Year 2016
VCE
Mathematical Methods
Trial Examination 2
Solutions

KILBAHA MULTIMEDIA PUBLISHING	**TEL: (03) 9018 5376**
PO BOX 2227	**FAX: (03) 9817 4334**
KEW VIC 3101	**kilbaha@gmail.com**
AUSTRALIA	**http://kilbaha.com.au**

SECTION A

ANSWERS

1	A	B	C	**D**	E
2	A	**B**	C	D	E
3	A	B	C	D	**E**
4	A	B	**C**	D	E
5	**A**	B	C	D	E
6	A	**B**	C	D	E
7	A	**B**	C	D	E
8	A	B	C	D	**E**
9	A	B	**C**	D	E
10	A	B	**C**	D	E
11	A	B	C	**D**	E
12	A	B	C	**D**	E
13	**A**	B	C	D	E
14	A	B	C	D	**E**
15	A	B	C	**D**	E
16	**A**	B	C	D	E
17	A	B	C	D	**E**
18	**A**	B	C	D	E
19	A	**B**	C	D	E
20	A	B	**C**	D	E

SECTION A

Question 1 **Answer D**

$f:[2a,\infty)\to R$, $f(x)=x(a-x)$, the maximum value occurs at the end-point

$f(2a)=2a(a-2a)=-2a^2$

Question 2 **Answer B**

$P(3,-2)$ lies on the graph of the function *f*. The graph of *f* is translated one unit away from the *y*-axis, the point becomes $(4,-2)$, then reflected in the *x*-axis it becomes $(4,2)$ then reflected in the line $y=x$, it becomes $(2,4)$.

Question 3 **Answer E**

$f(x)=a\sin\left(\frac{\pi x}{a}\right)$, $g(x)=a\tan\left(\frac{\pi x}{a}\right)$ and $h(x)=a\cos\left(\frac{\pi}{a}\left(x-\frac{\pi}{2}\right)\right)\neq f(x)$

Both f and h have amplitudes of a and periods $\frac{2\pi}{\frac{\pi}{a}}=2a$

g does not have an amplitude and has period $\frac{\pi}{\frac{\pi}{a}}=a$

Question 4 **Answer C**

Let *f* be a function with domain *R*. The function has the following properties

$f'(x)<0$ for $x<a$ and $f'(x)<0$ for $x>a$ and $f'(a)=0$.

Then at $x=a$ the graph of f has a stationary point of inflection

Question 5 **Answer A**

$f(x)=u(x)\log_e(2x)$ using the product rule

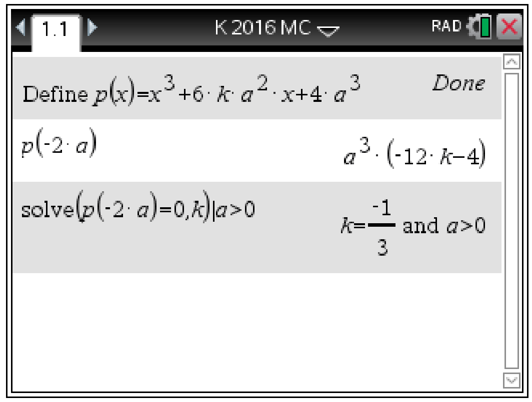
$$f'(x)=u(x)\frac{d}{dx}\left[\log_e(2x)\right]+u'(x)\log_e(2x)=\frac{u(x)}{x}+u'(x)\log_e(2x)$$

$$f'(2)=\frac{u(2)}{2}+u'(2)\log_e(4)=\frac{6}{2}+3\log_e(4)=3+3\log_e(4)$$

$$=3\left(1+\log_e(4)\right)=3\left(\log_e(e)+\log_e(4)\right)$$

$$=3\log_e(4e)$$

Question 6 **Answer B**



If $x+2a$ is a factor of

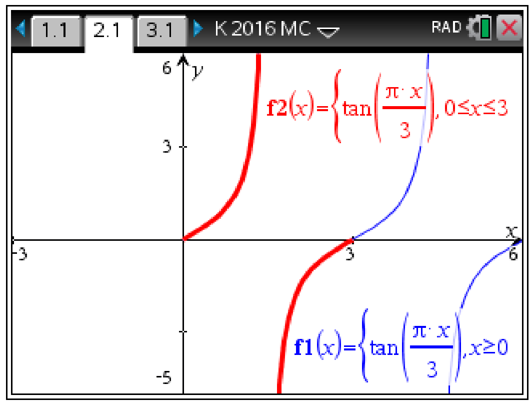
$$P(x)=x^3+6ka^2x+4a^3$$

by the factor theorem $P(-2a)=0$

$$P(-2a)=(-2a)^3+6ka^2(-2a)+4a^3$$
$$=-8a^3-12ka^3+4a^3$$
$$=-a^3(12k+4)=0$$

so that $k=-\frac{1}{3}$

Question 7 **Answer B**



$f(x)=\tan\left(\frac{\pi x}{3}\right)$ has period

$\frac{\pi}{\frac{\pi}{3}}=3$, since it has an inverse

function, it must be a one-one function.

The maximum possible value is $a=3$.

Question 8 **Answer E**

$f(x)=\frac{1}{x^2}$ and $g(x)=\frac{1}{\sqrt{x}}$ defined on their maximal domains.

domain $f=R\setminus\{0\}$ range $f=R\setminus\{0\}$

domain $g=(0,\infty)$ range $g=(0,\infty)$

$$f\ :\ y=\frac{1}{x^2}$$

$$f^{-1}:\ x=\frac{1}{y^2}\ \Rightarrow\ y^2=\frac{1}{x}\ \Rightarrow\ y=\pm\frac{1}{\sqrt{x}}$$

so $f^{-1}(x)=g(x)$ but only for $(0,\infty)$

Question 9 **Answer C**

by properties of the definite integral

$$\int_1^a \left(2f(x)-3g(x)+1\right)dx = 2\int_1^a f(x)\,dx - 3\int_1^a g(x)\,dx + [x]_1^a$$

$$= 2\int_1^a f(x)\,dx + 3\int_a^1 g(x)\,dx + [x]_1^a = 2\times 4 + 3\times {}^{-}3 + a - 1 = a-2$$

Question 10 **Answer C**

$y=e^x$ is transformed by a dilation from the *y*-axis by a scale factor of 2, the curve becomes $y=e^{0.5x}$, then a translation by one unit to the left in the *x*-direction the curve becomes $y=e^{0.5(x+1)}$, then a translation of two units downwards in the *y*-direction, the curve becomes $y=e^{0.5(x+1)}-2$.

