Neap:

Trial Examination 2015

VCE Mathematical Methods (CAS) Units 3&4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	**B**	C	D	E
2	A	B	C	**D**	E
3	A	**B**	C	D	E
4	A	B	C	**D**	E
5	A	B	**C**	D	E
6	A	B	**C**	D	E
7	A	B	C	D	**E**
8	A	**B**	C	D	E
9	A	B	C	**D**	E
10	A	B	**C**	D	E
11	**A**	B	C	D	E

12	A	B	C	**D**	E
13	**A**	B	C	D	E
14	A	B	**C**	D	E
15	A	B	C	**D**	E
16	A	B	C	**D**	E
17	A	**B**	C	D	E
18	A	**B**	C	D	E
19	A	B	C	**D**	E
20	A	B	C	D	**E**
21	A	**B**	C	D	E
22	A	B	C	**D**	E

SECTION 1

Question 1 B

The square-root function is defined when the argument is zero or positive. Therefore:

- $\sqrt{x+2}$ is defined when $x + 2 \geq 0$; that is, $x \geq -2$.
- $\sqrt{x+1}$ is defined when $x + 1 \geq 0$; that is, $x \geq -1$.
- $\sqrt{x-1}$ is defined when $x - 1 \geq 0$; that is, $x \geq 1$.

Hence for all three square roots to be defined, it is necessary that $x \geq 1$.

However, notice that $\sqrt{x-1}$ is in the denominator of $\frac{1}{\sqrt{x-1}}$, hence when $\sqrt{x-1} = 0$, $\frac{1}{\sqrt{x-1}}$ is undefined.

$\sqrt{x-1} = 0$ when $x = 1$, so we exclude when $x = 1$ from the domain and conclude that $\frac{\sqrt{x+2}}{\sqrt{x+1}} + \frac{1}{\sqrt{x-1}}$ is only defined when $x > 1$.

Therefore the required maximal domain is $(1, \infty)$.

Question 2 D

$$y'(x) = 3x^2$$

$$\begin{aligned} y'(1) &= 3 \times 1^2 \\ &= 3 \end{aligned}$$

Since $m_1 = 3$:

$$\begin{aligned} m_1 m_2 &= -1 \\ 3m_2 &= -1 \\ m_2 &= -\frac{1}{3} \end{aligned}$$

The gradient of the normal is $-\frac{1}{3}$.

Given that a line through the point (x_1, y_1) with gradient m is $y - y_1 = m(x - x_1)$, substitute $(x_1, y_1) = (1, 2)$ and gradient $m = -\frac{1}{3}$.

Thus $y - 2 = -\frac{1}{3}(x - 1)$, which is equivalent to $y - 2 = \frac{1}{3}(1 - x)$.

Question 3 B

Use CAS to find the derivative of $y = x^2(x - a) + b \Rightarrow \dfrac{dy}{dx} = 3x^2 - 2ax$

Turning points occur when $3x^2 - 2ax = 0$.

$$\therefore x = 0 \text{ or } x = \frac{2a}{3}$$

Since (2, 10) is a turning point:

$2 = \dfrac{2a}{3}$

$a = 3$

Hence the equation becomes $y = x^2(x - 3) + b$.

Since the turning point (2, 10) lies on the graph:

$10 = 2^2(2 - 3) + b$

$b = 14$

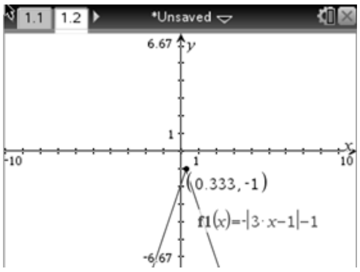
So $y = x^2(x - 3) + 14$.

Local maximum occurs when $x = 0$, so the value of y when $x = 0$ is given by $y = 0^2(0 - 3) + 14$.
Coordinates of the local maximum are thus (0, 14).

The slope of the line between the turning points is $\dfrac{10 - 14}{2 - 0} = -2$.

Question 4 D

Using CAS to sketch the graph:



It can be seen that the graph is continuous and has a vertex at $\left(\dfrac{1}{3}, -1\right)$, so **A** and **B** are true statements.

The graph is completely below the y-axis, therefore **C** is also true.

