Quality Assessment Tasks

NAME: ____________________________

UNITS 3 & 4 Practice Examination

VCE® Mathematical Methods

Written examination 1

Reading time: 15 minutes **Writing time: 1 hour**

QUESTION AND ANSWER BOOK
Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are **not** permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied
- A question and answer booklet of 13 pages.
- A double sided page of formulas.
- Working space is provided throughout the question answer booklet.

Instructions
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this examination paper are **not** drawn to scale

Question 1 (3 marks)

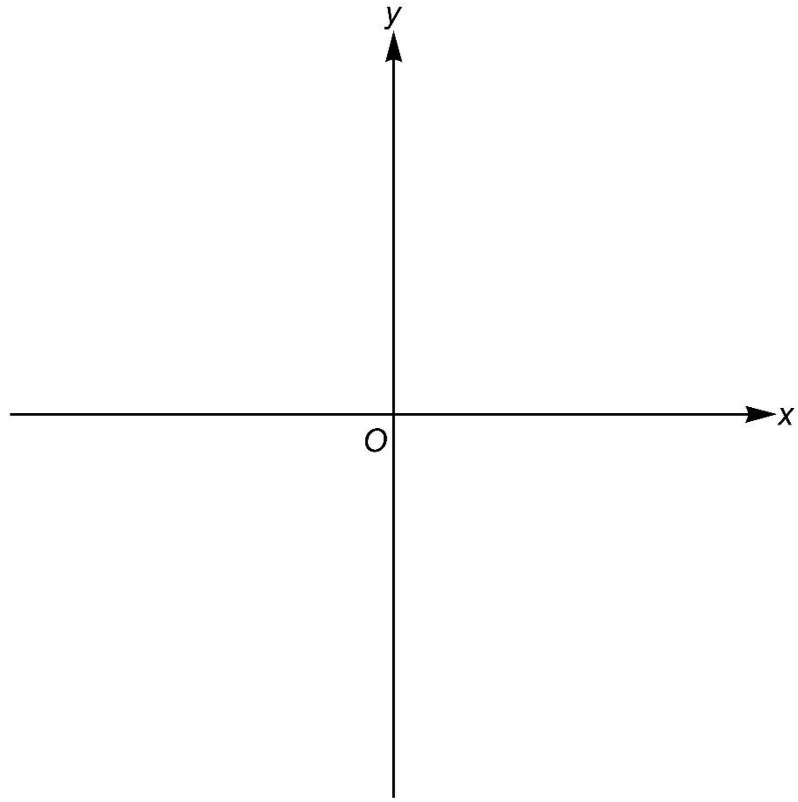
a. Differentiate $\sqrt{1-x^2}$ with respect to x. 1 mark

b. If $f(x) = \frac{x}{\cos(x)}$, find $f'(\pi)$. 2 marks

Question 2 (3 marks)

On the axes below, sketch the graph of $f: R\backslash\{-2\} \to R, f(x) = 1 - \frac{4}{x+2}$.

Label each axis intercept with its coordinates. Label each asymptote with its equation.



Question 3 (4 marks)

a. Find an antiderivative of $\frac{1}{(3x-1)^2}$ with respect to x. Express your answer in the form $\frac{1}{a-bx}$ where a and b are positive real numbers. 2 marks

b. The function with rule $g(x)$ has derivative $g'(x) = \cos(\pi x)$.
Given that $g(1) = \pi$, find $g(x)$. 2 marks

Question 4 (3 marks)

Let X be the random variable that represents the number of houses Clare sells on any given week with probability distribution given by the table below.

x	0	1	2
$\Pr(X = x)$	0.5	0.3	0.2

a. Find the mean of X. 2 marks

b. What is the probability that Clare sells **at least one** house on each of three consecutive weeks? 1 mark

Question 5 (3 marks)

The graphs of $y = a\cos(x)$ and $y = \sin(x)$ where a is a real constant, have a point of intersection at $x = \frac{\pi}{6}$.

a. Find the value of a. 2 marks

b. If $x \in [0, 2\pi]$, find the x −coordinate of the other point of intersection of the two graphs. 1 mark

Question 6 (5 marks)

a. Solve the equation $\log_2(x) + 3\log_2(6) = 2\log_2 12$ for x. 2 marks

b. Solve $2e^t = e^{-t} - 1$ for t. 3 marks

Question 7 (4 marks)

The *It tastes so bad it must be good for you company* wants to estimate the proportion p of customers in a large city that intend to purchase its new Brussel-sprout - flavored yoghurt.