Question 11 **Answer D**

$f: R \to R$, $f(x)=ax^3+\sqrt{b}\,x^2-x$ where $b>0$.

Since $f(x)=x\left(ax^2+\sqrt{b}\,x-1\right)$ crosses the *x*-axis once at the origin, and twice more if its discriminant $\Delta=\left(\sqrt{b}\right)^2+4a=b+4a>0$. **A.** and **C.** are incorrect

Now $f'(x)=3ax^2+2\sqrt{b}\,x-1$ and there are no turning points, if this discriminant

$\Delta=\left(2\sqrt{b}\right)^2+12a=4b+12a=4(b+3a)<0$ or $b+3a<0$ **D.** is correct

B. is incorrect $b+4a$ does not refer to turning points, and

E. is incorrect as $b+3a=0$ gives one turning point.

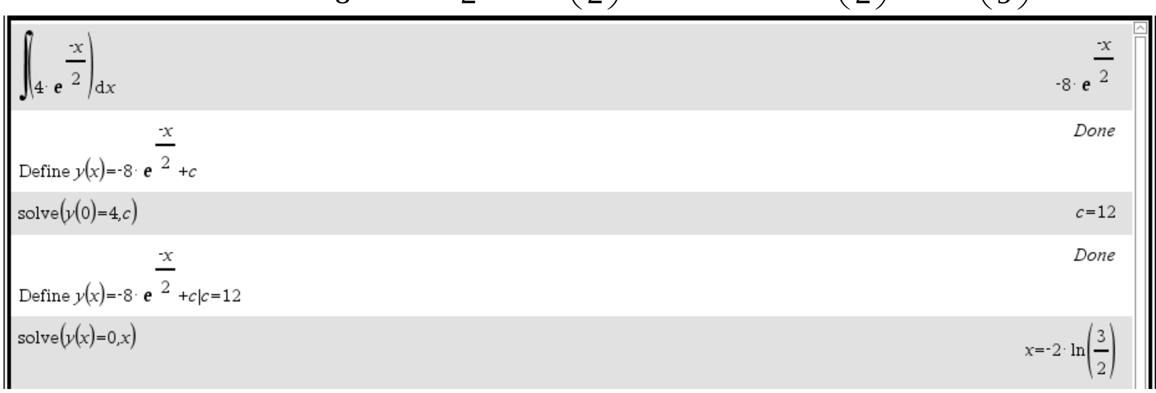
Question 12 **Answer D**

gradient $\dfrac{dy}{dx}=4e^{-\frac{x}{2}}$ integrating $y=\int 4e^{-\frac{x}{2}}\,dx=-8e^{-\frac{x}{2}}+c$.

To find *c*, use $x=0$ when $y=4$, so that $4=-8+c \Rightarrow c=12$.

The curve is $y=12-8e^{-\frac{x}{2}}$. This crosses the *x*-axis when $y=0$ so that

$$12-8e^{-\frac{x}{2}}=0 \Rightarrow e^{-\frac{x}{2}}=\frac{12}{8} \Rightarrow -\frac{x}{2}=\log_e\left(\frac{3}{2}\right) \Rightarrow x=-2\log_e\left(\frac{3}{2}\right)=\log_e\left(\frac{4}{9}\right).$$



Question 13 **Answer A**

Image curve $y' = 2 + \dfrac{8}{(x'+4)^2}$

$$\Rightarrow \ y' - 2 = \frac{8}{(x'+4)^2} \quad \Rightarrow \quad \frac{y'-2}{2} = \frac{4}{(x'+4)^2} = \frac{1}{\left(\frac{x'+4}{2}\right)^2}$$

The original curve is $y = \dfrac{1}{x^2}$ so that $y = \dfrac{y'-2}{2} \ \Rightarrow \ y' = 2y + 2$ and

$x = \dfrac{x'+4}{2} \ \Rightarrow \ x' = 2x - 4$, in matrix form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

Question 14 **Answer E**

$y = f(x) = \cos(x)$ using left-rectangles

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$f(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

$$L = \frac{\pi}{6}\left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{\pi}{12}\left(3 + \sqrt{3}\right) \approx 1.24$$

$$A = \int_0^{\frac{\pi}{2}} \cos(x)\,dx = \left[\sin(x)\right]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1$$

so that $L > A$ over-estimates, as is evident from the graph.

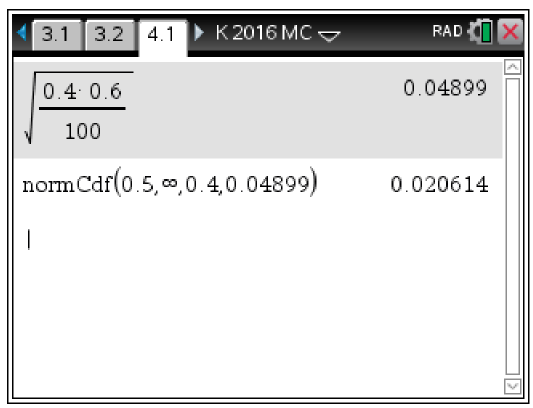
Question 15 **Answer D**

$p = 0.4$, $n = 100$

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.4 \times 0.6}{100}} = 0.04899$$