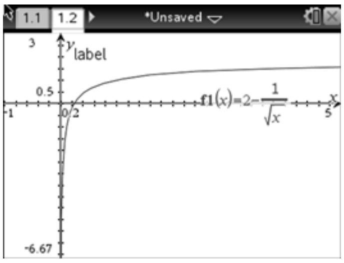
The gradient of the graph is positive to the left of $x = \dfrac{1}{3}$ and negative to the right of $x = \dfrac{1}{3}$, but is not defined at $x = 0$. Therefore **D** is false.

Question 5 **C**

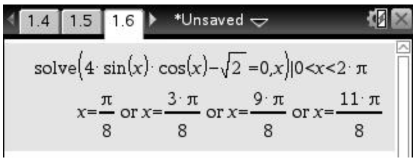
To find the inverse, interchange x and y:

The range of f will be the domain of f^{-1}.

From the graph, the range of f is $(-\infty, 2)$, so the domain of f^{-1} is $(-\infty, 2)$.



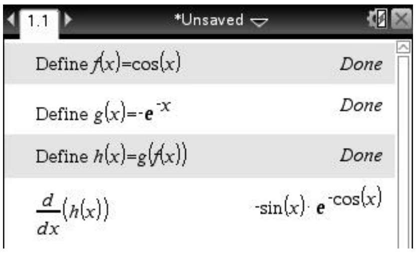
Question 6 **C**



We observe that the only solutions in the required interval are $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}$ and $\frac{11\pi}{8}$.

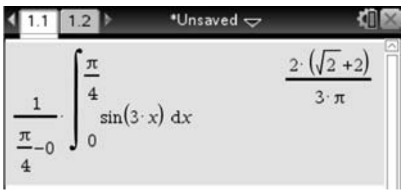
Question 7 **E**

Using CAS:



Question 8 **B**

Using CAS:



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Question 9 D

Using the rule for linear approximation: $f(x+h) \approx hf'(x) + f(x)$

$$f(x) = e^{-2x}$$
$$= \frac{1}{e^{2x}}$$

$\frac{1}{e^{1.96}} = \frac{1}{e^{2 \times 0.98}}$, so we require $f(0.98)$.

Therefore we use $x = 1$ and $h = -0.02$.

Applying these values to the rule, we get $f(1 - 0.02) \approx 0.02f'(1) + f(1)$.

Question 10 C

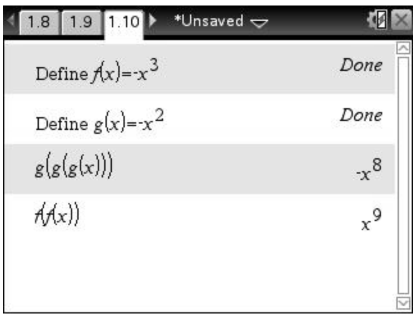
We require the graph of the antiderivative of *h*, so *h* is a gradient function. Since the graph of $h \geq 0$ for all values of *x*, the graph of the antiderivative must have a gradient ≥ 0. Hence **C** is the only correct option.

Question 11 A

$$\Pr(X > 13.5) = \Pr\left(Z > \frac{x - \mu}{\sigma}\right)$$
$$= \Pr\left(Z > \frac{13.5 - 12}{0.75}\right)$$
$= (Z > 2)$, which is equal to $\Pr(Z < -2)$ by symmetry

Question 12 D

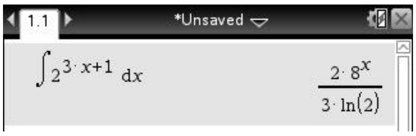
Using CAS:



These screenshots demonstrate that $f(f(x))$ does not equal $g(g(g(x)))$ in general, but only when $x = 0$ or 1.

Question 13 A

Since $f(x)$ is an antiderivative of $f'(x) = 2^{3x+1}$, using CAS:



We can use CAS to confirm that $\ln(8) = 3\ln(2)$.

Question 14 C

By the chain rule, $h'(x) = f'(g(x))g'(x)$.

So if $g'(a) = 0$, then $h'(a) = 0$.

Hence if g has a stationary point at $x = a$, h will have a stationary point at $x = a$.

