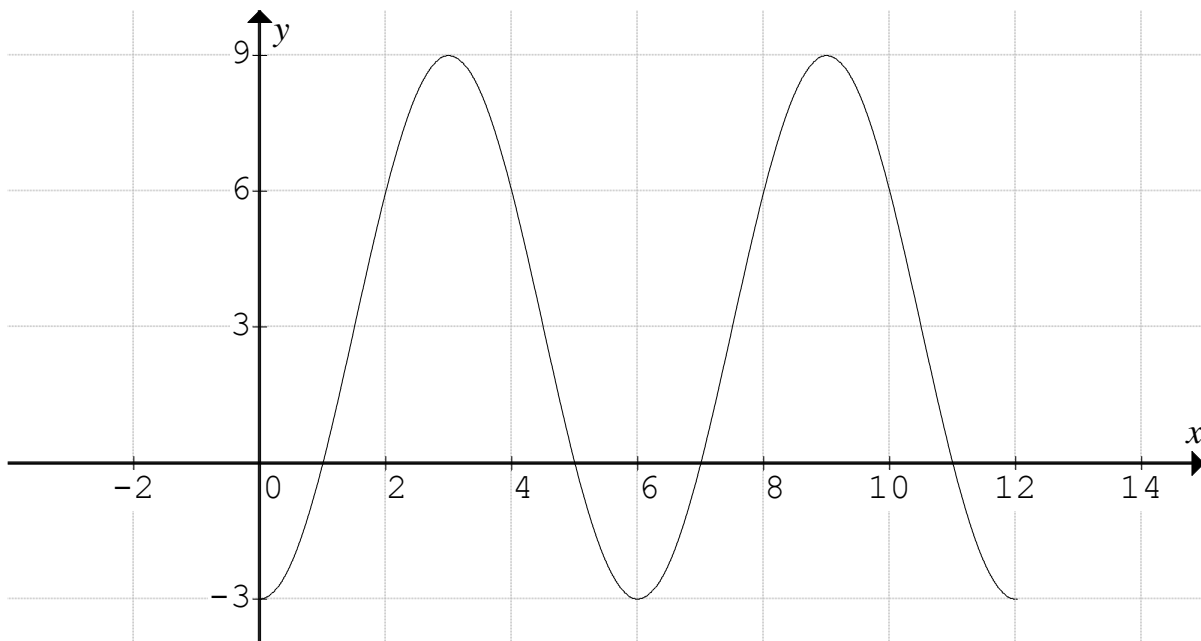
c. $f: [0, 12] \rightarrow R, f(x) = 3 - 6\cos\left(\frac{\pi x}{3}\right)$

crosses the x -axis $x = 6n \pm 1, x = 1, 5, 7, 11, (1, 0), (5, 0), (7, 0), (11, 0)$

