Neap:

Trial Examination 2016

VCE Mathematical Methods Units 3&4

Written Examination 2

Suggested Solutions

SECTION A

1	E
2	C
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6	E
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13	A
14	A
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16	E
17	D
18	B
19	E
20	B

Question 1 **E**

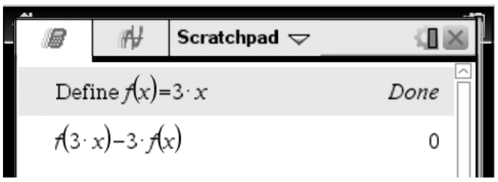
$$g(x) = 5\cos\left(\frac{1}{3}x - \pi\right) - 3$$

$$= 5\cos\frac{1}{3}(x - 3\pi) - 3$$

$$= f\left(\frac{1}{3}(x - 3\pi)\right) - 3$$

This has a horizontal translation of 3π units in the positive direction of the x-axis.

Question 2 **C**



Question 3 **E**

The population proportion, p, is $\dfrac{\text{number of population with attribute}}{\text{population size}}$.

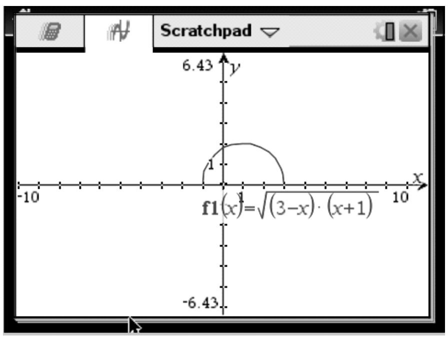
The population proportion is a population parameter; its value is constant.

Since 73% of the entire population have the required attribute (at least two children), then 73% is a population parameter.

Question 4 **B**

$h(x) = f(x)g(x)$

$\quad = \sqrt{(3 - x)(x + 1)}$ with a maximal domain of $[-1, 3]$



The function is not differentiable at the endpoints, therefore not at $x = -1$ or $x = 3$.

Question 5 **D**

$$\begin{aligned}E(X) &= \Sigma x p(x)\\ &= -10(2p-p^2)+0(p-1)(3p-1)+10(2p-2p^2)\\ &= -10p^2\end{aligned}$$

Question 6 **E**

The 95% approximate margin of error is calculated using the formula $1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

By increasing the sample size by a factor of 4, the new margin of error is calculated to be:

$$\begin{aligned}1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{4n}} &= \frac{1.96}{2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\\ &= \frac{1}{2}\times 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{4n}} \text{ as required.}\end{aligned}$$

(Halving the confidence level of 95% does not halve the related z-scores.)

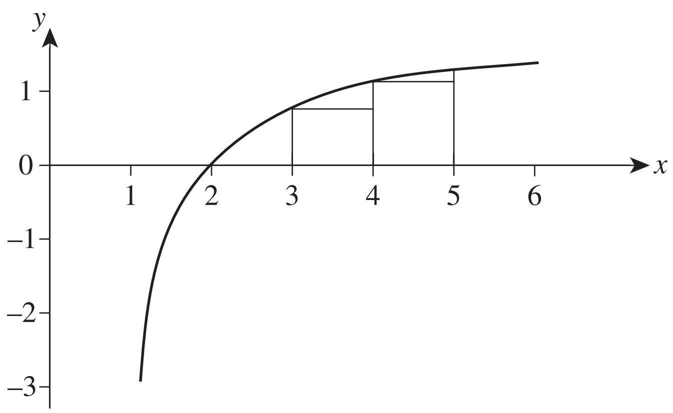
Question 7 **E**

There are 3 rectangles for the required area.

Using left rectangles, we have:

$$\begin{aligned}\text{area} &= \log_e(2-1)\times 1+\log_e(3-1)\times 1+\log_e(4-1)\times 1\\ &= \log_e(1)+\log_e(2)+\log_e(3)\end{aligned}$$

Since $\log_e(1)=0$ the answer comes down to be $\log_e(2)+\log_e(3)$.