The company wants to be 95% confident that they have estimated this proportion to within .05. Preliminary market research indicated that 25 out every 49 people surveyed across the city intended to purchase the new yoghurt when it became available.

a. How many customers should they sample? 2 marks
Hint: $1.96 = \frac{49}{25}$.

Each of the twenty store managers across the city perform independent surveys to calculate an approximate 95% confidence interval for p using the sample size determined in part **a**. It is subsequently found that exactly one of the twenty confidence intervals calculated by the store managers does **not** contain the value of p. The CEO of the company randomly selects two of these stores without replacement.

b. Find the probability that exactly one of the two selected stores has a confidence interval that does **not** contain the value of p. 2 marks

Question 8 (4 marks)

A continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \frac{\pi}{2}\sin(\pi x) & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

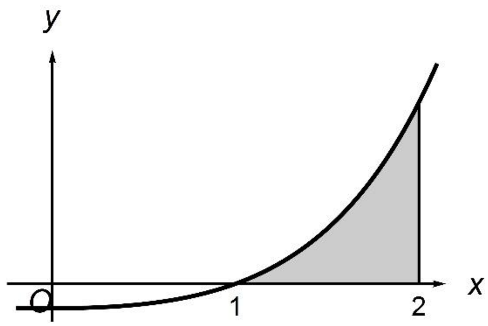
The median of X is m.

a. Determine the value of m. 2 marks

b. The value of m is a number greater than $\frac{1}{4}$.
Find Pr $\left(X < \frac{1}{4} | x < m\right)$ 2 marks

Question 9 (4 marks)

Part of the graph of $f: R \to R, f(x) = (x-1)e^x$ is shown below.



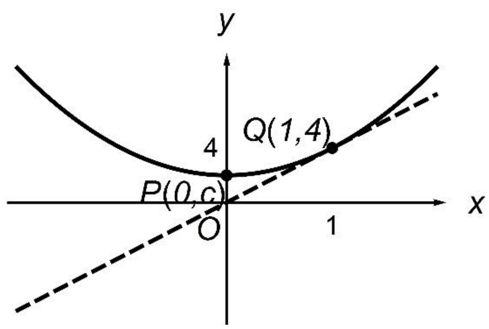
a. Find the derivative of $x\mathrm{e}^x$ 1 mark

b. Use your answer to part **a**, to the find the area of the shaded region in the form ae^a, where a is a nonzero real constant. 3 marks

Question 10 (7 marks)

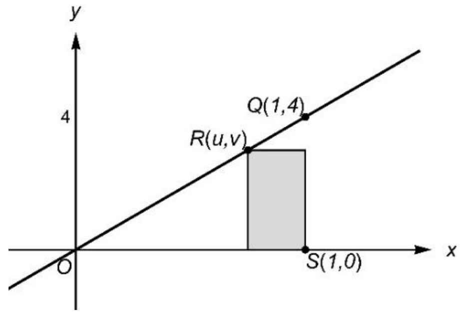
a. Define the equation for the discriminant of the quadratic equation $ax^2 - mx + c = 0$ in terms of the nonzero real constants a, m, c. 1 mark

A parabola $y = ax^2 + c$ has its minimum at point P with coordinates $(0, c)$ where c is a positive real constant. A line $y = mx$ containing the coordinate origin O and the point Q with coordinates $(1,4)$ is tangent to the graph of this parabola at point Q, as shown below.



