# The Mathematical Association of Victoria
# MATHEMATICAL METHODS (CAS)
## SOLUTIONS: Trial Exam 2015

## Written Examination 1

**Question 1**

**a.** Let $y = xe^{2x}$

Using the Product Rule

$\frac{dy}{dx} = \left(e^{2x} \times 1\right) + \left(x \times 2e^{2x}\right)$ **1M**

$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$ **1A**

**b.** Find $\int \left(xe^{2x}\right) dx$

From **part a.** we know that $\int \left(e^{2x} + 2xe^{2x}\right) dx = xe^{2x} + c$ **1M**

$\int \left(e^{2x}\right) dx + \int \left(2xe^{2x}\right) dx = xe^{2x} + c$

$\int \left(2xe^{2x}\right) dx = xe^{2x} - \int \left(e^{2x}\right) dx + c$

$\int \left(2xe^{2x}\right) dx = xe^{2x} - \frac{1}{2}e^{2x} + c_1$

$\int \left(xe^{2x}\right) dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c_1$ **1A**

**Question 2**

$(k-1)x + 2y = 1$

$x + (k-1)y = -k$

Let $A = \begin{bmatrix} k-1 & 2 \\ 1 & k-1 \end{bmatrix}$

$\det(A) = (k-1)^2 - 2 = 0$ **1M**

$(k-1)^2 - 2 = 0$

$(k-1)^2 = 2$

$k - 1 = \pm\sqrt{2}$

$k = 1 + \sqrt{2}$ or $k = 1 - \sqrt{2}$

For a unique solution

$k \in R \setminus \left\{1 \pm \sqrt{2}\right\}$ **1A**

**OR**

$y = \frac{1-(k-1)x}{2}$ (1)

$y = \frac{-k-x}{k-1}$ (2)

If there is a unique solution, the lines must not be parallel hence have different gradients. Therefore, equate gradients to find the case where the lines <u>are</u> parallel.

$\frac{-(k-1)}{2} = \frac{-1}{k-1}$ **1M**

$-(k-1)^2 = -2$

$(k-1)^2 = 2$

$\therefore k = 1+\sqrt{2}$ or $k = 1-\sqrt{2}$

For a unique solution

$k \in R \setminus \{1 \pm \sqrt{2}\}$ **1A**

**Question 3**

**a.** $f : \left[-\frac{1}{3}, \infty\right) \to R, f(x) = \log_e(3x+2)$ and $g : [0, \infty) \to R, g(x) = |x-1|$.

$g(f(x)) = |\log_e(3x+2) - 1|$ **1M**

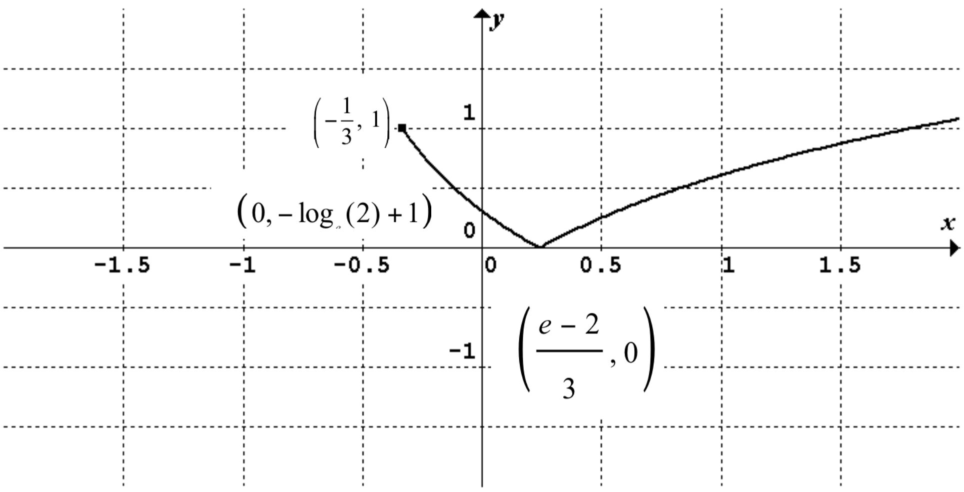
Dom $g(f(x))$ = dom $f(x)$ = $\left[-\frac{1}{3}, \infty\right)$ **1A**