endpoint $x = 0, y = 3 - 6\cos(0) = -3, (0, -3)$

endpoint $x = 12, y = 3 - 6\cos(4\pi) = -3, (12, -3)$ A1

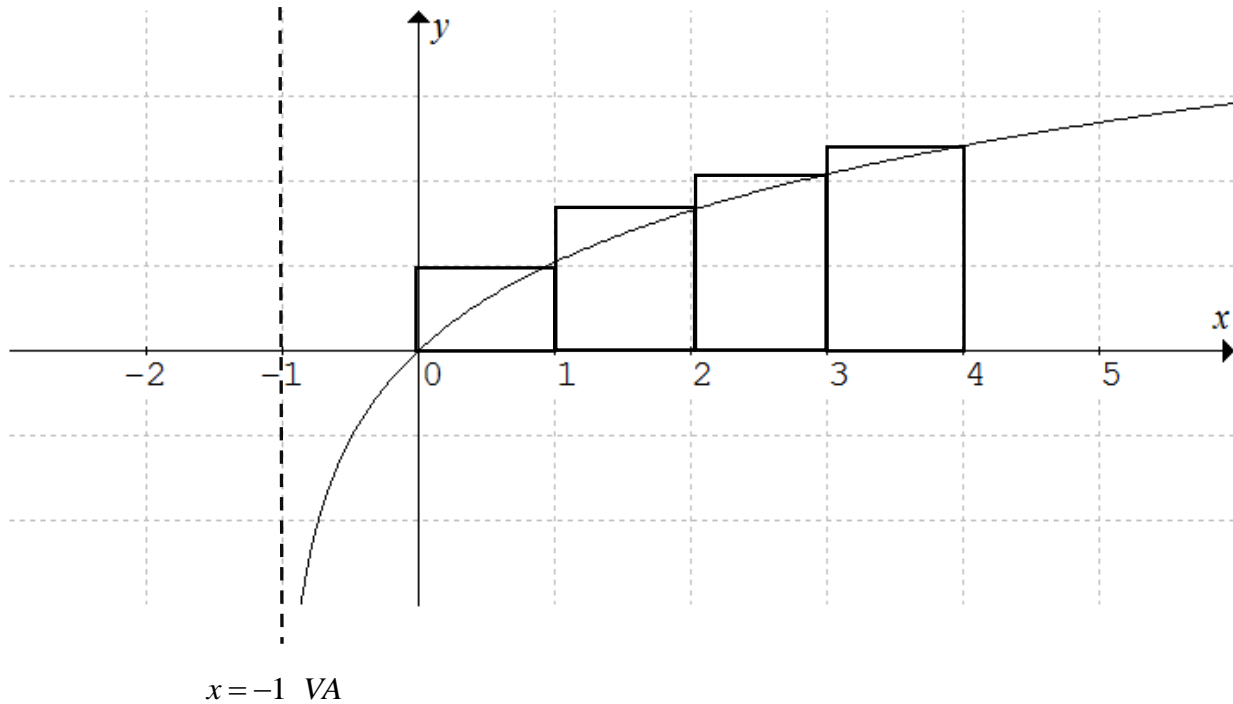
maximum $(3, 9), (9, 9)$ minimum $(6, -3)$ G1



Question 4

$f : D \rightarrow R, f(x) = 3\log_e(x+1)$

- a. $x = -1$ vertical asymptote, domain $(-1, \infty)$, range R A1
 graph, shape, passes through the origin $(0,0)$ G1



- b.i Drawing the rectangles above G1

ii.

x	0	1	2	3	4
$y = 3\log_e(x+1)$	0	$3\log_e(2)$	$3\log_e(3)$	$3\log_e(4)$	$3\log_e(5)$

$$A_u = 1[3\log_e(2) + 3\log_e(3) + 3\log_e(4) + 3\log_e(5)]$$

$$= 3\log_e(2 \times 3 \times 4 \times 5)$$

$$= 3\log_e(120) \quad b = 3 \quad c = 120 \quad \text{A1}$$

- c. $f : y = 3\log_e(x+1)$ swap x and y
 $f^{-1} : x = 3\log_e(y+1)$
 $\frac{x}{3} = \log_e(y+1) \Rightarrow y+1 = e^{\frac{x}{3}}$ M1
 $y = f^{-1}(x) = e^{\frac{x}{3}} - 1$ A1

d.i.
$$\int_0^{\log_e(125)} \left(e^{\frac{x}{3}} - 1 \right) dx$$

$$= \left[3e^{\frac{x}{3}} - x \right]_0^{\log_e(125)}$$

$$= \left(3e^{\frac{1}{3}\log_e 5^3} - 3\log_e(5) \right) - (3e^0 - 0)$$

$$= 15 - 3\log_e(5) - 3$$

$$= 12 - 3\log_e(5)$$
A1

ii. Now the area of the inverse function $A_1 = \int_0^{\log_e(125)} \left(e^{\frac{x}{3}} - 1 \right) dx = 12 - 3\log_e(5)$ units²
 and the area of original function $A_2 = \int_0^4 3\log_e(x+1) dx$ equals the area of the rectangle $A_1 + A_2 = 4 \times 3\log_e(5)$, so that

$$A_2 = \int_0^4 3\log_e(x+1) dx = 12\log_e(5) - (12 - 3\log_e(5))$$

$$= 15\log_e(5) - 12 \text{ units}^2$$
A1

Question 5 $f : [0, \infty) \rightarrow R, f(x) = k\sqrt{3x+4}$

a. $f(4) = k\sqrt{16} = 4k$ $f(0) = 2k$
 average rate of change $\frac{f(4) - f(0)}{4 - 0} = \frac{4k - 2k}{4} = \frac{k}{2} = 9$ A1
 $k = 18$ A1