$$N\left(\mu = 0.4, \sigma^2 = 0.04899^2\right)$$

$$\Pr\left(\hat{p} > 0.5\right) = 0.0206$$



Question 16 **Answer A**

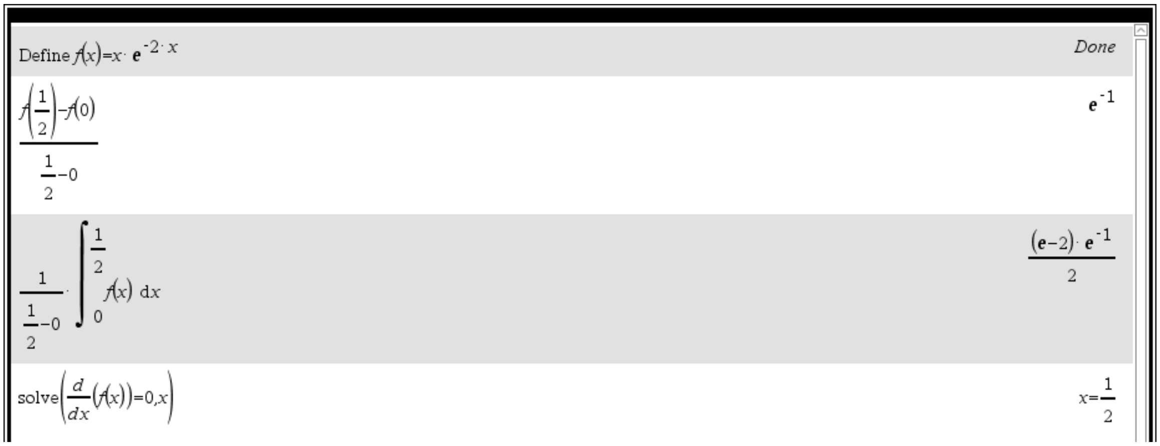
$Z \overset{d}{=} N\left(\mu = 0, \sigma^2 = 1\right)$, $\Pr(Z < a) = A$, $\Pr(Z > b) = B$, where $b > a$

$$\Pr(a < Z < b \mid Z < b) = \frac{\Pr(a < Z < b)}{\Pr(Z < b)} = \frac{\Pr(Z < b) - \Pr(Z < a)}{\Pr(Z < b)}$$

$$= \frac{1 - \Pr(Z > b) - \Pr(Z < a)}{1 - \Pr(Z > b)} = \frac{1 - B - A}{1 - B}$$

Question 17 **Answer E**

$f:[0,\infty)\to R,\ f(x)=xe^{-2x}$



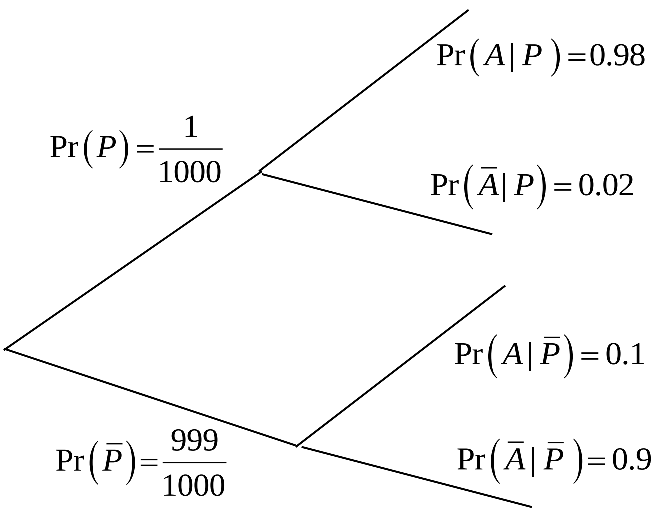
Albert is correct the average rate of change of the function over $0\le x\le\frac{1}{2}$ is $\frac{1}{e}=e^{-1}$.

Ben is correct the average value of the function over $0\le x\le\frac{1}{2}$ is $\frac{e-2}{2e}=\frac{(e-2)e^{-1}}{2}$.

Colin is correct the gradient of the function is zero when $x=\frac{1}{2}$.

Question 18 **Answer A**

P the suitcase contains prohibited substances, A activates alarm, using a tree diagram



$$\Pr(P\mid A)=\frac{\Pr(P\cap A)}{\Pr(A)}$$

$$=\frac{\Pr(A\mid P)\Pr(P)}{\Pr(A\mid P)\Pr(P)+\Pr(A\mid\bar{P})\Pr(\bar{P})}$$

$$=\frac{0.98\times\frac{1}{1000}}{0.98\times\frac{1}{1000}+0.1\times\frac{999}{1000}}$$

$$=0.0097$$

Question 19 **Answer B**

$\Pr(A) = p$ and $\Pr(B) = 2p$,

consider when A and B are mutually exclusive events $\Pr(A \cap B) = 0$

	A	A'	
B	0	$2p$	$2p$
B'	p	$1-3p$	$1-2p$
	p	$1-p$	

$\Pr(A' \cup B') = \Pr(A') + \Pr(B') - \Pr(A' \cap B') = 1 - p + 1 - 2p - (1-3p) = 1$

C. D. and **E.** are all correct

consider when A and B are independent events $\Pr(A \cap B) = \Pr(A)\Pr(B) = 2p^2$

	A	A'	
B	$2p^2$	$2p-2p^2$	$2p$
B'	$p-2p^2$	$1-3p+2p^2$	$1-2p$
	p	$1-p$	

A. is true, **B.** is false $\Pr(A' \cap B') = 2p^2 - 3p + 1$ not $2p^2 - 2p + 1$

Question 20 **Answer C**

The confidence interval is

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \,,\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