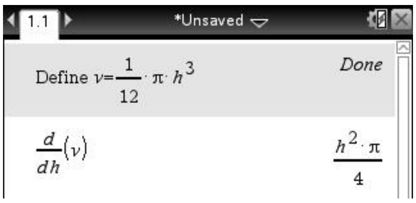
Question 15 D

The chain rule applies here: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

Since the diameter equals the height of the cone, then $2r = h$, where h is the depth of the water and r is the radius of the water surface. This relation will still hold as the level of the water drops because of similar triangles.

$$\begin{aligned} \text{Hence } V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \\ &= \frac{1}{12}\pi h^3 \end{aligned}$$

Using CAS to find the derivative:



Since water leaks out at a rate proportional to the depth, $\frac{dV}{dt} = kh$.

Substituting into the chain rule:

$$kh = \frac{h^2\pi}{4} \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4k}{\pi h}$$

Question 16 D

Using mapping coordinates:

$(x, y) \rightarrow (3x, y)$ under a dilation of factor 3 from y-axis, and $(x, y) \rightarrow (3x, -y)$ under a reflection on the x-axis

$x = 3x'$ gives $x' = \frac{x}{3}$, and $y = -y'$ gives $y' = y$.

The image equation becomes $-y = \cos\left(3 \times \frac{x}{3}\right) + 1$

$$y = -\cos(x) - 1$$

Since there is a dilation from the y-axis, the domain is affected: $\left[0, \frac{\pi}{3}\right] \rightarrow \left[0 \times 3, \frac{\pi}{3} \times 3\right]$

Hence the domain of the image function is $[0, \pi]$.

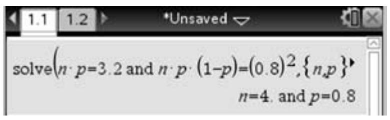
Question 17 B

$$\begin{aligned}\int_1^4 3(f(x) - 1)dx &= \int_1^4 3(f(x) - 3)dx \\ &= \int_1^4 3f(x)dx - \int_1^4 3dx \\ &= \int_1^4 3f(x)dx - [3x]_1^4 \\ &= 3 \times 6 - (12 - 3) \\ &= 9\end{aligned}$$

Question 18 B

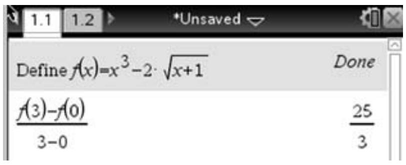
Since distribution is binomial, $\mu = np$ and $\sigma^2 = np(p - 1)$.

Solving simultaneously gives:



Question 19 D

The average rate of change over this interval is given by $\frac{f(3) - f(0)}{3 - 0}$.



Question 20 E

Since $\Pr(A) = a$, $\Pr(A') = 1 - a$. Similarly, $\Pr(B') = 1 - b$.

Since A and B are independent, $\Pr(A' \cap B') = \Pr(A') \times (B')$.

Using the addition rule: $\Pr(A' \cup B') = \Pr(A') + \Pr(B') - \Pr(A' \cap B')$

$$= (1 - a) + (1 - b) - (1 - a)(1 - b)$$

$$= 2 - a - b - (1 - a - b + ab)$$

$$= 1 - ab$$

Question 21 B

We observe that the system of equations are non-linear, since $y - 4x = x^2$ is equivalent to $y = x^2 + 4x$, and so y is a quadratic function of x.

The graph of the equation $2y - 2ax = -b$ will be from a family of straight lines. We know that in general, a line can intersect the parabola in 0, 1 or 2 places. It will touch in 1 place only if it is a tangent to the parabola or if it is a vertical line.

Making y the subject in both equations leads to $y = x^2 + 4x$ and $y = ax - \frac{b}{2}$.

Eliminating y from these equations gives:

$$x^2 + 4x = ax - \frac{b}{2}$$

$$x^2 + (4 - a)x + \frac{b}{2} = 0$$

The discriminant of this equation is $(4 - a)^2 - 4 \times \frac{b}{2} = (4 - a)^2 - 2b$.

The equation will have one solution when the discriminant is zero.

$$(4 - a)^2 - 2b = 0$$

$$(4 - a)^2 = 2b$$

$$a = 4 \pm \sqrt{2b}$$

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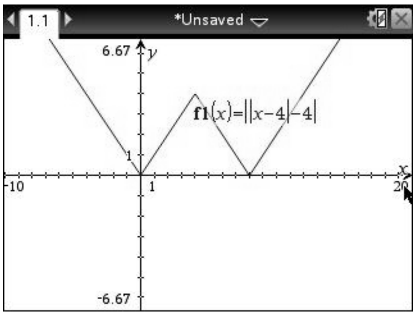
Question 22 D

Use CAS to plot a graph of $y = ||x - 4| - 4|$.



