The Mathematical Association of Victoria
MATHEMATICAL METHODS (CAS)
SOLUTIONS: Trial Exam 2015

Written Examination 2

SECTION 1

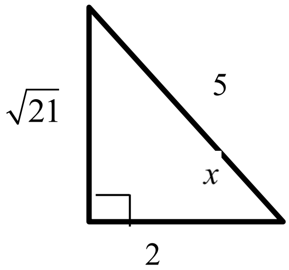
1. A 2. B 3. D 4. C 5. D 6. B 7. B 8. E 9. D 10. A 11. C

12. E 13. C 14. A 15. E 16. B 17. C 18. E 19. D 20. D 21. B 22. A

MULTIPLE CHOICE - WORKED SOLUTIONS

Question 1

x is in the fourth quadrant



$\tan(x) = -\frac{\sqrt{21}}{2}$ **A**

Question 2

$f : R \to R, f(x) = -3\sin\left(\frac{x}{2}+1\right)+1$

The amplitude is $|-3| = 3$

The period = $\frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$ **B**

Question 3

$g(x) = \tan(3x-1)+\frac{\pi}{2}$

The period = $\frac{\pi}{3}$

Let $3x-1 = \frac{\pi}{2}$

$x = \frac{\pi}{6}+\frac{1}{3}$

General solution

$x = \frac{\pi}{6}+\frac{1}{3}+\frac{\pi}{3}k,\ k \in Z$

$x = \frac{(2k+1)\pi+2}{6}, k \in Z$ **D**

Question 4

$A\cos^2(x) - B\cos(x) = 0,\ x \in [0, 2\pi]$

$\cos(x)\left(A\cos(x) - B\right) = 0$

Solve $\cos(x) = 0$

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ 2 solutions

Solve $A\cos(x) - B = 0$

$\cos(x) = \frac{B}{A}$

$-1 \le \frac{B}{A} \le 1$

To get no solutions $|B| > |A|$ **C**

Question 5

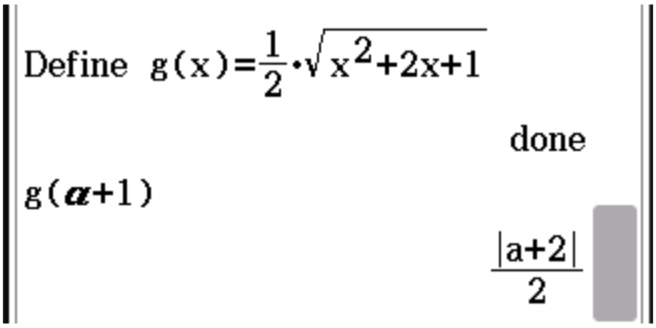
$g(x) = \frac{1}{2}\sqrt{x^2 + 2x + 1}$

$= \frac{1}{2}\sqrt{\left(x+1\right)^2}$

$= \frac{1}{2}|x+1|$

then $g(a+1) = \frac{1}{2}|a+1+1| = \frac{1}{2}|a+2|$,

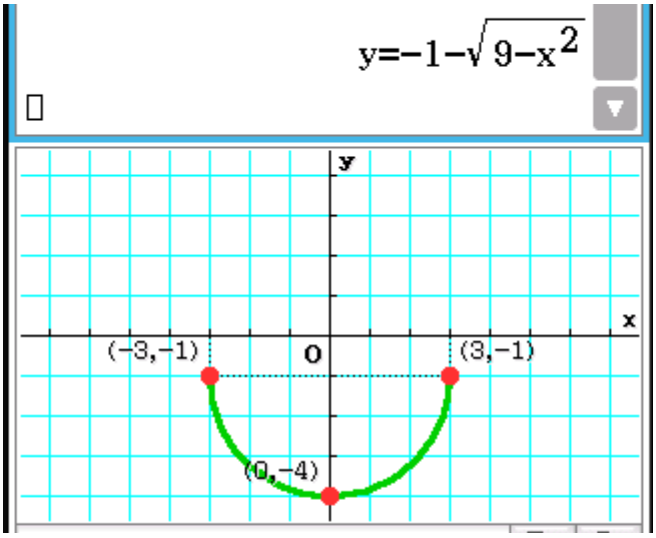
where $a > 2$, $g(a+1) = \frac{1}{2}\left(a+2\right)$ **D**