Question 8 C

$V = \frac{4}{3}\pi r^3$

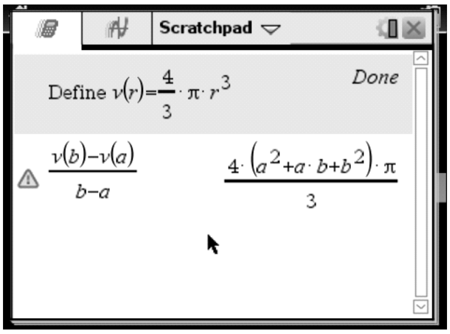
$$\text{average rate of change of volume} = \frac{V(b) - V(a)}{b - a}$$

$$= \frac{\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3}{b - a}$$

$$= \frac{\frac{4}{3}\pi(b^3 - a^3)}{b - a}$$

$$= \frac{\frac{4}{3}\pi(b - a)(b^2 + ba + a^2)}{b - a}$$

$$= \frac{4}{3}\pi(a^2 + ab + b^2)$$



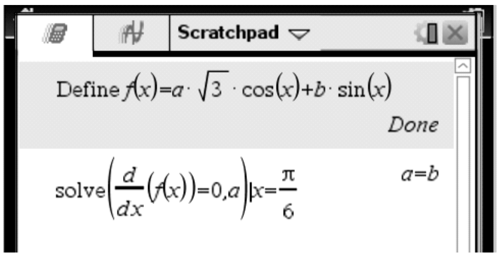
Question 9 D

Doubling the (positive) mean would result in a translation of the normal graph in the positive direction of the x-axis by the value of the mean itself.

Halving the standard deviation would result in a dilation of the normal graph from the mean by factor of $\frac{1}{2}$. This would make the normal graph appear narrower.

Question 10 D

For turning point, derivative = 0, so $f'(x) = 0$ at $x = \frac{\pi}{6}$.



Question 11 C

The probability density function for X is

x	0	1	2	3	4
$\Pr(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

50% of the data lie below the median.

As $\Pr(X \le 1) = 0.4$ the median is greater than 1.

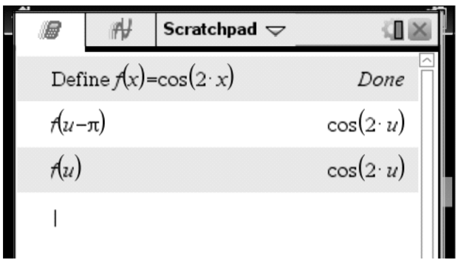
As $\Pr(X \le 2) = 0.6$, the median is less than or equal to 2.

Hence the median is 2.

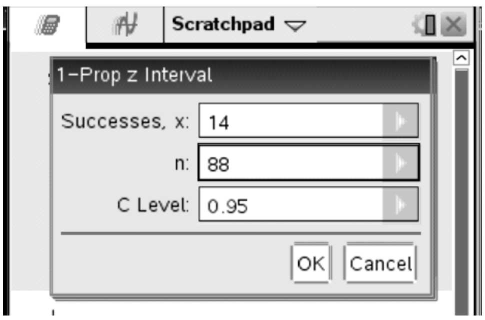
Question 12 E

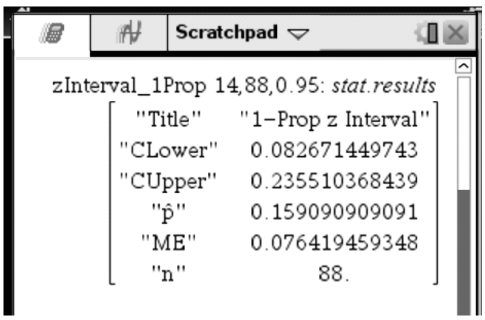
To satisfy $f(u - \pi) = f(u)$, f must have period of π. $\cos(2x)$ has a period of $\frac{2\pi}{2} = \pi$ and is the only one of the given functions that does, hence it is the only one that satisfies the functional equation.

$$\begin{aligned} f(u) &= \cos(2u) \\ f(u - \pi) &= \cos(2(u - \pi)) \\ &= \cos(2u - 2\pi) \\ &= \cos(2u) \\ &= f(u) \end{aligned}$$



Question 13 A





CLower = 0.083 and CUpper = 0.236, correct to 3 decimal places.

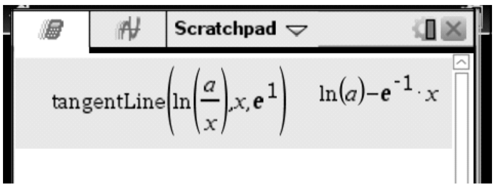
Question 14 A

The equation of the tangent can be found using $y = f'(e)(x - e) + f(e)$, where $f(x) = \log_e\left(\frac{a}{x}\right)$.