**b.** $g(f(x)) = |\log_e(3x+2) - 1|$

Shape **1A**

Correct intercepts $\left(\frac{e-2}{3}, 0\right)$ and $(0, -\log_e(2)+1)$ **1A**

Correct endpoint $\left(-\frac{1}{3}, 1\right)$ **1A**



**Question 4**

**a.** $h:[0,14]\rightarrow R, h(t)=2\sin\left(\frac{\pi}{30}(t+1)\right)+2$

$h'(t)=\frac{\pi}{15}\cos\left(\frac{\pi}{30}(t+1)\right)$ **1A**

**b.** Given $\frac{dV}{dt}=2$ cm$^3$/s

$\frac{dV}{dh}=\frac{dV}{dt}\times\frac{dt}{dh}$ **1M**

$\frac{dV}{dh}=2\times\frac{15}{\pi\cos\left(\frac{\pi}{30}(t+1)\right)}$

When $h=3$,

$2\sin\left(\frac{\pi}{30}(t+1)\right)+2=3$

$\sin\left(\frac{\pi}{30}(t+1)\right)=\frac{1}{2}$

$\frac{\pi}{30}(t+1)=\frac{\pi}{6}$

$t=4$ **1A**

$\frac{dV}{dh}=2\times\frac{15}{\pi\cos\left(\frac{\pi}{30}(4+1)\right)}$

$\frac{dV}{dh}=\frac{30}{\pi\cos\left(\frac{\pi}{6}\right)}$

$\frac{dV}{dh}=\frac{20\sqrt{3}}{\pi}$ cm$^3$/cm **1A**

**Question 5**

**a.** $f(x)=\frac{1}{2}\log_e\left(x(x+1)\right)\log_e(2x-1)$

Using the product and chain rules

$f'(x)=\left[\log_e(2x-1)\times\frac{1}{2x(x+1)}\times(2x+1)\right]+\left[\frac{1}{2}\log_e\left(x(x+1)\right)\times\frac{2}{2x-1}\right]$

$\therefore f'(x)=\left[\frac{2x+1}{2x(x+1)}\log_e(2x-1)\right]+\left[\frac{1}{2x-1}\log_e\left(x(x+1)\right)\right]$ **1A**

**b.** $f'(2)=\left[\frac{5}{12}\log_e(3)\right]+\left[\frac{1}{3}\log_e(6)\right]$ **1M**

in the form of $\log_e\left(a^m b^n\right)$

$$\therefore f'(2) = \left[\log_e\left(3^{\frac{5}{12}}\right)\right] + \left[\log_e\left(6^{\frac{1}{3}}\right)\right]$$