Question 6

A possible equation for the graph shown is $y=-\sqrt{9-x^2}-1$ **B**

It is a semicircle with radius 3 and centre $(0,-1)$

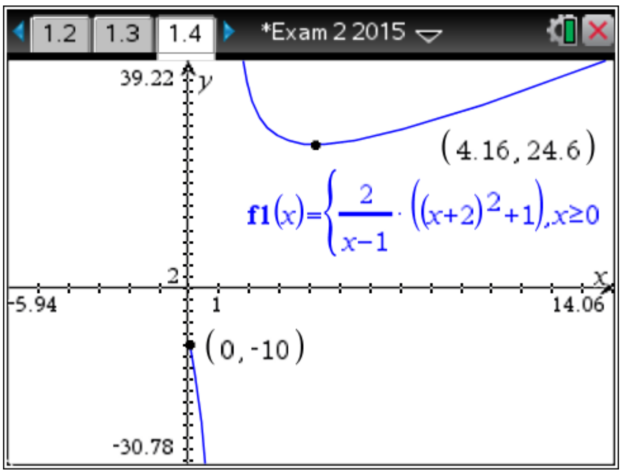


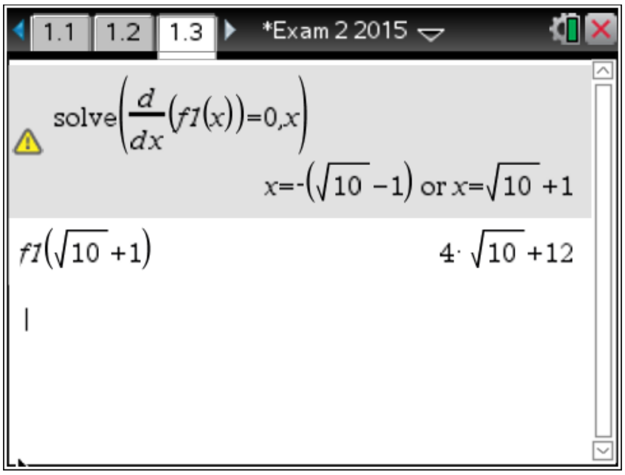
Question 7

$g: R\backslash\{1\} \to R, g(x)=\dfrac{2}{x-1}$ and $f:[0,\infty) \to R, f(x)=(x+2)^2+1$

The domain of $f \times g$ is $[0,1)\cup(1,\infty)$.

The range is $(-\infty,-10]\cup\left[12+4\sqrt{10},\infty\right)$ **B**

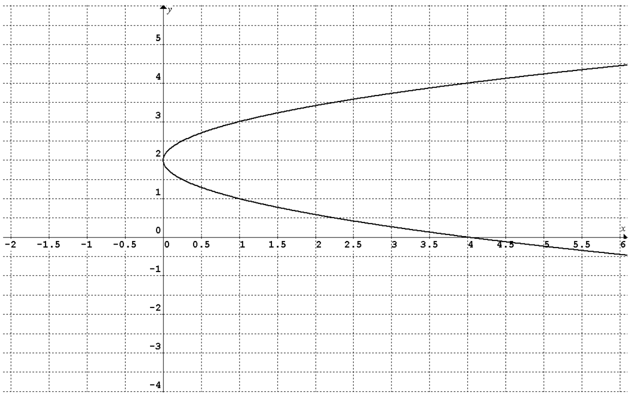




Question 8

This is the graph of a one to many relation (horizontal:vertical line test).

It is not a function as a vertical line hits the graph more than once for $x > 0$. **E**



Options A – D are correct

Question 9

The image of the function $g(x) = 2x^4$ is $y = -\frac{2}{3}\left(\frac{x}{3}+1\right)^4$.