$\therefore f(e) = \log_e\left(\frac{a}{e}\right)$

$f'(x) = -\frac{1}{x}, f'(e) = -\frac{1}{e}$

$$\begin{aligned}\therefore y &= -\frac{1}{e}(x - e) + \log_e\left(\frac{a}{e}\right)\\ &= -\frac{x}{e} + 1 + \log_e(a) - \log_e(e)\\ &= -\frac{x}{e} + 1 + \log_e(a) - 1\\ &= -\frac{x}{e} + \log_e(a)\end{aligned}$$



Question 15 E

The two equations are parallel when their gradients are the same,

or when $\text{Det}\begin{bmatrix} m-1 & 7 \\ 6 & 3m+2 \end{bmatrix} = 0$

$$\therefore (m-1)(3m+2) - 42 = 0$$

$$m = -\frac{11}{3}, 4$$

When $m = 4$, the equations become $3x + 7y = 12$ and $6x + 14y = -24n$, which are the same equation and have infinite solutions when $n = -1$.

Question 16 E

The population size cannot be determined by taking a single sample of 100 bats without knowing what proportion the sample is of the whole population.

A table of exact proportions is best suited to small samples ($n < 10$), so not appropriate.

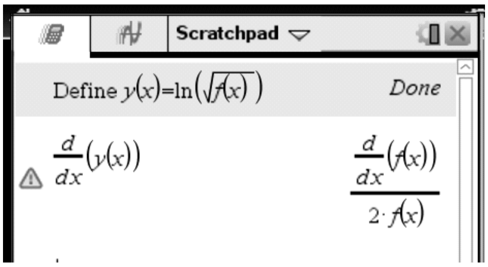
A point estimate (such as the sample proportion) is a statistic calculated on one sample and would be appropriate.

A confidence interval can also be calculated on one sample, giving us an interval where the population proportion is likely to lie, given the point estimate sample proportion.

Question 17 D

Using chain rule: $y = \log_e(u)$, where $u = \sqrt{f(x)}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{u} \times \left(\frac{1}{2\sqrt{f(x)}} \times f'(x)\right)\\ &= \frac{1}{\sqrt{f(x)}} \times \frac{f'(x)}{2\sqrt{f(x)}}\\ &= \frac{f'(x)}{2f(x)}\end{aligned}$$



Question 18 B

$ax^4 + 4x^3 + 2x^2 = 0$

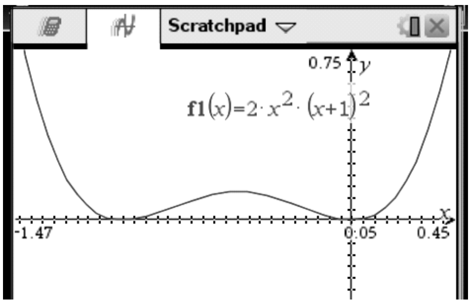
$x^2(ax^2 + 4x + 2) = 0$

$x = 0$ is one solution, therefore $ax^2 + 4x + 2$ must be a perfect square (or could use discriminant equals zero) for there to be the required two solutions only.

$2\left(\frac{a}{2}x^2 + 2x + 1\right)$ is a perfect square if $a = 2$.

$2x^2(x + 1)^2 = 0$

Since a is positive, this is a positive quartic function and the graph of this function has x-intercepts at $x = -1$ and $x = 0$, which are both turning points and therefore local minimums.



Question 19 E

$$\int_0^8 f\left(\frac{1}{4}x\right) + 2dx = \int_0^8 f\left(\frac{1}{4}x\right)dx + \int_0^8 (2)dx$$

$$= 4 \times \int_0^2 f(x)dx + [2x]_0^8$$

$$\left(\text{as} \int_0^8 f\left(\frac{1}{4}x\right)dx \text{ is a dilation by a factor 4 of } \int_0^2 f(x)dx \text{ from the } y\text{-axis}\right)$$

$$= (4 \times 3) + (2 \times 8 - 2 \times 0)$$

$$= 28$$

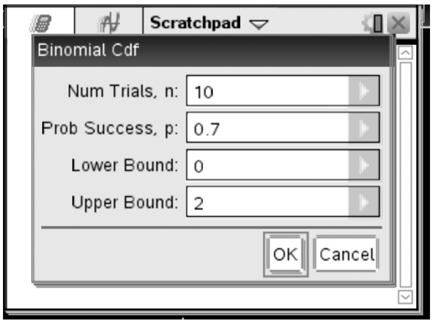
Question 20 B

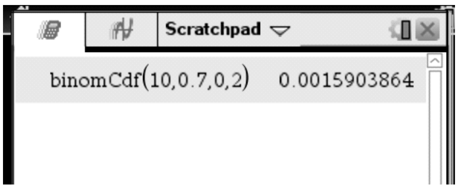
This is a binomial distribution with $n = 10$. If we consider **not** becoming ill (as the question requires) as a success, then $p = 0.7$.