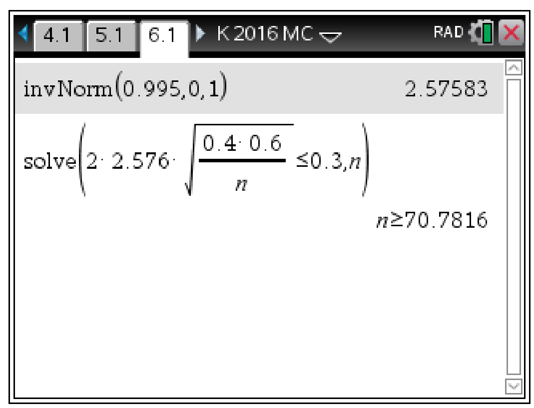
and has a width of $2z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Now $\hat{p} = 0.4$, $z = 2.576$,

solving $2 \times 2.576\sqrt{\frac{0.4 \times 0.6}{n}} \le 0.3$

$n \ge \left(\frac{2 \times 2.576}{0.3}\right)^2 \times 0.4 \times 0.6 = 70.78$

so $n = 71$



END OF SECTION A SUGGESTED ANSWERS

SECTION B

Question 1

a. $f(x)=4+\frac{x}{5}+\sin\left(\frac{\pi x}{5}\right)$

$f(5)=4+\frac{5}{5}+\sin(\pi)=5$, $f(-5)=4-\frac{5}{5}+\sin(-\pi)=3$

$A(-5,3)$, $B(5,5)$ A1

b. solving $f'(x)=\frac{1}{5}+\frac{\pi}{5}\cos\left(\frac{\pi x}{5}\right)=0$ with $0<x<5 \Rightarrow x=3.01558$,

$f(3.01558)=5.551$

$(3.016, 5.551)$ A1

c. $C(5,-2)$, $D(-5,-4)$ $m(CD)=\frac{-4+2}{-5-5}=\frac{1}{5}$ A1

line segment CD $y+2=\frac{1}{5}(x-5) \Rightarrow y=\frac{x}{5}-3$

$g:[-5,5]\to R,\ g(x)=\frac{x}{5}-3$ A1

d. $w(x)=f(x)-g(x)=\sin\left(\frac{\pi x}{5}\right)+7$ A1

when $\sin\left(\frac{\pi x}{5}\right)=1$, $x=\frac{5}{2}$ $w\left(\frac{5}{2}\right)=8$

when $\sin\left(\frac{\pi x}{5}\right)=-1$, $x=-\frac{5}{2}$ $w\left(-\frac{5}{2}\right)=6$ M1

the maximum width is 8 metres and the minimum width is 6 metres. A1

e. $A=\int_{-5}^{5}\left(f(x)-g(x)\right)dx=\int_{-5}^{5}w(x)\,dx=\int_{-5}^{5}\left(\sin\left(\frac{\pi x}{5}\right)+7\right)dx$ A1

f. $V=1.5\int_{-5}^{5}\left(\sin\left(\frac{\pi x}{5}\right)+7\right)dx$

$V=1.5\left[-\frac{5}{\pi}\cos\left(\frac{\pi x}{5}\right)+7x\right]_{-5}^{5}$ M1

$=1.5\left[\left(-\frac{5}{\pi}\cos(\pi)+7\times 5\right)-\left(-\frac{5}{\pi}\cos(-\pi)+7\times{}^{-}5\right)\right]=1.5\times 70$

$V=105$ metres^3 A1

Define $f(x)=4+\frac{x}{5}+\sin\left(\frac{\pi\cdot x}{5}\right)$	*Done*
$f(-5)$	3
$f(5)$	5
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)\|0<x<5$	$x=3.01558$
$f(3.0155762421361)$	5.5511
Define $g(x)=\frac{x}{5}-3$	*Done*
Define $w(x)=f(x)-g(x)$	*Done*
$\text{fMax}(w(x),x)\|-5<x<5$	$x=\frac{5}{2}$
$w\left(\frac{5}{2}\right)$	8
$\text{fMin}(w(x),x)\|-5<x<5$	$x=\frac{-5}{2}$
$w\left(\frac{-5}{2}\right)$	6
$1.5\cdot\int_{-5}^{5} w(x)\,dx$	105.

Question 2

a. there is a turning point at $S(600,-6)$, so $h=600$, $c=-6$ A1

$f(x)=a(x-600)^2-6$ at $B(400,-4)$

$f(400)=-4 \Rightarrow -4=a(-200)^2-6$ M1

$$a=\frac{2}{(-200)^2}=\frac{1}{20,000}$$

b.

$$f(x)=\frac{1}{20,000}(x-600)^2-6$$

$$f'(x)=\frac{x}{10,000}-\frac{3}{50}$$

$f'(400)=\frac{400}{10,000}-\frac{3}{50}=-\frac{1}{50}$ A1

c. $g(x) = px^2 + qx + r$

$g(0) = 0 \quad \Rightarrow \quad r = 0$

$g(400) = -4 \quad \Rightarrow \quad (1) \quad -4 = p(400)^2 + 400q$ A1

$g'(x) = 2px + q$

$g'(400) = -\dfrac{1}{50} \quad \Rightarrow \quad (2) \quad -\dfrac{1}{50} = 800p + q$ A1

d. $r = 0$ solving (1) and (2) $q = 0$, $p = \dfrac{-1}{40,000}$ A1

e. $h(x) = mx^4 + nx^3$

$h(400) = -4 \quad \Rightarrow \quad (3) \quad -4 = m(400)^4 + n(400)^3$ A1

$h'(x) = 4mx^3 + 3nx^2$

$h'(400) = -\dfrac{1}{50} \quad \Rightarrow \quad (4) \quad -\dfrac{1}{50} = 4m(400)^3 + 3n(400)^2$ A1

f. solving (3) and (4) $m = \dfrac{1}{6,400,000,000}$, $n = \dfrac{-1}{8,000,000}$ A1

g. $g(x) = -\dfrac{x^2}{40,000}$

$A_1 = \displaystyle\int_{400}^{0} \left(-\frac{x^2}{40,000} \right) dx = \frac{1,600}{3} = 533\frac{1}{3} \text{ m}^2$ M1

$h(x) = \dfrac{x^4}{6,400,000,000} - \dfrac{x^3}{8,000,000}$

$A_2 = \displaystyle\int_{400}^{0} \left(\frac{x^4}{6,400,000,000} - \frac{x^3}{8,000,000} \right) dx = 480 \text{ m}^2$ M1

since $A_2 < A_1$ decide on option 2 A1

alternatively graphically $h(x) > g(x)$ over $x \in (0, 400)$

so choose option 2 as less excavation.