The transformations that have been applied are:

<u>reflection in the *x*-axis</u>

gives $y_1 = -2x^4$

<u>then a dilation from the *y*-axis by a factor of 3</u>

gives $y_2 = -2\left(\frac{x}{3}\right)^4$

<u>then a translation in the negative direction of the *x* axis by 3</u>

gives $y_3 = -2\left(\frac{x}{3}+1\right)^4$

<u>followed by a dilation from the *x*-axis by a factor of $\frac{1}{3}$</u>

gives $y_4 = -\frac{2}{3}\left(\frac{x}{3}+1\right)^4$ **D**

Question 10

$f:(-\infty,-2)\to R, f(x)=\dfrac{-3}{x+2}$

The inverse is $f^{-1}(x)=-\dfrac{3}{x}-2$

```
define f(x)=-3/(x+2)
                              done
solve(f(y)=x, y)
                      {y=-3/x-2}
```

Domain f^{-1} = range $f=(0,\infty)$

$f^{-1}:(0,\infty)\to R, f^{-1}(x)=-\dfrac{3}{x}-2$ **A**

Question 11

The graph with equation $y=(x+1)^2$ is transformed to its image equation $y=5(x-3)^2+2$.

Option A is incorrect as the translation is incorrect.

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}$$

Option B is incorrect as the dilation of a factor of 5 from the y-axis should be a factor of 5 from the x-axis.

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}5&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}$$

Test Option C

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}$$

Gives $\begin{aligned}-x+2=x_1&\Rightarrow x=2-x_1\\5y+2=y_1&\Rightarrow y=\dfrac{y_1-2}{5}\end{aligned}$

In equation $y=(x+1)^2$

We get $\dfrac{y_1-2}{5}=(2-x_1+1)^2$

$y_1=5(3-x_1)^2+2$

Giving $y=5(x-3)^2+2$ **C**

Question 12

$f(x) = -\frac{3}{x} \Rightarrow f'(x) = \frac{3}{x^2}$

$\therefore f'(3) = \frac{1}{3}$ **E**

$\text{diff}\left(\frac{-3}{x}, x, 1, 3\right)$

$\frac{1}{3}$

Question 13

$h(x) = x^2 + 2x + 1$

$\therefore h'(x) = 2x + 2$

For $\{x : h'(x) = 1\}$

Solve $2x + 2 = 1$

Gives $x = -\frac{1}{2}$ **C**

Question 14

$$f(x) = \begin{cases} x+1, x \in [2, \infty) \\ \sqrt{2x+1}, x \in (0, 2) \\ x^{\frac{2}{3}}, x \in (-\infty, 0] \end{cases}$$

$x = \frac{1}{2}$. Hence differentiate the middle rule of the function.