$X \sim \text{Bi}(10, 0.7)$

$\Pr(\text{at most 2 successes}) = \Pr(X \le 2)$

$= 0.0016$ (correct to four decimal places)





SECTION B

Question 1 (10 marks)

a. $h = A\sin(nx) + B$, where A, n and B are real constants.

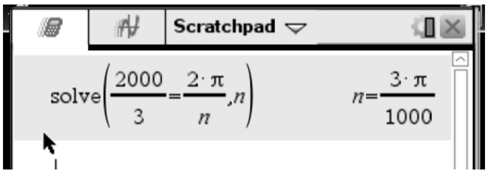
The amplitude of the graph is 125, therefore $A = 125$.

The graph has been translated 125 units up, therefore $B = 125$. *both correct* A1

From the graph, $\frac{3}{4}$ of a period = 500 so $\text{period} = 500 \times \frac{4}{3}$

$$= \frac{2000}{3}$$

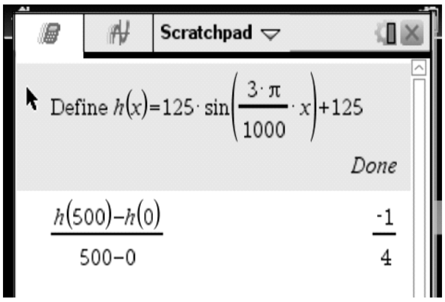
Since $\text{period} = \frac{2\pi}{n}$, $n = \frac{3\pi}{1000}$. A1



b. First touches the ground at $x = 500$.

$$\text{average rate of change} = \frac{h(500) - h(0)}{500 - 0}$$ M1

$$= -\frac{1}{4}$$ A1



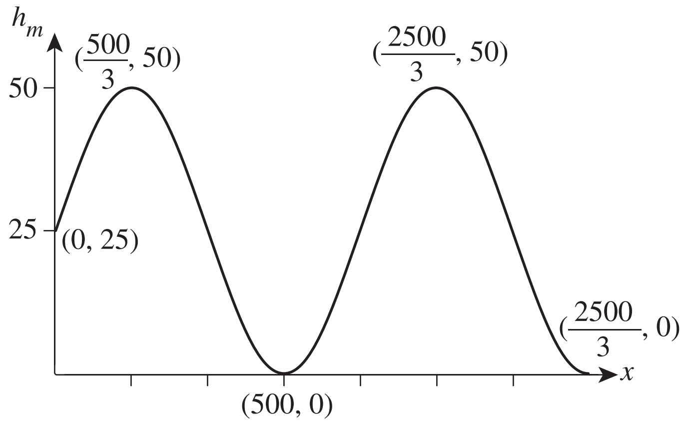
c. $\frac{50}{2} = 25$ mm

$\frac{500}{3} \pm 25 = \frac{575}{3}$ and $\frac{425}{3}$ M1

So cuboid structure touches curve at $x = \frac{575}{3}$ and $x = \frac{425}{3}$. M1

So maximum height of cuboid is 247 mm. A1

d.



correct shape A1
all coordinates labelled A1

e. $h_m = 25\sin\left(\frac{3\pi}{1000}x\right) + 25$ A1

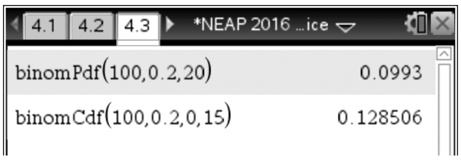
Question 2 (16 marks)

a. **i.** Let D be the number of children found to have tooth decay.