Inspection of the graph reveals that it has a minimum value of 0 when $x = 0$ and $x = 8$. There is also a local maximum of 4 when $x = 4$.

We now consider the lines $y = k$ for various values of k and, in particular, consider how many intercepts it will have with the graph of $y = ||x - 4| - 4|$.

The solutions for x of the simultaneous equations $y = ||x - 4| - 4|$ and $y = k$ will be the same as the solutions of $||x - 4| - 4| = k$.

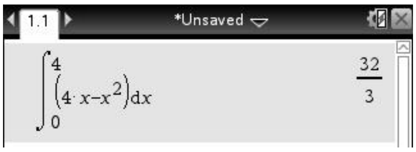
To sum up the cases:

$k < 0$	0 solutions
$k = 0$	2 solutions
$0 < k < 4$	4 solutions
$k = 4$	3 solutions
$k > 4$	2 solutions

SECTION 2

Question 1 (12 marks)

a. $\int_0^4 (4x - x^2)dx = \frac{32}{3}$ units^2 A1

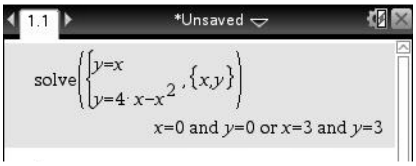


b. Solving:

$y = x$

$y = 4x - x^2$

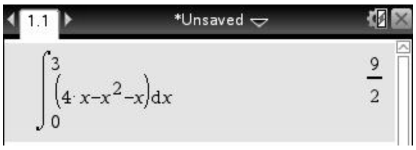
simultaneously gives the coordinates of the points of intersection.



The required points are (0, 0) and (3, 3). A1

c. area $= \int_0^3 (4x - x^2 - x)dx$ M1

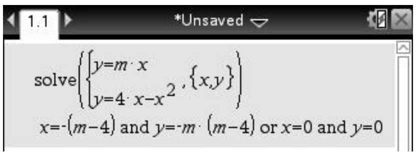
$= 4.5$ units^2 A1



d. The required coordinates of P_m are the solutions of the following simultaneous equations:

$y = mx$

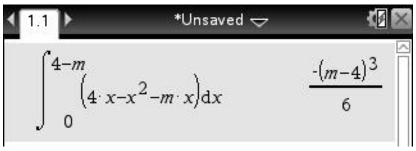
$y = 4x - x^2$



M1

As P_m is on the line $y = mx$ and is not located at the origin, it has coordinates $(4 - m, m(4 - m))$. A1

e. $\displaystyle\int_0^{4-m}(4x - x^2 - mx)\,dx = \frac{(4-m)^3}{6}\text{ units}^2$ M1 A1



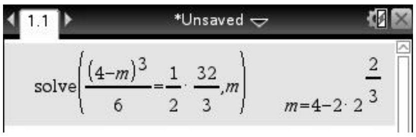
f. area of C_m = area of A – area of B_m M1

$$= \frac{32}{3} - \frac{(4-m)^3}{6}$$ A1

g. area of C_m = area of B_m when the area of B_m is half the area of region A.

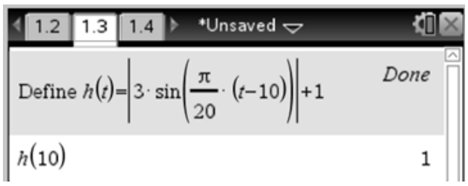
$$\frac{(4-m)^3}{6} = \frac{1}{2} \times \frac{32}{3}$$ M1

$$m = -\sqrt[3]{32}\ \left(\text{also } 4 - 2^{\frac{5}{3}}\right)$$ A1

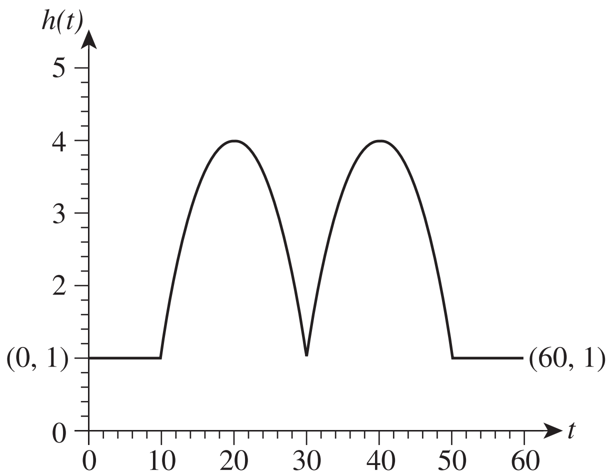


Question 2 (14 marks)

a. $h(t)$ is a continuous function, therefore at $t = 10$, $k = h(10)$.
$h(10) = 1$
$\therefore\ k = 1$ A1



b.