$\frac{d}{dx}\left(\sqrt{2x+1}\right) = \frac{1}{\sqrt{2x+1}}$

$f'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$ **A**

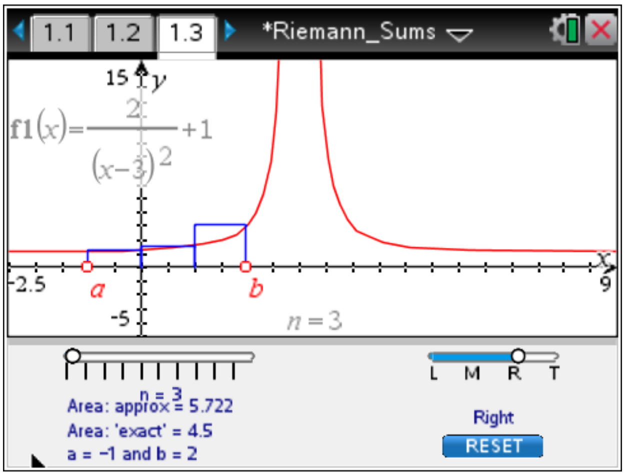
Question 15

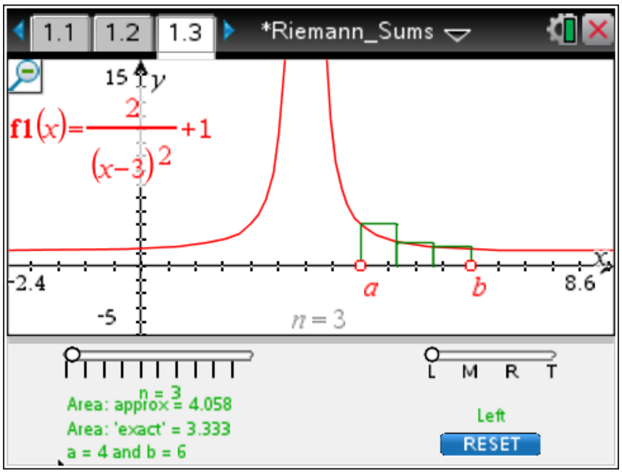
Area = $f(0)+f(1)+f(2)$

By symmetry this is the same as

$f(4)+f(5)+f(6)$

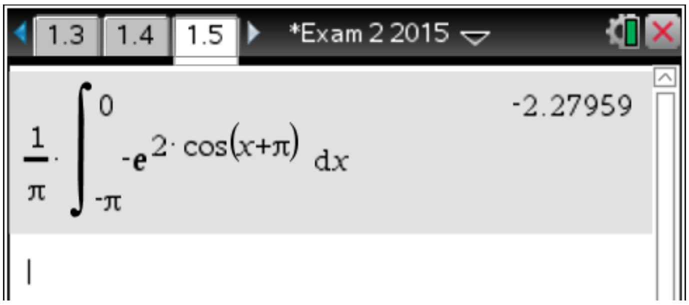
as the asymptote is at $x=3$ **E**





Question 16

The average value = $\frac{1}{\pi}\int_{-\pi}^{0}\left(-e^{2\cos(x+\pi)}\right)dx=-2.3$ correct to one decimal place **B**



Question 17

$$\int_{-1}^{3}h(x)dx=5$$

$$\int_{3}^{-1}\left(1+5h(x)\right)dx$$

$$=-\int_{-1}^{3}\left(1+5h(x)\right)dx$$

$$=-\int_{-1}^{3}\left(1\right)dx-\int_{-1}^{3}\left(5h(x)\right)dx$$

$$=-\int_{-1}^{3}\left(1\right)dx-5\int_{-1}^{3}h(x)dx$$

$$=-\left[x\right]_{-1}^{3}-5\times5$$

$$=-\left(3+1\right)-25$$

$$=-4-25=-29$$ **C**

Question 18

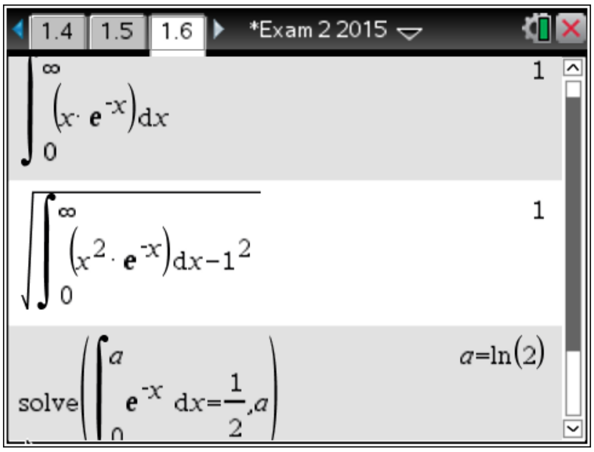
$$E(X)=\int_0^{\infty}\left(x\times e^{-x}\right)dx=1$$

$$SD(X)=\sqrt{\int_0^{\infty}\left(x^2\times e^{-x}\right)dx-\left(\int_0^{\infty}\left(x\times e^{-x}\right)dx\right)^2}=1$$