b. Hence, show that the line has equation $y = 4x$ and the parabola has equation $y = 2x^2 + 2$ 3 marks

c. A rectangle has diagonally opposite vertices at point S with coordinates $(1,0)$ and point R with coordinates (u, v), as shown below.



i. Find an expression for v in terms of u. 1 mark

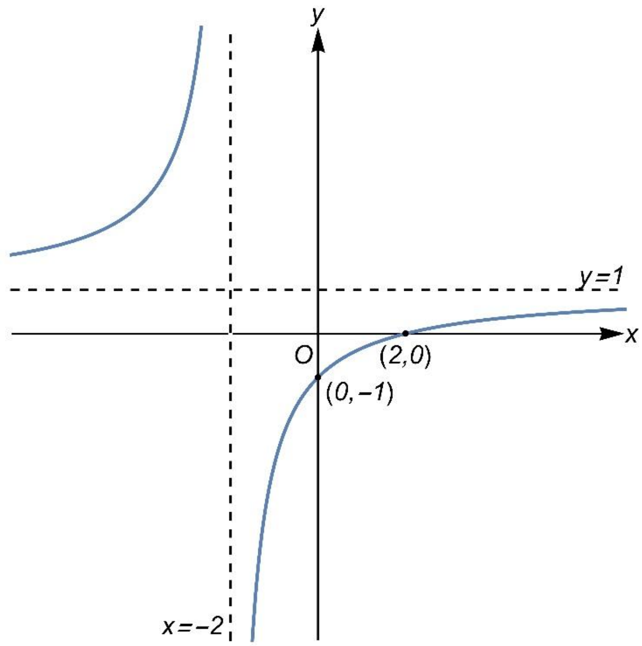
ii. Hence, find the **maximum** possible value for the shaded area in the figure above. 2 marks

END OF QUESTION AND ANSWER BOOK

Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

Mathematical Methods Exam 1: SOLUTIONS

1(a)	$-\dfrac{x}{\sqrt{1-x^2}}$	(A1)
1(b)	$f'(x) = \dfrac{\cos(x) + x\sin(x)}{\cos^2(x)}$ $f'(\pi) = -1$	(A1) (A1)
2		(A1) Correctly positioned and labelled axial intercepts. (A1) Correctly positioned and labelled asymptotes. (A1) Correct graph shape relative to the top left and bottom right of the asymptotes. Graph must not "hook away".
3(a)	$\int (3x-1)^{-2}dx = \dfrac{(3x-1)^{-1}}{-3}$ $= \dfrac{1}{3-9x}$	(M1) r.a. at anti-differentiation, $+c$ not required, but dx required (A1).
3(b)	$g(x) = \dfrac{1}{\pi}\sin(\pi x) + c$ $g(1) = \pi \Rightarrow c = \pi$	(A1) $+c$ required. (C1) solved correctly for **their** constant.
4(a)	$E(X) = 0 \times 0.5 + 1 \times 0.3 + 2 \times 0.2$ $= 0.7$	(M1) (A1)
(b)	$\left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$	(A1) accept 0.125.
5(a)	$a\cos\left(\dfrac{\pi}{6}\right) = \sin\left(\dfrac{\pi}{6}\right) \Rightarrow \tan\left(\dfrac{\pi}{6}\right) = a$ $a = \dfrac{1}{\sqrt{3}}$	(M1) (A1) or $\frac{\sqrt{3}}{3}$

