MATHEMATICAL METHODS

Units 3 & 4 – Written examination 2

(**TSSM's** ***2012 trial exam updated for the current study design)***

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: B

Explanation:

Amplitude = |-2| = 2
Period = $\frac{2\pi}{n} = \frac{2\pi}{3}$

Question 2

Answer: C

Explanation:

$$\frac{kx-4}{x+1} = x$$
$$kx - 4 = x^2 + x$$
$$x^2 + (1-k)x + 4 = 0$$

For unique solution discriminant = 0

$(1-k)^2 - 16 = 0 \Rightarrow k = 5, -3 \ but \ as \ k \ is \ positive \ k = 5$

Question 3

Answer: E

Explanation:

Sketch graph on CAS: Read the range of this function, as range of the function is the same as domain of the inverse.

Question 4

Answer: B

Explanation:

$$(f(x))^3 = (e^x - e^{-x})^3 = e^{3x} - e^{-3x} - 3(e^x - e^{-x}) = f(3x) - 3f(x)$$

Question 5

Answer: E

Explanation:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} 4h + 8 = 8$$

Question 6

Answer: D

Explanation:

$$Solve\ the\ following\ simultaneous\ equations$$

$$3 = 4a + 2b \quad and \quad -b = 4a$$

Question 7

Answer: A

Explanation:

$$\int \frac{dy}{dx} = \int f(x)dx$$
$$y = F(x) + c$$
$$Applying\ the\ limits, y = F(3) - F(2)$$

Question 8

Answer: E

Explanation:

$y - 2 = m(x - 1)\ or\ y = mx - m + 2,$

The x-intercept of the line is $\left(\frac{m-2}{m}, 0\right)$ and the y-intercept is$(0, -m + 2)$

Area = $\frac{1}{2} \times (-m + 2) \times \left(\frac{m-2}{m}\right) = -\frac{1}{2}\left(m - 4 + \frac{4}{m}\right)$

$$A' = 0\ gives\ m = \pm 2\ and\ the\ area\ is\ minimum\ at\ m = -2$$

Question 9

Answer: A

Explanation:

Use CAS: $solve\left(\int_0^a (3x - 6)dx = 0, x\right)$

Question 10

Answer: C

Explanation:

Solve $\mu - 2\sigma = 42\ and\ \mu + 2\sigma = 58$ on CAS

Question 11

Answer: B

Explanation:

Solve $log_e(5a + 3) = 4$ on CAS

Question 12

Answer: C

Explanation:

$$Pr(X < 4.5) = \Pr(Z < -1) = \Pr(Z > 1)$$

Question 13

Answer: B

Explanation:

Read all the sequences carefully to determine the correct choice.

Question 14

Answer: B

Explanation:

Graph $f\left(g(x)\right) = e^{sinx}$ on CAS

Question 15

Answer: D

Explanation:

Use CAS to find the value of $\left(\frac{1}{\frac{\pi}{4}-\frac{\pi}{8}}\right)\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \tan x \, dx$

Question 16

Answer: B

Explanation:

Graph on CAS.

Question 17

Answer: D

Explanation:

Solve on CAS: $solve(\tan(2x) = \sqrt{3}, x)$ and then add the two solutions.

Question 18

Answer: B

Explanation:

Solve $\int_0^k 4e^{-4x}dx = 0.8$ on CAS.

Question 19

Answer: B

Explanation:

Solve the equations: $np = 80 \;\; and \;\; np(1-p) = 16$

Question 20

Answer: C

Explanation:

Use CAS to sketch the graph and read the turning point.
Alternatively, $\frac{dy}{dx} = \frac{1}{x} - 2 = 0 \; implies \; x = \frac{1}{2} \;\; and \; y = -1$

Question 21

Answer: E

Explanation:

Solve on CAS for x.

Question 22

Answer: C

Explanation:

Note that the function is not differentiable at the points x = -5, -1, 1, 6

SECTION 2: Analysis Questions

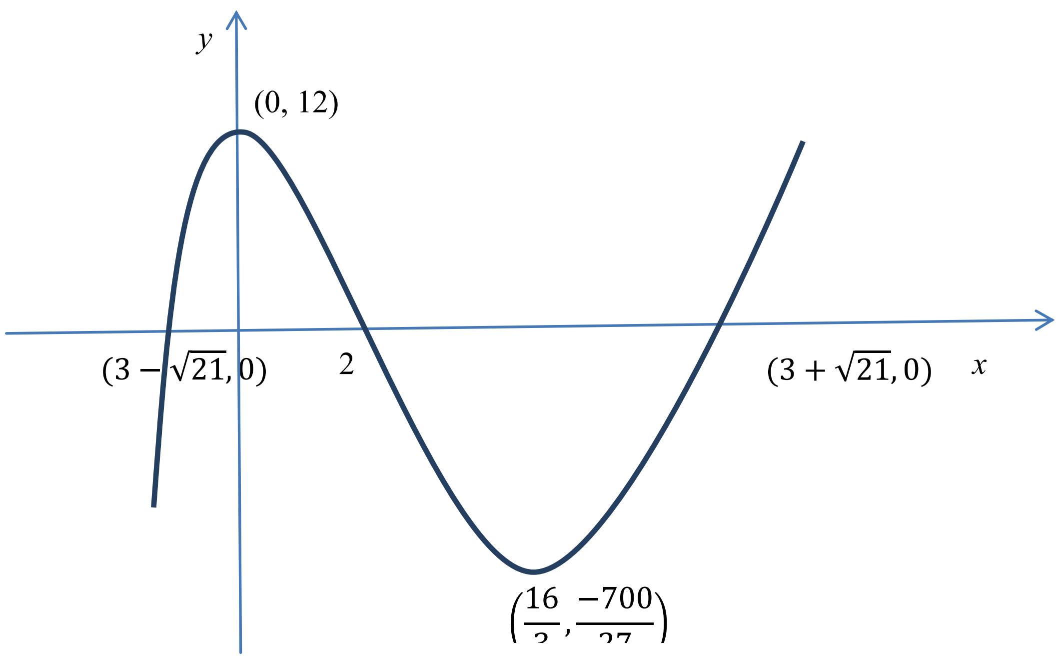
Question 1

a. $f(x) = \frac{1}{2}x^3 - 4x^2 + 12$

x-intercepts are (-1.58, 0), (2, 0) and (7.58, 0)

y-intercept is (0, 12)

Turning points are (0, 12) and (5.33, -25.93)



M2+A1
3 marks

b. $\int_{3-\sqrt{21}}^{2} (0.5x^2(x-8)+12)\,dx - \int_{2}^{\sqrt{21}+3} (0.5x^2(x-8)+12)\,dx = \frac{362}{3}$

M1+A1
2 marks

c. $g(x) = ax^3 - 8ax^2$

$$g'(x) = 3ax^2 - 16ax$$

$$gradient\ of\ the\ tangent\ at\ x = 1 = 3a - 16a$$

$$= -13a\ Equation\ of\ the\ tangent\ is{:}\ y + 7a = -13a(x - 1)$$

$$y = -13ax + 13a - 7a \quad or \quad y = -13ax + 6a$$

M2+A1

3 marks

d. On CAS: $solve\ (ax^3 - 8ax^2 = -13ax + 6a, x)$

$$x = 1, 6$$

$$\left|\int_1^6 ax^2(x-8) - (6a - 13ax)\ dx\right| = \frac{625a}{12}\ \ square\ units$$

M2+A1

3marks

e. The point is (6, -72a)

Equation of tangent at x = 6 is:

$$y + 72a = 12a(x - 6)$$

For the two lines to be perpendicular $12a\ \times -13a = -1$

$$a = \frac{\sqrt{39}}{78}$$

M2+A1

3 marks

Question 2

a. Let $X \sim Bi(61, 8^2)$

$\Pr(X > 67) = 0.2266$ (on CAS use $normcdf(67, \infty, 61{,}8)$)

A1

1 mark

b. $\Pr(X > 67 | X > 61) = \frac{\Pr(X>67)}{\Pr\ (X>61)} = 0.4533$ $\left(CAS{:}\ \frac{normcdf(67,\infty,61,8)}{normcdf(61,\infty,61,8)}\right)$

M1+A1

1 mark

c. $z = \frac{59-61}{8} = -\frac{1}{4}$

M1
1 mark

d. $\Pr(X < 59) = \Pr(Z < -0.25) = 0.4013$

A1
1 mark

e. Binomial, $n = 6$, Probability of success = 0.2266

$$Pr(X \geq 2) = 1 - \Pr(X \leq 1) = 0.4098$$

M1
1 mark

f. $Pr(X > 67) = 0.98$

$$Pr\left(Z > \frac{67-\mu}{\sigma}\right) = 0.98$$

$$invnorm(0.02, 0, 1) = -2.05375$$

$$\frac{67-\mu}{2} = -2.05375 \ \ which\ gives\ \mu = 71.1075$$

M1+A1
2 marks

Question 3

a. $\hat{p} = \frac{7}{10}\ or\ 0.7$

M1
1 mark

b. $\sqrt{\frac{a(1-a)}{4}} = 0.2$

$0.04 = \frac{a(1-a)}{4}$

$0.16 = a - a^2$

$a^2 - a + 0.16 = 0$

$a = 0.8$ (as a improved)