Let median = a

Solve $\int_0^{a}\left(e^{-x}\right)dx=\frac{1}{2}$ for a

$a=\log_e(2)$ **E**



Question 19

$0.1+a+b+c=1$

$a+b+c=0.9$ B True

Using $E(X)$

$0.1+3a+5b+7c=4$

$3a+5b+7c=3.9$ C True

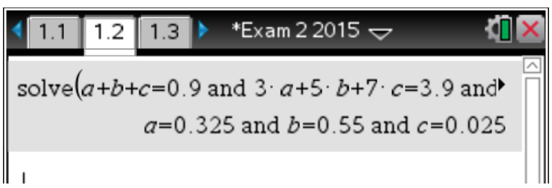
Using $\text{Var}(X)$

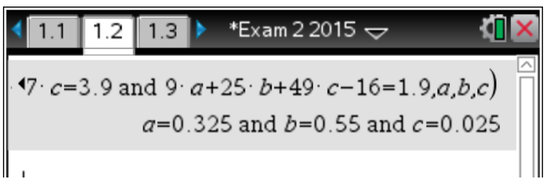
$0.1+9a+25b+49c-4^2=2$

Hence $9a+25b+49c=1.9$ D false **D**

Solving the three equations shows A is correct

0.1 + 0.325 = 0.425 The median is 5. E is correct.

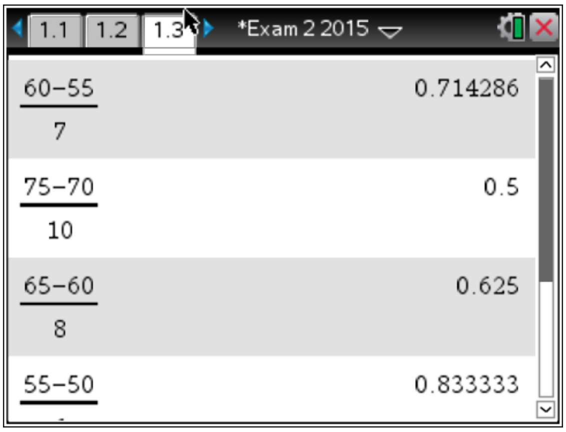


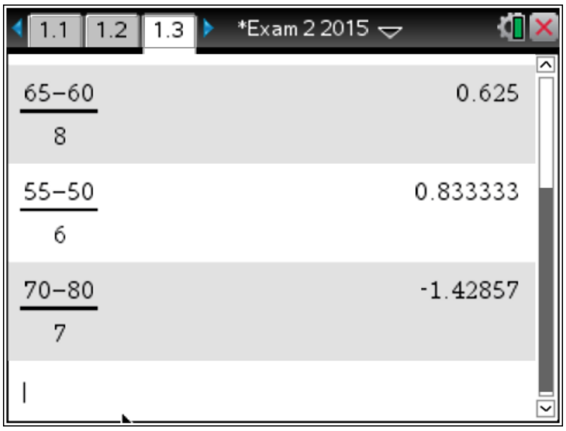


Question 20

Using z scores, $z = \dfrac{x - \mu}{\sigma}$

Physics is the highest. **D**





Question 21

$\Pr\left(X < 135 \mid X > 130\right)$

$= \dfrac{\Pr\left(130 < x < 135\right)}{\Pr(x > 130)}$

$= \dfrac{\Pr\left(130 < x < 135\right)}{\frac{1}{2}}$

$= 2\Pr\left(130 < X < 135\right)$

$= 2\Pr\left(125 < X < 130\right)$ **B**

Question 22

A score of 15 can be obtained by a 5 then 10 or a 10 then 5.