$E(D) = np$, $n = 100$, $p = 0.2$. Therefore, he would expect 20 children with tooth decay. A1

ii. $D \sim \text{Bi}(100, 0.2)$ M1

$\Pr(D = 20) = 0.099$ A1



b. $\Pr(D \le 15) = 0.129$ (as above) A1

c. **i.** standard deviation $= \sqrt{np(1-p)}$ $n = 100$, $p = 0.2$

$= 4$

mean $= np = 20$

$20 - 2 \times 4 = 12$

$20 + 2 \times 4 = 28$

2 standard deviation limit = (12, 28) A1

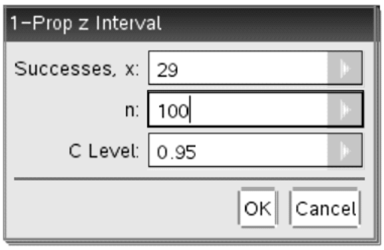
29 children having tooth decay is more than 2 standard deviations away from the expected number of children with tooth decay. The chance of more than 28 children having tooth decay is less than approximately 2.5%. Xavier could have cause to doubt the accuracy of the information but should realise that one sample is not enough to be a reliable estimate.

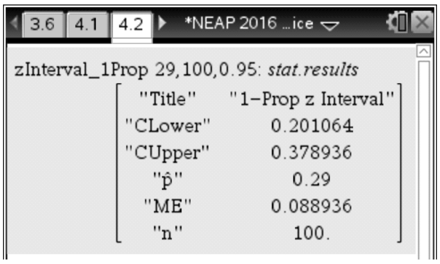
feasible answer with justification A1

ii. 0.29 A1

iii. $0.29 \pm 1.96\sqrt{\dfrac{0.29 \times (1 - 0.29)}{100}} = 0.29 \pm 0.0889\ldots$

$= (0.201, 0.379)$ A1





d. i. $0.95 \times 200 = 190$ A1

ii. a normal distribution A1

iii. mean $\mu = 0.2$, standard deviation $\sigma = \sqrt{\dfrac{0.2(1 - 0.2)}{200}} = 0.03$ *both correct* A1

iv. $\hat{p} \sim N(0.2, 0.03^2)$

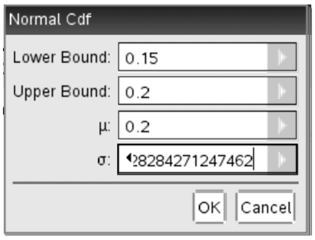
$$\Pr(0.15 < \hat{p} < 0.2 | \hat{p} < 0.25) = \frac{\Pr(0.15 < (\hat{p} < 0.2 \cap \hat{p} < 0.25)}{\Pr(\hat{p} < 0.25)}$$

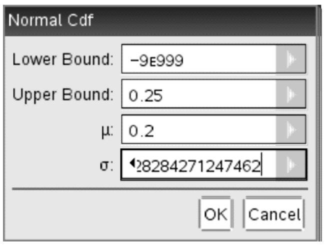
recognising conditional probability M1

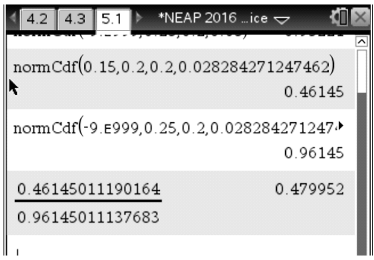
$$= \frac{\Pr(0.15 < \hat{p} < 0.2)}{\Pr(\hat{p} < 0.25)}$$ M1

$$= \frac{0.4615}{0.9615}$$

$= 0.480$ correct to three decimal places A1







e. $n = \left(\dfrac{1.96}{0.03}\right)^2 \times 0.2 \times 0.8$ M1

$= 682.95$

$= 683$ A1

He will need to sample 683 children.

Question 3 (8 marks)

a. domain: $f:[1000, 1200] \to R$ A1

rule: $f(x) = 400 - \frac{1}{100}(x - 1200)^2$ A1

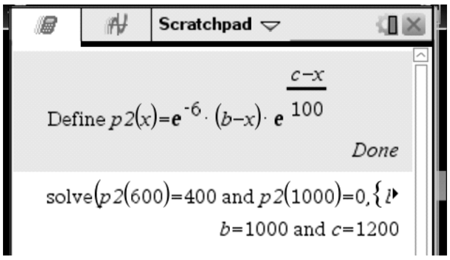
b. A reflection in the y-axis maps $(x, y) \to (-x, y)$.