M1+A1
2 marks

c. $(0.8 - 1.96 \times 0.2, 0.8 + 1.96 \times 0.2)$
$(0.408, 1)$

A1
1 mark

d. $\Pr(X > 80) = 0.04$
$Pr\left(Z > \frac{80-55}{\sigma}\right) = 0.04$
$\frac{80-55}{\sigma} = 1.75069 \ \ which\ gives\ \sigma = 14.28$

M1+A1
2 marks

e. $\Pr(X < 60|X \geq 30) = \frac{\Pr(30<X<60)}{\Pr\ (X\geq 30)} = 0.62$

M1+A1
2 marks

Question 4

a. $Minimum = 120 - 40 = 80 \quad and\ Maximum = 120 + 40 = 160$

M2
2 marks

b. Solve on CAS: $120 + 40\sin\left(\frac{\pi}{3}\left(5 - \frac{3}{2}\right)\right) = 100$

A1
1 mark

c. Solve $120 + 40\sin\left(\frac{\pi}{3}\left(t - \frac{3}{2}\right)\right) = 140$

$$\sin\left(\frac{\pi}{3}\left(t - \frac{3}{2}\right)\right) = \frac{1}{2}$$

$$\frac{\pi}{3}\left(t - \frac{3}{2}\right) = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$t = 2, 4, 8, 10$$

M2+A1
3 marks

d. Solve on CAS : $120+40\sin\left(\frac{\pi}{3}\left(t-\frac{3}{2}\right)\right)\geq 150$

which gives t $=$ 2.3 to 3.7 and 8.3 to 9.7

The farmer stays away for 2.76 weeks.

M1+A1

2 marks

e. Graph on CAS : By symmetry, if we consider one week either side of $t=6$, the maximum number of mice will occur when $t=5$. It follows that the maximum number is 100

M1+A1

2 marks

f. **i.** $M'(t)=\frac{40\pi}{3}\cos\left(\frac{\pi}{3}\left(t-\frac{3}{2}\right)\right)$

$=\frac{40\pi}{3}\cos\left(\frac{\pi}{3}t-\frac{\pi}{2}\right)$

$=\frac{40\pi}{3}\sin\left(\frac{\pi}{3}t\right)$

M1

ii. solving $M'(t)=\frac{20\pi}{3}$ gives $t=\frac{1}{2},\frac{5}{2},\frac{13}{2},\frac{17}{2}$ A2

3 marks

Question 5

a. $4h+16x+4x=240$

$4h+20x=240 \quad or \quad h=60-5x$

M2

2 marks

b. $V=x\times 4x\times h=4x^2(60-5x)=240x^2-20x^3$

M1+A1

2 marks

c. $V=2420\text{cm}^3$

A1

1 mark

d. $60-5x>0$

$-5x>-60$

$0<x<12$

M1+A1

2 marks

e. Solve on CAS :$solve(240x^2 - 20x^3 = 1620)$
$x = 3.00,\ 11.37$

A1
1 mark

f. $\frac{dV}{dx} = 480x - 60x^2 = 0$
$x = 8,\ 0$

The gradient of the curve changes from positive to negative as x passes through 8, hence x = 8 is a point of local maxima.
Max Volume = $240 \times 8^2 - 20 \times 8^3 = 5120\ m^3$

M1+A1
2 marks

Question 6

a. $solve\ (2\cos(3x) = 1, x)|0 \le x \le \pi$

$$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$

Points of intersection are $\left(\frac{\pi}{9}, 1\right)$, $\left(\frac{5\pi}{9}, 1\right)$ and $\left(\frac{7\pi}{9}, 1\right)$

M1+A1
2 marks

b. $x^2 - 2x + 1 = x - 2k$

$$x^2 - 3x + (1 + 2k) = 0$$

No intersection point means the determinant of the above quadratic equation is less than 0

$$9 - 4(1 + 2k) < 0$$

$$k > \frac{5}{8}$$

M2+A1
3 marks