Define $f(x)=a\cdot(x-600)^2-6$	*Done*
$\text{solve}(f(400)=-4,a)$	$a=\frac{1}{20000}$
Define $f(x)=a\cdot(x-600)^2-6 \mid a=\frac{1}{20000}$	*Done*
$\frac{d}{dx}(f(x)) \mid x=400$	$\frac{-1}{50}$
Define $g(x)=p\cdot x^2+q\cdot x+r$	*Done*
$\text{solve}(g(0)=0,r)$	$r=0$
Define $g(x)=p\cdot x^2+q\cdot x$	*Done*
$eq1:=g(400)=-4$	$160000\cdot p+400\cdot q=-4$
$\frac{d}{dx}(g(x))$	$2\cdot p\cdot x+q$
$eq2:=\frac{d}{dx}(g(x))=\frac{-1}{50} \mid x=400$	$800\cdot p+q=\frac{-1}{50}$
$\text{linSolve}\left(\begin{cases}eq1\\eq2\end{cases},\{p,q\}\right)$	$\left\{\frac{-1}{40000},0\right\}$
Define $h(x)=m\cdot x^4+n\cdot x^3$	*Done*
$eq3:=h(400)=-4$	$25600000000\cdot m+64000000\cdot n=-4$
$\frac{d}{dx}(h(x))$	$4\cdot m\cdot x^3+3\cdot n\cdot x^2$
$eq4:=\frac{d}{dx}(h(x))=\frac{-1}{50} \mid x=400$	$256000000\cdot m+480000\cdot n=\frac{-1}{50}$
$\text{linSolve}\left(\begin{cases}eq3\\eq4\end{cases},\{m,n\}\right)$	$\left\{\frac{1}{6400000000},\frac{-1}{8000000}\right\}$
Define $g(x)=p\cdot x^2+q\cdot x \mid p=\frac{-1}{40000}$ and $q=0$	*Done*
$\int_{400}^{0} g(x)\,dx$	$\frac{1600}{3}$
Define $h(x)=m\cdot x^4+n\cdot x^3 \mid m=\frac{1}{6400000000}$ and $n=\frac{-1}{8000000}$	*Done*
$\int_{400}^{0} h(x)\,dx$	480

Question 3

a.i. $A \stackrel{d}{=} N\left(\mu = 45{,}000, \sigma^2 = 8{,}000^2\right)$

$\Pr\left(A > 40{,}000\right) = 0.7340$ A1

ii. $Y \stackrel{d}{=} Bi\left(n = 5, p = 0.7340\right)$

$\Pr\left(Y \geq 3\right) = 0.8789$ A1

normCdf(40000,∞,45000,8000)	0.734015
binomCdf(5,0.7340153,3,5)	0.878912

b. $B \stackrel{d}{=} N\left(\mu = ?, \sigma^2 = ?\right)$

$\Pr\left(B > 61{,}000\right) = 0.11 \Rightarrow \Pr\left(B < 61{,}000\right) = 0.89 \quad \Rightarrow \dfrac{61{,}000 - \mu}{\sigma} = 1.2265$ M1

$\Pr\left(B < 42{,}500\right) = 0.20 \quad \Rightarrow \dfrac{42{,}500 - \mu}{\sigma} = -0.8416$ M1

solving $\mu = 50{,}028$, $\sigma = 8{,}945$

the mean is 50,000 km, the standard deviation is 9,000 km. A1

invNorm(0.89,0,1)	1.2265
invNorm(0.2,0,1)	-0.8416
$\frac{61000-m}{s}=1.2265281204161$	$\frac{61000-m}{s}=1.2265$
$\frac{42500-m}{s}=-0.84162123346456$	$\frac{42500-m}{s}=-0.8416$
solve$\left(\frac{61000-m}{s}=1.2265 \text{ and } \frac{42500-m}{s}=-0.8416,\{m,s\}\right)$	$s=8945.4088$ and $m=50028.4561$

c. $\hat{p} = \dfrac{26}{36} = 0.7\dot{2}$, $n = 36$, $z = 1.96$

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}\ ,\ \hat{p} + z\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}\right)$$

$$= \left(0.722 - 1.96\sqrt{\frac{0.722 \times \left(1 - 0.722\right)}{36}}, 0.722 + 1.96\sqrt{\frac{0.722 \times \left(1 - 0.722\right)}{36}}\right)$$

$= \left(0.576, 0.869\right)$ A1

$\frac{26}{36}$	0.722222
$0.7222-1.96\cdot\sqrt{\frac{0.7222\cdot(1-0.7222)}{36}}$	0.576
$0.7222+1.96\cdot\sqrt{\frac{0.7222\cdot(1-0.7222)}{36}}$	0.869
zInterval_1Prop 26,36,0.95: *stat.results*	"Title" "1−Prop z Interval"; "CLower" 0.576; "CUpper" 0.869; "p̂" 0.722; "ME" 0.146; "n" 36.000

d.i $E(\hat{P}) = p = 0.7$ A1

$\text{var}(\hat{P}) = \frac{p(1-p)}{n} = \frac{0.7 \times 0.3}{36} = \frac{7}{1200}$ A1

ii. $\mu = np = 36 \times 0.7 = 25.2$

$\sigma = \sqrt{npq} = \sqrt{36 \times 0.7 \times 0.3} = 2.75$

$\mu \pm 2\sigma = 25.2 \pm 2 \times 2.75 = (19.7, 30.7)$ M1

$Y \overset{d}{=} Bi(n = 36, p = 0.7)$

$\Pr(20 \le Y \le 30) = 0.956$ A1

$\frac{7}{10}\cdot\frac{3}{10}\cdot\frac{1}{36}$	$\frac{7}{1200}$
$36\cdot 0.7$	25.200
$\sqrt{\frac{7}{1200}}$	$\frac{\sqrt{21}}{60}$
$\sqrt{36\cdot 0.7\cdot 0.3}$	2.750
$25.2+2\cdot 2.7495454169735$	30.699
$25.2-2\cdot 2.7495454169735$	19.701
binomCdf(36,0.7,20,30)	0.956

e.i. Since the graph is continuous at $t = 50$

$$f(50) = a \times 50^2 = b(100-50) \quad \Rightarrow \quad b = 50a$$ A1

Since the total area under the curve is equal to one.