Hence $y = e^{-6}(x - 200)e^{\frac{x}{100}} \to y = e^{-6}(-x - 200)e^{\frac{-x}{100}}$. A1

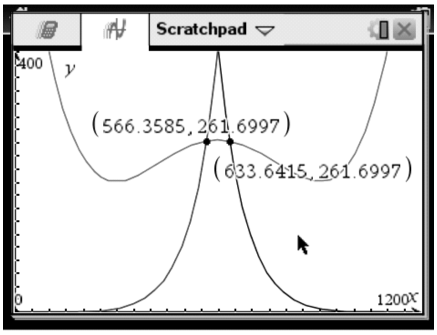
Since the reflection is in the line $x = 600$, there is also a horizontal translation.

Using the points (600, 400) and (1000, 0) we can solve for b and c.

$b = 1000$ and $c = 1200$ A1

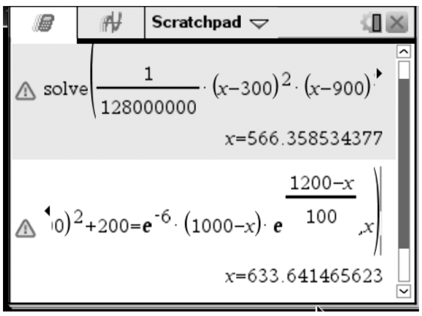


c. Method 1: graphically



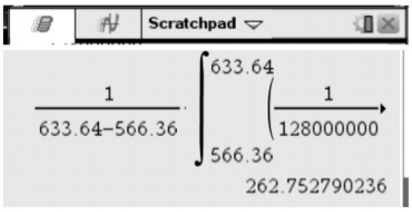
Coordinates are (566.4, 261.7) and (633.6, 261.7). A1 A1

Method 2: algebraically



Coordinates are (566.4, 261.7) and (633.6, 261.7). A1 A1

d. average value of function over interval [566.36, 633.64]



correct integral M1

= 262.8 A1

Question 4 (18 marks)

a. **i.** driver A: 100 – 30 = 70 km; driver B: 100 – 40 = 60 km *both correct* A1

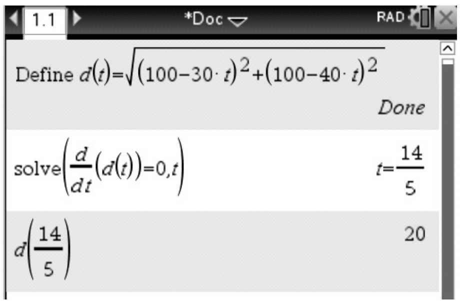
ii. distance = $\sqrt{70^2 + 60^2}$

= 92.2 km A1

b. **i.** $d_A = 100 - 30t$, $d_B = 100 - 40t$ A1

ii. $d(t) = \sqrt{(100 - 30t)^2 + (100 - 40t)^2}$ *using Pythagoras* M1 A1

iii. $d'(t) = 0$ for the minimum distance M1



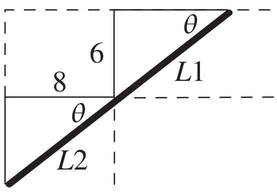
$t = 2.8$ hours A1

iv. $d(2.8) = 20$ km A1

c. **i.** Splitting the diagonal:

$L = L1 + L2$, where = $L1$ and $L2$ are the diagonals which cross the respective roads.

splitting the diagonal A1



M1

From the diagram we see $\sin(\theta) = \frac{6}{L1}$ and $\cos(\theta) = \frac{8}{L2}$.

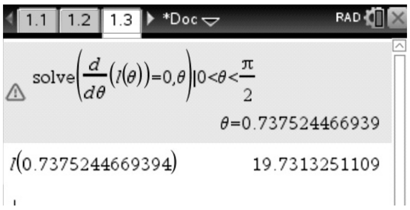
So $L1 = \frac{6}{\sin(\theta)}$ and $L2 = \frac{8}{\cos(\theta)}$.