5(b)	$\tan\left(\pi+\frac{\pi}{6}\right)=\tan\left(\frac{\pi}{6}\right)\Rightarrow x=\frac{7\pi}{6}$	(A1)
6(a)	$\log_2(6^3x)=\log_2(12^2)$ $x=\frac{12^2}{6^3}=\frac{2}{3}$	(M1) or equivalent. (A1) must cancel down.
6(b)	$u=e^t\Rightarrow 2u^2+u-1=0$ $(u+1)(2u-1)=0\Rightarrow u=\frac{1}{2}\ only$ $\therefore t=\log_e\left(\frac{1}{2}\right)$ or $t=-\log_e(2)$	(M1) r.a. to write as a quadratic equation. (A1) must explicitly exclude $u=-1$. (A1)
7(a)	margin of error $=0.05$ in 95% C.I. for population proportion $\Rightarrow \frac{1}{20}$ $=\frac{49}{25}\sqrt{\frac{\left(\frac{25}{49}\right)\left(\frac{24}{49}\right)}{n}}$ $n=20^2\times\frac{49^2}{25^2}\times\left(\frac{25}{49}\right)\left(\frac{24}{49}\right)\Rightarrow n=384$	(M1) define equation for sample size n (may employ decimal equivalents). (A1) must cancel down.
7(b)	Let $A=$ "p ouitside C.I." Require $\Pr\left(A'A\cup AA'\right)\equiv 1-\Pr\left(A'A'\right)$ $\Pr\left(A'A\cup AA'\right)=1-\frac{19}{20}\times\frac{18}{19}=\frac{1}{10}$	(M1) suitable statement in terms of an explicitly defined event. (A1)
8(a)	$\frac{1}{2}=\int_0^m\frac{\pi}{2}\sin(\pi x)\,dx$ $\cos(\pi m)=0\Rightarrow m=\frac{1}{2}$	(M1) correctly defined statement for m. (A1)
8(b)	$\Pr\left(X<\frac{1}{4}\middle\vert x<\frac{1}{2}\right)=\frac{\Pr\left(X<\frac{1}{4}\right)}{\Pr\left(X<\frac{1}{2}\right)}$ $=1-\frac{1}{\sqrt{2}}$	(C1) using their answer from part a. (A1) or a numerically equivalent representation. No need to rationalize.
9(a)	$\frac{d}{dx}(xe^x)=e^x+xe^x$	(A1)
9(b)	Shaded area $=\int_1^2(x-1)e^x dx$ $=\int_1^2\left(\frac{d}{dx}(xe^x)-2e^x\right)dx$ $=e$	(M1) correctly expressed area (with dx). (M1) r.a. to express integrand in terms of $\frac{d}{dx}(xe^x)$. (A1)
10(a)	$\Delta=m^2-4ac$	(A1)

10(b)	Line $y = mx$ passes through $Q(1,4) \Rightarrow m = 4$. Minimum of parabola at $(0, c) \Rightarrow y = ax^2 + c$ parabola through $(1,4) \Rightarrow a + c = 4$ At Q the line $y = mx$ is tangent to $y = ax^2 + c$ So $ax^2 - mx + c = 0$ has $\Delta = m^2 - 4ac = 0 \Rightarrow ac = 4$ But $a = 2$ and $c = 2$ satisfy this pair of equations in a and c.	(A1) (M1) (M1)
10(c)i	Either $R(u, v)$ is on the line $y = 4x \Rightarrow v = 4u$. Or The slope of the line between R and Q is 4. $\frac{4 - v}{1 - u} = 4 \Rightarrow v = 4u$	 (M1)
10(c)ii	$A = v \times (1 - u) \Rightarrow A = 4u(1 - u)$ Max occurs at $u = \frac{1}{2}$ so maximum area is $A = 1$	(M1) find an equation for shaded area in terms of a single variable. (A1)

Mathematical Methods formulas

Mensu+ration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$	
$\frac{d}{dx}((ax+b)^n) = an(ax+b)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}(e^{ax}) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c, x > 0$	
$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$		$\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$		$\int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A\|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = E(x)$	variance	$\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

Probability distribution		**Mean**	**Variance**
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma x p(x)$	$\sigma^2 = \Sigma(x-\mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\ dx$	$\mu = \int_{-\infty}^{\infty} x f(x)\ dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$E(\hat{P}) = p$
standard deviation	$\text{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{P} - z\sqrt{\dfrac{\hat{P}(1-\hat{P})}{n}}, \hat{P} + z\sqrt{\dfrac{\hat{P}(1-\hat{P})}{n}}\right)$