$$a\int_0^{50} t^2\,dt + b\int_{50}^{100}(100-t)\,dt = 1$$

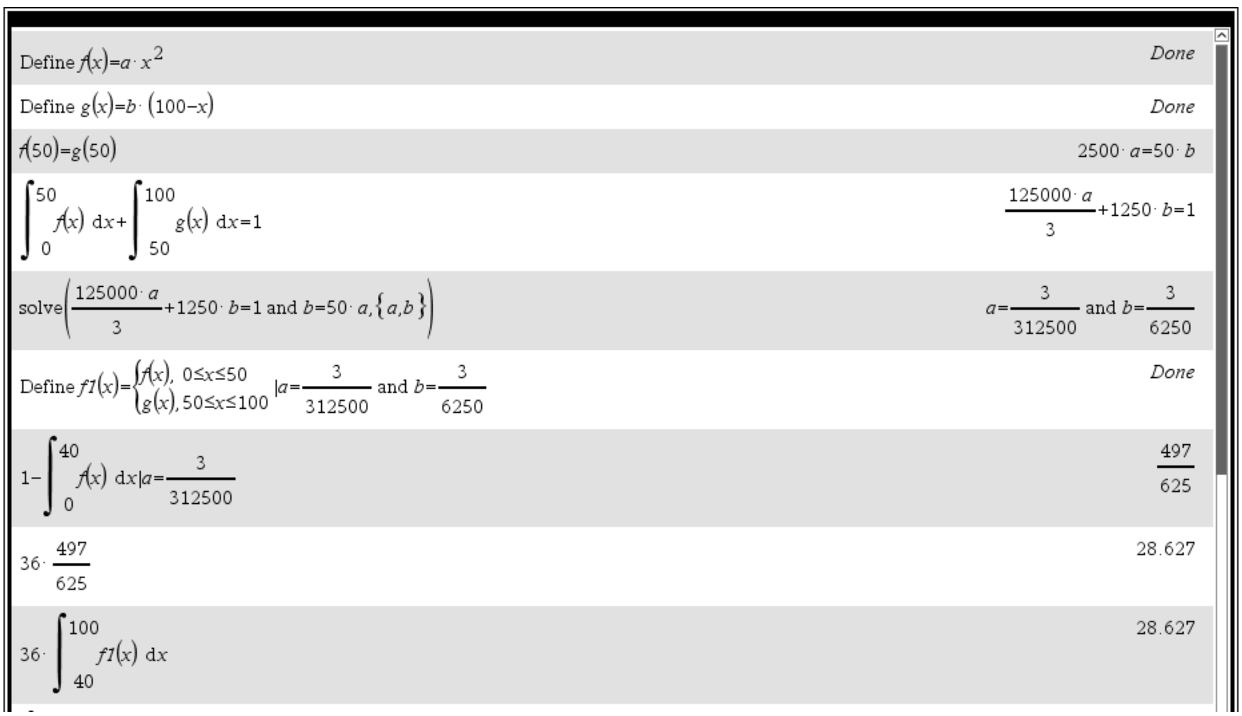
$$a\left[\frac{1}{3}t^3\right]_0^{50} + 50a\left[100t - \frac{1}{2}t^2\right]_{50}^{100} = 1$$

$$a\left[\left(\frac{1}{3}\times 50^3\right) + 50\left[\left(100\times 100 - \frac{1}{2}\times 100^2\right) - \left(100\times 50 - \frac{1}{2}\times 50^2\right)\right]\right] = 1$$ M1

$$\frac{312{,}500a}{3} = 1 \quad , \quad \text{so} \quad a = \frac{3}{312{,}500} \quad \text{and} \quad b = \frac{3}{6{,}250}$$

ii. $$\Pr(T \ge 40) = 1 - \Pr(T \le 40) = 1 - \frac{3}{312{,}500}\int_0^{40} t^2\,dt = \frac{497}{625}$$

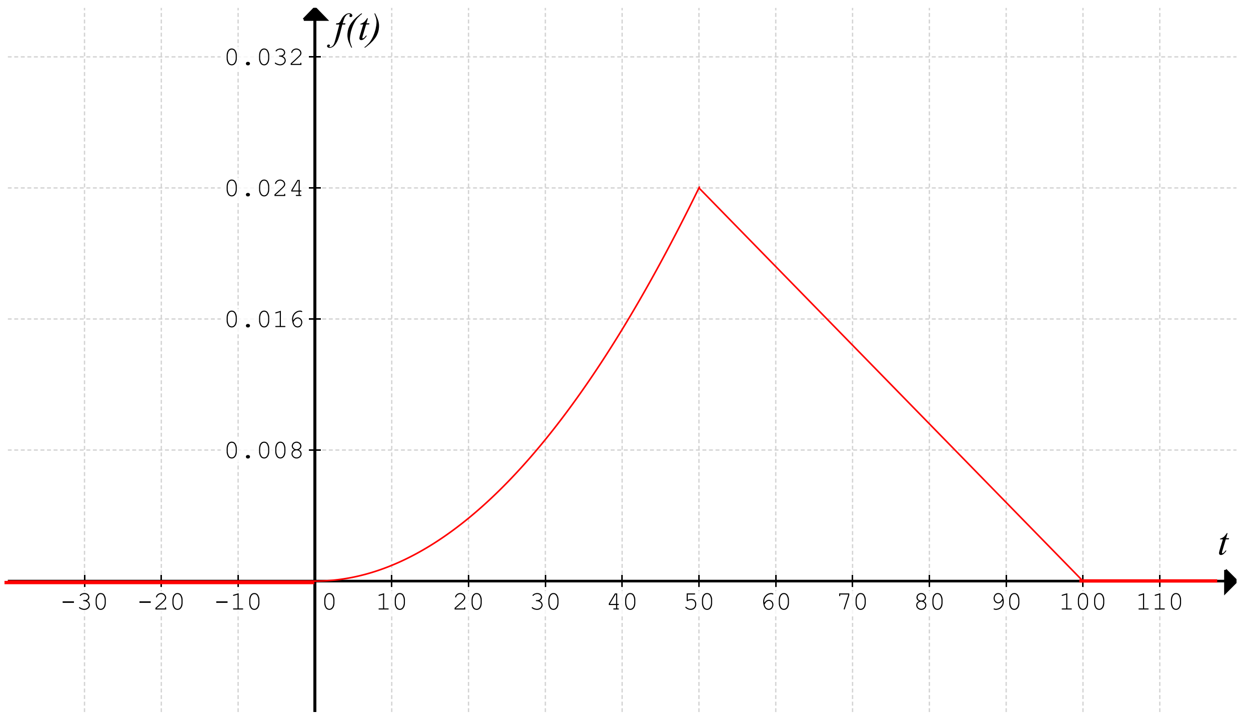
expect $36 \times \frac{497}{625} \approx 28.6$ accept 28 or 29 A1