$$\therefore f'(2) = \log_e\left(3^{\frac{5}{12}}6^{\frac{1}{3}}\right)$$ **1A**

$$= \log_e\left(3^{\frac{3}{4}}2^{\frac{1}{3}}\right)$$ **1A**

**Question 6**

$$2\sqrt{3}\cos(2x) = -3$$

$$\cos(2x) = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}$$

$$2x = \frac{5\pi}{6}, \frac{7\pi}{6}...$$ **1A**

$$x = \frac{5\pi}{12}, \frac{7\pi}{12}...$$

$$x = \frac{5\pi}{12} + k\pi, k \in Z$$ **1A**

$$x = \frac{7\pi}{12} + k\pi, k \in Z$$ **1A**

**Question 7**

**a.** $f(x+h) \approx hf'(x) + f(x)$

$f(x) = 2(x-1)^{\frac{1}{3}}$

Let $x = 28$, $f(28) = 6$ **1A**

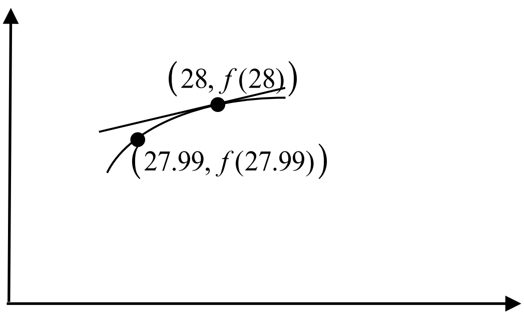
$f'(x) = \frac{2}{3}(x-1)^{-\frac{2}{3}}$, $f'(28) = \frac{2}{27}$ **1A**

$h = -0.01$

$f(27.99) \approx -\frac{1}{100} \times \frac{2}{27} + 6 = 5\frac{1349}{1350}$ **1A**

**b.** It will be an overestimate. **1A**

The tangent to the graph of $f$ at $x = 28$ will be above the graph of $f$ at $x = 27.99$. **1A**



**Question 8**

$$f(x)=\begin{cases} -(x-1)(x-2) & 1\le x\le 2 \\ \frac{1}{2}x-1 & 2<x\le a \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_1^2 \left(-(x-1)(x-2)\right)dx$$

$$=\int_1^2 \left(-x^2+3x-2\right)dx$$

$$=\left[-\frac{x^3}{3}+\frac{3x^2}{2}-2x\right]_1^2$$ **1M**

$$=-\frac{8}{3}+6-4+\frac{1}{3}-\frac{3}{2}+2$$

$$=-\frac{7}{3}-\frac{3}{2}+4$$

$$=-\frac{23}{6}+4$$

$$=\frac{1}{6}$$ **1A**

Solve $\int_2^a \left(\frac{1}{2}x-1\right)dx=\frac{5}{6}$ for $a$ **1M**

$$\left[\frac{x^2}{4}-x\right]_2^a=\frac{5}{6}$$

$$\frac{a^2}{4}-a-1+2=\frac{5}{6}$$ **1M**

$$3a^2-12a+2=0$$

$$a=\frac{12\pm\sqrt{120}}{6}$$

$$a=\frac{6+\sqrt{30}}{3},\ a>2$$ **1A**

**Note: some students might solve for *a* using the following equation**

$$\int_1^2 \left(-(x-1)(x-2)\right)dx+\int_2^a \left(\frac{1}{2}x-1\right)dx=1$$

**Question 9**

**a.** $\left(\frac{1}{5}\right)^3 = \frac{1}{125}$ **1A**

**b.** ${}^5C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3$ **1M**

$= 10 \times \frac{1}{25} \times \frac{64}{125}$

$= \frac{128}{625}$ **1A**

**c.** $X \sim \text{Bi}\left(22, \frac{1}{5}\right)$

$\mu = np = \frac{22}{5},\ \sigma = \sqrt{npq} = \sqrt{22 \times \frac{1}{5} \times \frac{4}{5}} = \frac{2\sqrt{22}}{5}$ **1M**

$(\mu - 2\sigma, \mu + 2\sigma)$

$= \left(\frac{22}{5} - \frac{4\sqrt{22}}{5}, \frac{22}{5} + \frac{4\sqrt{22}}{5}\right)$ **1A**

We are approximately 95% certain that Max will get between and including 1 and 8 correct. The data is skewed. So it is only an approximation. **1A**

**d.** $1 - \Pr(X = 0) > \frac{369}{625}$ **1M**

${}^nC_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^n < \frac{256}{625}$

$\left(\frac{4}{5}\right)^n < \left(\frac{4}{5}\right)^4$ **1M**

$n > 4$

$n = 5$ **1A**