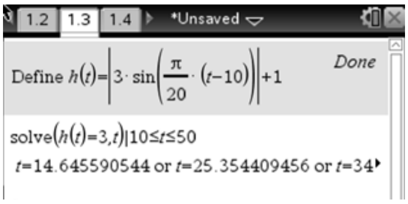


correct shape/period of sine graph A1
correct amplitude and effect of absolute value A1
correct linear 'branches' shown A1

c. From the graph (and due to the symmetry of the sine curve) we can determine that the maximum value occurs at $t = 20$ and $t = 40$. Since $t = 0$ is 6:00 am, the gate reaches its maximum height at 6:20 am and 6:40 am on Saturday. A1

d. $h(t) = 3,\ t \in [0, 60]$: $t = 14.65, 25.35, 34.65$ and 45.35 M1

Corresponding to the hour after 9:00 am, the gate would first reach a height of 3 m at 14.65 minutes after 9:00 am. Since the delivery van arrives at 9:10 am, the driver will have to wait 4.65 minutes. A1

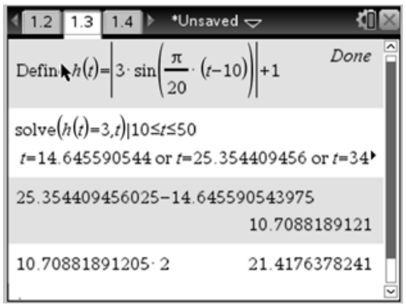


e. Each hour, the boom gate is at least 3 m high between 14.65 and 25.35 minutes past the hour, and again between 34.65 and 45.35 minutes past the hour.

$25.35 - 14.65 = 10.71$ M1

Twice every hour the boom gate is over 3 m high.

$\therefore\ 10.71 \times 2 = 21.42$ minutes each hour A1

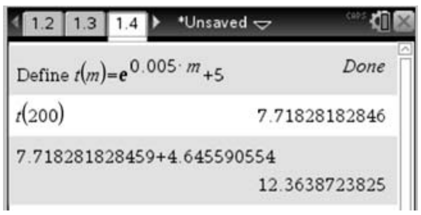


f. The driver arrives at 9:10 am and cannot enter for 4.65 minutes.

The time taken to deliver his 200 kg load $= t(200)$

$= 7.72$ minutes A1

The total time elapsed is 12.36 minutes. A1



g. The driver arrives at 9:28 am and cannot enter for 6.65 minutes.

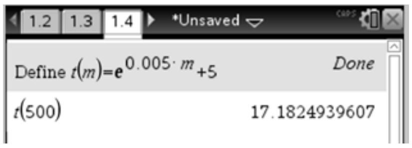
The time taken to deliver his 500 kg load $= t(500)$

$= 17.18$ minutes **A1**

By this time the boom gate has dropped below a height of 3 m, so he cannot get out.
The boom gate is only above 3 m for 10.71 minutes at a time, which is not enough time for the driver to unload his delivery. He will have to wait until the boom gate next reaches a height of 3 m to exit, which will be at 14.65 minutes past the next hour.

Therefore the time he will exit = 14.65 minutes after 10:00 am **M1**

= 10:15 am **A1**



Question 3 (16 marks)

a. **i.** $\Pr(\text{paddles}|\text{paddles previous morning}) = 0.4$ (from question)

Therefore $\Pr(\text{paddles next 2 mornings}) = 0.4 \times 0.4$

$= 0.16$ **A1**

ii. P = paddles, S = swims

$\Pr(\text{paddles once over next 3 mornings}) = \Pr(\text{PSS}) + \Pr(\text{SPS}) + \Pr(\text{SSP})$ **M1**

$= 0.4 \times 0.6 \times 0.3 + 0.6 \times 0.7 \times 0.6 + 0.6 \times 0.3 \times 0.7$

$= 0.45$ **A1**

iii. The transition matrix is $T = \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}$.