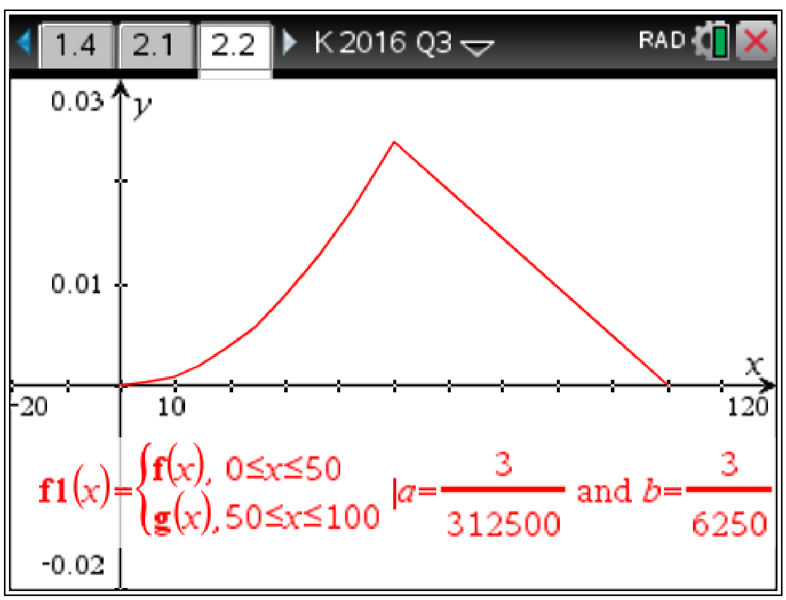


Define $f(x)=a\cdot x^2$	*Done*
Define $g(x)=b\cdot(100-x)$	*Done*
$f(50)=g(50)$	$2500\cdot a=50\cdot b$
$\int_0^{50} f(x)\,dx+\int_{50}^{100} g(x)\,dx=1$	$\frac{125000\cdot a}{3}+1250\cdot b=1$
solve$\left(\frac{125000\cdot a}{3}+1250\cdot b=1 \text{ and } b=50\cdot a,\{a,b\}\right)$	$a=\frac{3}{312500}$ and $b=\frac{3}{6250}$
Define $f1(x)=\begin{cases}f(x), & 0\le x\le 50\\ g(x), & 50\le x\le 100\end{cases} \mid a=\frac{3}{312500}$ and $b=\frac{3}{6250}$	*Done*
$1-\int_0^{40} f(x)\,dx \mid a=\frac{3}{312500}$	$\frac{497}{625}$
$36\cdot\frac{497}{625}$	28.627
$36\cdot\int_{40}^{100} f1(x)\,dx$	28.627

iii. graph, correct scale, shape, continuous at $\left(50, \frac{3}{125}\right) = (50, 0.024)$, G1

must show zero for $t \geq 100$ and $t \leq 0$ A1





iv. $E(T)=\dfrac{3}{312,500}\displaystyle\int_0^{50} t^3\,dt + \dfrac{3}{6,250}\int_{50}^{100} t(100-t)\,dt = 55$ A1

$E(T^2)=\dfrac{3}{312,500}\displaystyle\int_0^{50} t^4\,dt + \dfrac{3}{6,250}\int_{50}^{100} t^2(100-t)\,dt = 3350$

$\text{var}(T)=E(T^2)-(E(T))^2 = 3350-55^2 = 325$ A1

v. $\text{sd}(T)=\sqrt{325}=5\sqrt{13}$

$\mu \pm 2\sigma = 55 \pm 10\sqrt{13} = (18.944, 91.056)$

$\Pr(\mu-2\sigma \le T \le \mu+2\sigma) = \Pr(18.944 \le T \le 91.056)$ M1

$=\dfrac{3}{312,500}\displaystyle\int_{18.94}^{50} t^2\,dt + \dfrac{3}{6,250}\int_{50}^{91.056}(100-t)\,dt$

$=0.959$ A1

vi. Since $\dfrac{3}{312,500}\displaystyle\int_0^{50} t^2\,dt = \dfrac{2}{5} = 0.4$ M1

the median satisfies $\dfrac{3}{6,250}\displaystyle\int_{50}^{m}(100-t)\,dt = 0.1$

solving gives the median as 54.356 thousand km A1

$\int_0^{100} (x\cdot f1(x))\,dx$	55.000
$\int_0^{100} (x^2\cdot f1(x))\,dx$	3350.000
$3350.-(55.)^2$	325.000
$t1:=55.-2\cdot\sqrt{325.}$	18.944
$t2:=55.+2\cdot\sqrt{325.}$	91.056
$\int_{t1}^{t2} f1(x)\,dx$	0.9590
$\text{solve}\left(\int_0^{m} f1(x)\,dx=0.5,m\right)$	$m=54.3565$
$\int_0^{50} f(x)\,dx \mid a=\frac{3}{312500}$	$\frac{2}{5}$
$\text{solve}\left(\int_{50}^{m} g(x)\,dx=0.1,m\right) \mid b=\frac{3}{6250} \text{ and } 50<m<100$	$m=54.356$