b. average value $\frac{1}{4-0} \int_0^4 18\sqrt{3x+4} dx$

$$= \frac{9}{2} \times \frac{2}{3} \times \frac{1}{3} \left[(3x+4)^{\frac{3}{2}} \right]_0^4$$
A1

$$= 16^{\frac{3}{2}} - 4^{\frac{3}{2}} = 64 - 8$$

$$= 56$$
A1

Question 6

$$y = 5 - 2\sqrt{55 - 6x - x^2} = 5 - 2\sqrt{55 - (x^2 + 6x + 9) + 9}$$

$$y - 5 = -2\sqrt{64 - (x + 3)^2}$$

$$y - 5 = -4\sqrt{\frac{64 - (x + 3)^2}{4}} = -4\sqrt{16 - \left(\frac{x + 3}{2}\right)^2} \quad \text{M1}$$

image curve $\frac{y' - 5}{-4} = \sqrt{16 - \left(\frac{x' + 3}{2}\right)^2}$ original curve $y = \sqrt{16 - x^2}$ M1

$$x = \frac{x' + 3}{2}, \quad y = \frac{y' - 5}{-4}$$

$$x' = 2x - 3, \quad y' = -4y + 5$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix} \quad a = 2, \quad b = -4, \quad h = -3, \quad k = 5 \quad \text{A1}$$

Question 7 $f(x) = 4 - x^2$

$$s = d(OP) = \sqrt{p^2 + (f(p))^2} \quad \text{M1}$$

$$= \sqrt{p^2 + (4 - p^2)^2}$$

$$= \sqrt{p^2 + 16 - 8p^2 + p^4}$$

$$= \sqrt{16 - 7p^2 + p^4} = (16 - 7p^2 + p^4)^{\frac{1}{2}} \quad \text{A1}$$

$$\frac{ds}{dp} = \frac{1}{2} \times (4p^3 - 14p)(16 - 7p^2 + p^4)^{-\frac{1}{2}}$$

$$\frac{ds}{dp} = \frac{p(2p^2 - 7)}{\sqrt{16 - 7p^2 + p^4}} = 0 \quad \text{solving, since } 0 \leq p \leq 2 \text{ gives } p = 0, \sqrt{\frac{7}{2}} \quad \text{M1}$$

when $p = 0$, $s = 4$ this is a maximum distance

the minimum distance occurs when $p = \sqrt{\frac{7}{2}} = \frac{\sqrt{14}}{2}$ A1

Question 8**a.** Using the product rule

$$\frac{d}{dx}[x \cos(2x)] = \cos(2x) - 2x \sin(2x) \quad \text{A1}$$

b.i. $\int (\cos(2x) - 2x \sin(2x)) dx = x \cos(2x)$

$$\int \cos(2x) dx - 2 \int x \sin(2x) dx = x \cos(2x)$$

$$2 \int x \sin(2x) dx = \int \cos(2x) dx - x \cos(2x)$$

$$2 \int x \sin(2x) dx = \frac{1}{2} \sin(2x) - x \cos(2x) + C \quad \text{M1}$$

$$\int x \sin(2x) dx = \frac{1}{4} \sin(2x) - \frac{x}{2} \cos(2x) + c$$

$$k \int_0^{\frac{\pi}{2}} x \sin(2x) dx = k \left[\frac{1}{4} \sin(2x) - \frac{x}{2} \cos(2x) \right]_0^{\frac{\pi}{2}} = 1 \quad \text{since it is a pdf} \quad \text{M1}$$

$$= k \left[\left(\frac{1}{4} \sin(\pi) - \frac{\pi}{4} \cos(\pi) \right) - \left(\frac{1}{4} \sin(0) - 0 \right) \right] = 1$$