Since Timothy paddles on Monday morning, the initial state matrix is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\Pr(\text{swims on Friday}) = T^4 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ **M1**

$= 0.458$ **A1**

b. Method 1:

$$S_{\infty} = T_{\infty}\begin{bmatrix}0\\1\end{bmatrix}$$

He trains for the paddle on 53.85% of mornings. A1

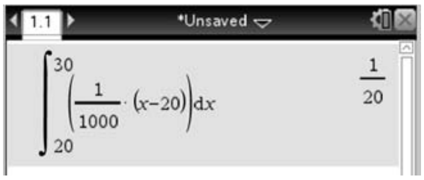
Method 2:

$$\frac{0.7}{0.6+0.7} = 0.53846$$

$$= 53.85\%$$ A1

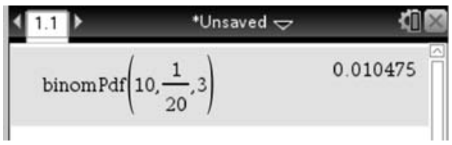
c. **i.** $\Pr(\text{paddles} < 30 \text{ minutes}) = \int_{20}^{30} \frac{1}{100}(x-20)\,dx$ M1

$$= \frac{1}{20}$$ A1



ii. Let Y = number of days he paddles less than 30 minutes.

This represents a binomial distribution: $Y \sim \text{Bi}\left(10, \frac{1}{20}\right)$. M1



$\Pr(Y = 3) = 0.010$ A1

d. This is conditional probability: $\Pr(X > 60 | X > 30)$. M1

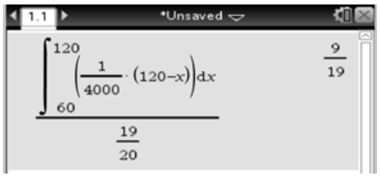
$$\frac{\Pr(X > 60 \cap X > 30)}{\Pr(X > 30)} = \frac{\Pr(X > 60)}{\Pr(X > 30)}$$

$$= \frac{\Pr(X > 60)}{1 - \Pr(X < 30)}$$

$$= \frac{\int_{60}^{120} f(x)dx}{\frac{19}{20}}$$ M1

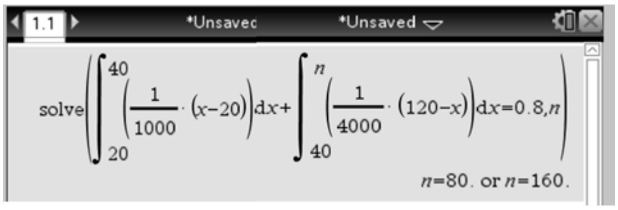
Must give either correct numerator or denominator for M1.

$\Pr(\text{paddles} > 60 \text{ minutes on a given weekend}) = \frac{9}{19}$ A1



e. $\Pr(X < n) = 0.8$ M1

$$\int_{20}^{40} \frac{1}{1000}(x - 20)dx + \int_{40}^{n} \frac{1}{4000}(120 - x)dx = 0.8$$ M1



Since $n < 120$, $n = 80$. A1

Question 4 (8 marks)

a. R_2 is the distance of P2 from the origin (0, 0).

The coordinates of P2 are $\left(4\cos\left(\frac{\pi t}{4}\right), 4\sin\left(\frac{\pi t}{4}\right)\right)$, hence using the distance formula:

$$R_2 = \sqrt{\left(4\cos\left(\frac{\pi t}{4}\right) - 0\right)^2 + \left(4\sin\left(\frac{\pi t}{4}\right) - 0\right)^2}$$

$= 4$ A1

To find the period of the orbit we need to find the period of the functions $4\cos\left(\frac{\pi t}{4}\right)$ and $4\sin\left(\frac{\pi t}{4}\right)$.

$$T_2 = \frac{2\pi}{\frac{\pi}{4}}$$

$= 8$ A1

b. **i.** $R_2 = kT_2{}^q$ gives $4 = k8^q$.