Therefore $L(\theta) = \frac{6}{\sin(\theta)} + \frac{8}{\cos(\theta)}$ as required.

ii. Maximum length occurs when $L'(\theta) = 0$. M1

$L'(\theta) = 0$ gives $\theta = 0.7375$. A1

$L(0.7375) = 19.73$ m A1



d. **i.** Following part **c. i.**:

$$L = \frac{b}{\sin(\theta)} + \frac{a}{\cos(\theta)}$$

$$L'(\theta) = \frac{a\sin(\theta)}{\cos^2(\theta)} - \frac{b\cos(\theta)}{\sin^2(\theta)}$$

$$= \frac{a\sin^3(\theta) - b\cos^3(\theta)}{\cos^2(\theta)\sin^2(\theta)}$$ M1

$$L'(\theta) = 0$$

$$\Rightarrow \frac{a\sin^3(\theta) - b\cos^3(\theta)}{\cos^2(\theta)\sin^2(\theta)} = 0$$

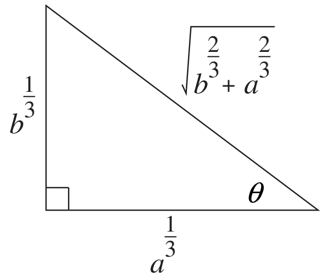
$$\Rightarrow a\sin^3(\theta) - b\cos^3(\theta) = 0$$ A1

Note: Must have correct answer for both marks.

ii. Maximum occurs when $a\sin^3(\theta) - b\cos^3(\theta) = 0$.

$$\Rightarrow \frac{\sin^3(\theta)}{\cos^3(\theta)} = \frac{b}{a}$$

$$\Rightarrow \tan(\theta) = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$$ A1



By considering the right-angled triangle where $\tan(\theta) = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$, we see $\sin(\theta) = \frac{b^{\frac{1}{3}}}{\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}$ and $\cos(\theta) = \frac{a^{\frac{1}{3}}}{\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}$.

Substituting into $L = \frac{b}{\sin(\theta)} + \frac{a}{\cos(\theta)}$ gives: *substitution* M1

$$L = \frac{b}{\frac{b^{\frac{1}{3}}}{\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}} + \frac{a}{\frac{a^{\frac{1}{3}}}{\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}}$$

$$= \frac{b\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}}{a^{\frac{1}{3}}}$$

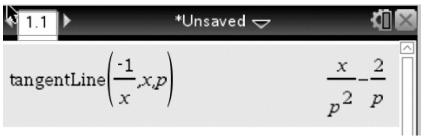
$$= b^{\frac{2}{3}}\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}} + a^{\frac{2}{3}}\sqrt{b^{\frac{2}{3}} + a^{\frac{2}{3}}}$$ A1

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Question 5 (8 marks)

a. $f'(x) = \frac{1}{x^2}$



At the point $P\left(p, -\frac{1}{p}\right)$ the gradient of the tangent is $\frac{1}{p^2}$.

Substitution into $y - y_1 = m(x - x_1)$ gives:

$$y + \frac{1}{p} = \frac{1}{p^2}(x - p)$$

$$y = \frac{1}{p^2}(x - p) - \frac{1}{p}$$

$$y = \frac{1}{p^2}x - \frac{2}{p}$$ A1

b. **i.** Coordinates of midpoint of $T\left(t, -\frac{1}{t}\right)$ and $S\left(s, -\frac{1}{s}\right) = \left(\frac{t+s}{2}, \frac{\left(-\frac{1}{t} - \frac{1}{s}\right)}{2}\right)$

$$= \left(\frac{t+s}{2}, \frac{\left(\frac{-s-t}{st}\right)}{2}\right)$$

$$= \left(\frac{t+s}{2}, \frac{-(s+t)}{2st}\right)$$ A1

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ii. Gradient of segment *OM*:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\left(\frac{-s-t}{2st}\right) - 0}{\left(\frac{t+s}{2}\right) - 0}$$

$$= \frac{-s-t}{2st} \times \frac{2}{t+s}$$

$$= -\frac{1}{st}$$ A1

Gradient of segment *OP*:

$$m = \frac{-\frac{1}{p} - 0}{p - 0}$$

$$= -\frac{1}{p^2}$$

Equating gradient gives $\left(-\frac{1}{p^2}\right) = -\frac{1}{st}$

$\Rightarrow p^2 = st$ *final solution with reasoning* A1

c. $A = \int_t^s -\frac{1}{x}dx$

$= \log_e\left(\frac{t}{s}\right)$ A1

d. $B = \int_s^t f(x)dx + \Delta OSS' - \Delta OTT'$ M1

$= -\int_s^t \frac{1}{x}dx + \frac{1}{2} \times s \times -\frac{1}{s} - \frac{1}{2} \times t \times -\frac{1}{t}$

$= -\int_s^t \frac{1}{x}dx - \frac{1}{2} + \frac{1}{2}$ A1

$= -\int_s^t \frac{1}{x}dx$

$= f(x)$

Therefore $g(x) = f(x)$. A1