$$\frac{k\pi}{4} = 1 \Rightarrow k = \frac{4}{\pi}$$

ii. the median m , satisfies $\frac{4}{\pi} \int_0^m x \sin(2x) dx = \frac{4}{\pi} \left[\frac{1}{4} \sin(2x) - \frac{x}{2} \cos(2x) \right]_0^m = \frac{1}{2}$

$$= \frac{4}{\pi} \left[\left(\frac{1}{4} \sin(2m) - \frac{m}{2} \cos(2m) \right) - \left(\frac{1}{4} \sin(0) - 0 \right) \right] = \frac{1}{2} \quad \text{M1}$$

$$= \frac{1}{\pi} (\sin(2m) - 2m \cos(2m)) = \frac{1}{2}$$

$$\sin(2m) - 2m \cos(2m) = \frac{\pi}{2}, \quad p = \frac{\pi}{2} \quad \text{A1}$$

iii. the mode M , satisfies $f'(M) = 0$

$$f(x) = x \sin(2x)$$

$$f'(x) = \sin(2x) + 2x \cos(2x)$$

$$f'(M) = \sin(2M) + 2M \cos(2M) = 0$$

M1

$$\sin(2M) = -2M \cos(2M)$$

$$\frac{\sin(2M)}{2M \cos(2M)} = -1$$

$$\frac{\tan(2M)}{2M} = -1, \quad q = -1$$

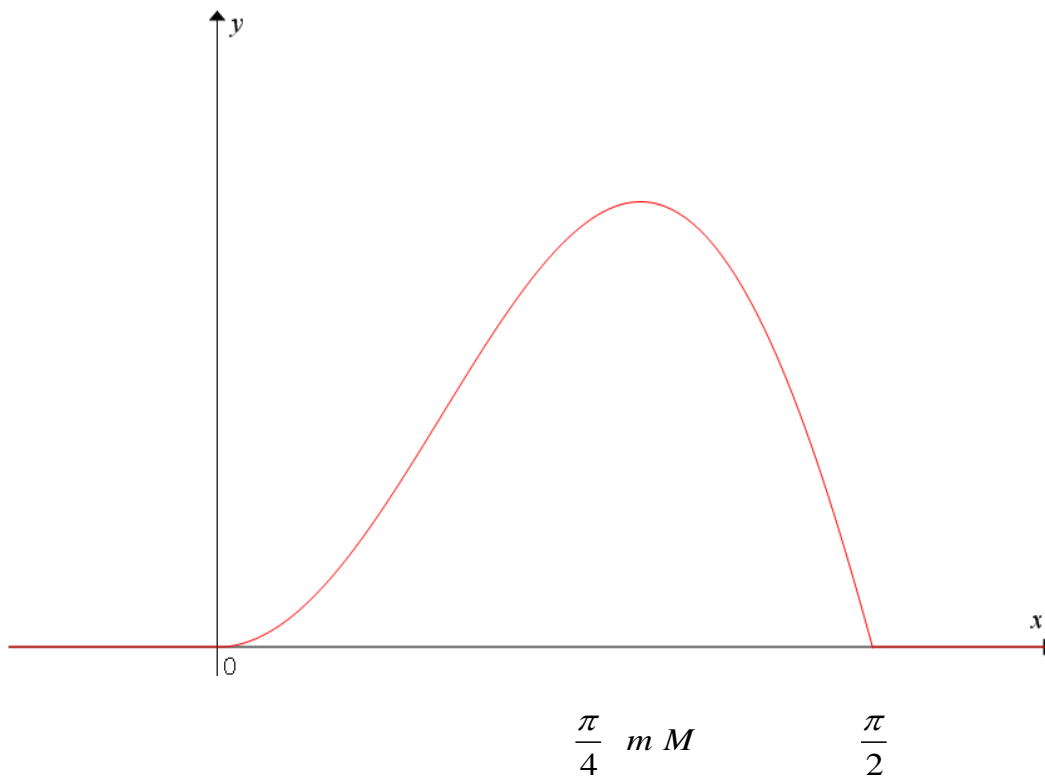
A1

iv. graph, correct shape, zero elsewhere

G1

$$\text{correct ordering } 0 < \frac{\pi}{4} < m < M < \frac{\pi}{2}$$

A1



END OF SUGGESTED SOLUTIONS