$R_1 = kT_1{}^q$ gives $1 = k1^q$. A1

ii. $k = 1$ (since $1^q = 1$ for all values of q)

Therefore $4 = 8^q$

$q = \frac{2}{3}$

Hence the required law is $R = T^{\frac{2}{3}}$. A1

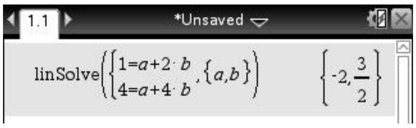
c. $R_1 = a + b \times 2^1$

$R_2 = a + b \times 2^2$ M1

$1 = a + 2b$

$4 = a + 4b$

Solving simultaneously gives:



Thus $a = -2,\ b = \frac{3}{2}$. A1

d. Substituting $n = 3$ into $R_n = -2 + \frac{3}{2} \times 2^n$ gives:

$R_3 = -2 + \frac{3}{2} \times 2^3$

$= 10$ A1

$R = T^{\frac{2}{3}}$

$10 = (T_3)^{\frac{2}{3}}$

$T_3 = 10\sqrt{10}$ A1

Question 5 (8 marks)

a. $f(-1) = 6$ and $f(2) = 24$.

Therefore coordinates of endpoints are $(-1, 6)$ and $(2, 24)$. A1

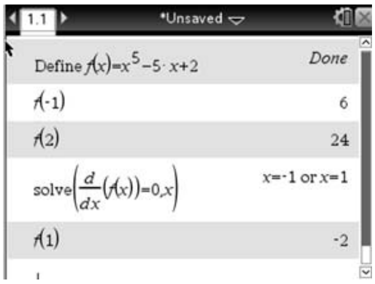
The stationary points are given by $f'(x) = 0$

$$x = \pm 1$$ M1

Since there can be no stationary point at an endpoint, $x = -1$ is not a stationary point.

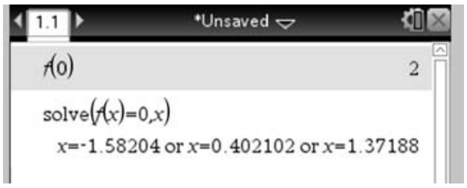
$f(1) = -2$, so there is a stationary point at $(1, -2)$. A1

No mark to be awarded if two stationary points are given.

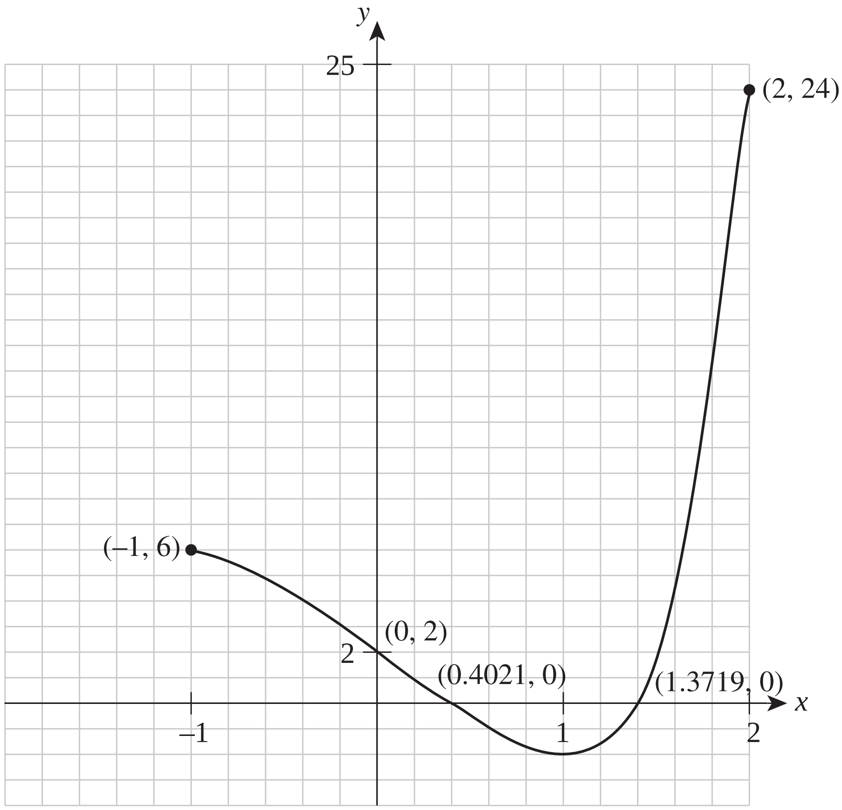


b. y-intercept is given by $f(0)$: $(0, 2)$ A1

x-intercept is given by $f(x) = 0$: $(0.4021, 0)$ and $(1.3719, 0)$ A1



c.



shape A1
all coordinates A1

d. range of f over given domain: [–2, 24] A1