Question 4

a.i. $f(x)=x(x+a)(x-a)=x\left(x^2-a^2\right)=x^3-a^2x$, where $a>0$

$f(k)=k^3-a^2k \quad f'(x)=3x^2-a^2 \quad , \quad f'(k)=3k^2-a^2$

Tangent $y-f(k)=f'(k)(x-k)$

$y-\left(k^3-a^2k\right)=\left(3k^2-a^2\right)(x-k)$ or $y=\left(3k^2-a^2\right)x-2k^3$ A1

ii. crosses the *x*-axis at $x=a$ when $y=0$ so $\left(3k^2-a^2\right)a-2k^3=0$

$3k^2a-a^3-2k^3=0$

solving for *k* $\Rightarrow \ k=-\dfrac{a}{2}$ since $k<0$ A1

b. $f'(x)=3x^2-a^2=0 \ \Rightarrow \ x=\pm\dfrac{\sqrt{3}\,a}{3}$

increasing $f'(x)>0 \ \Rightarrow \ \left(-\infty,-\dfrac{\sqrt{3}\,a}{3}\right)\cup\left(\dfrac{\sqrt{3}\,a}{3},\infty\right)$ A1

c. a translation by *d* units parallel to the *y*-axis (or away from the the *x*-axis) A1

d.i turning points of $f(x)$ are at $\left(-\dfrac{\sqrt{3}\,a}{3},\dfrac{2a^3\sqrt{3}}{9}\right)$ and $\left(\dfrac{\sqrt{3}\,a}{3},-\dfrac{2a^3\sqrt{3}}{9}\right)$

turning points of $g(x)$ are at $\left(-\dfrac{\sqrt{3}\,a}{3},d+\dfrac{2a^3\sqrt{3}}{9}\right)$ and $\left(\dfrac{\sqrt{3}\,a}{3},d-\dfrac{2a^3\sqrt{3}}{9}\right)$

for one *x* intercept, we require

$d-\dfrac{2a^3\sqrt{3}}{9}>0 \ \Rightarrow \ d>\dfrac{2a^3\sqrt{3}}{9}$ or $d+\dfrac{2a^3\sqrt{3}}{9}<0 \ \Rightarrow \ d<-\dfrac{2a^3\sqrt{3}}{9}$ A1

ii. for three *x* intercepts, we require $d-\dfrac{2a^3\sqrt{3}}{9}<0$ and $d+\dfrac{2a^3\sqrt{3}}{9}>0$

$-\dfrac{2a^3\sqrt{3}}{9}<d<\dfrac{2a^3\sqrt{3}}{9}$ A1

e. by symmetry $A=\int_{-a}^{0}\left(x^3-a^2x\right)dx=64$ and $A=\int_{0}^{a}\left(x^3-a^2x\right)dx=-64$

$\left[\dfrac{1}{4}x^4-\dfrac{1}{2}a^2x^2\right]_{-a}^{0}=0-\left(\dfrac{1}{4}a^4-\dfrac{1}{2}a^4\right)=\dfrac{1}{4}a^4=64$ M1

$a^4=256$

$a=4$ A1

Define $f(x)=x\cdot(x-a)\cdot(x+a)$	*Done*
$\text{tangentLine}(f(x),x,k)$	$-(a^2-3\cdot k^2)\cdot x-2\cdot k^3$
$\text{solve}(-(a^2-3\cdot k^2)\cdot x-2\cdot k^3=0,k)\mid x=a$	$k=\frac{-a}{2}$ or $k=a$
$\frac{d}{dx}(f(x))$	$3\cdot x^2-a^2$
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$x=\frac{a\cdot\sqrt{3}}{3}$ or $x=\frac{-a\cdot\sqrt{3}}{3}$
$\text{solve}\left(\frac{d}{dx}(f(x))\geq 0,x\right)\mid a>0$	$x\leq\frac{-a\cdot\sqrt{3}}{3}$ and $a>0$ or $x\geq\frac{a\cdot\sqrt{3}}{3}$ and $a>0$
$f\left(\frac{-a\cdot\sqrt{3}}{3}\right)$	$\frac{2\cdot a^3\cdot\sqrt{3}}{9}$
$f\left(\frac{a\cdot\sqrt{3}}{3}\right)$	$\frac{-2\cdot a^3\cdot\sqrt{3}}{9}$
$\text{solve}\left(d+\frac{2\cdot a^3\cdot\sqrt{3}}{9}>0 \text{ and } d-\frac{2\cdot a^3\cdot\sqrt{3}}{9}<0,d\right)$	$\frac{-2\cdot a^3\cdot\sqrt{3}}{9}<d<\frac{2\cdot a^3\cdot\sqrt{3}}{9}$
$\text{solve}\left(\int_{-a}^{0} f(x)\,dx=64,a\right)\mid a>0$	$a=4$

Question 5

a. by similar triangles $\dfrac{a-r}{h}=\dfrac{r-b}{L-h}$ M1

$h(r-b)=(a-r)(L-h)$

$hr-hb=aL-Lr-ah+hr$ M1

$ah-bh=aL-rL$

$h(a-b)=L(a-r)$

$h=\dfrac{L(a-r)}{a-b}$ A1

b. $V=\pi r^2 h=\dfrac{\pi r^2 L(a-r)}{a-b}=\dfrac{\pi L}{a-b}\left(ar^2-r^3\right)$ A1

$\dfrac{dV}{dr}=\dfrac{\pi L}{a-b}\left(2ar-3r^2\right)$ A1

c. for maximum volume $\dfrac{dV}{dr}=0 \Rightarrow 2ar-3r^2=r(2a-3r)=0$

since $a>0$, $b>0$, $r>0 \Rightarrow r=\dfrac{2a}{3}$ A1

also $r-b>0 \Rightarrow b<\dfrac{2a}{3}$ A1

d. Now when $r=\dfrac{2a}{3} \Rightarrow h=\dfrac{L\left(a-\frac{2a}{3}\right)}{a-b}=\dfrac{aL}{3(a-b)}$ M1

$V=\pi r^2 h=\pi\left(\dfrac{4a^2}{9}\right)\left(\dfrac{aL}{3(a-b)}\right)$

$V=\dfrac{4a^3\pi L}{27(a-b)}$ A1

END OF SUGGESTED SOLUTIONS