Area of the inner triangle = $4\sqrt{3}$

Area of the middle section = $25\sqrt{3} - 4\sqrt{3} = 21\sqrt{3}$

Area of the board = $100\sqrt{3}$

Probability of a score of 15 = $2 \times \dfrac{4\sqrt{3}}{100\sqrt{3}} \times \dfrac{21\sqrt{3}}{100\sqrt{3}} = 2 \times \dfrac{4}{100} \times \dfrac{21}{100}$ **A**

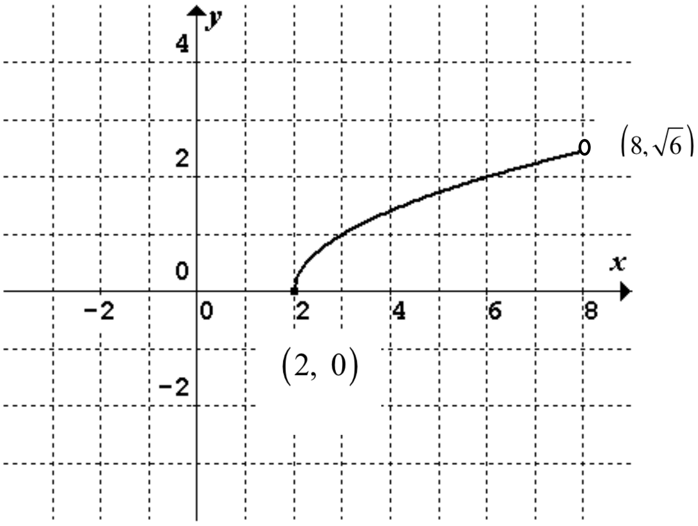
SECTION 2
EXTENDED RESPONSE QUESTIONS

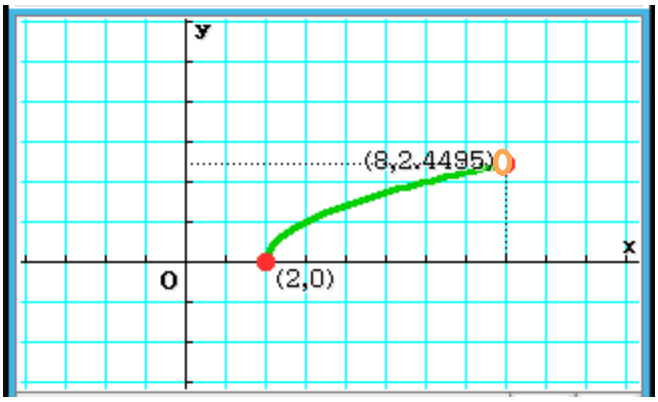
Question 1 (13 marks)

a. Shape **1A**

$(2, 0)$ closed circle **1A**

$\left(8, \sqrt{6}\right)$ open circle **1A**



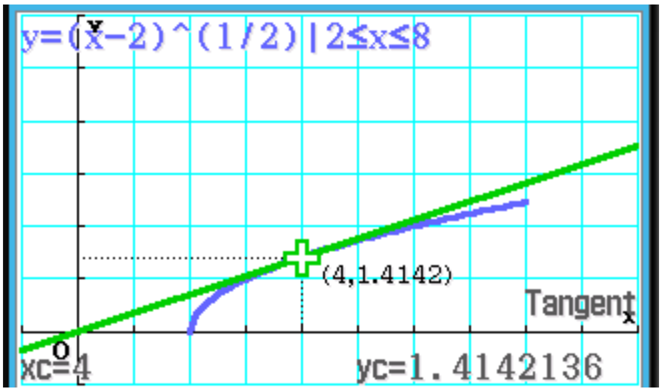


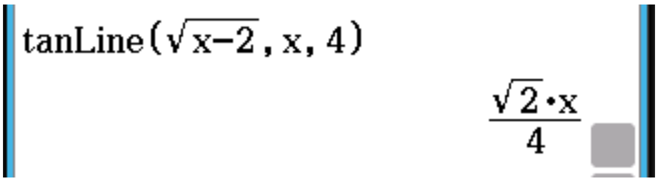
b. $f(4) = \sqrt{2}$ **1A**

c. Solve $\sqrt{x-2} = \frac{1}{2}$

$x = \frac{9}{4}$ **1A**

d. Equation of the tangent at $x = 4$ is $y = \frac{\sqrt{2}}{4}x$ **1A**





(Note that this tangent goes through the origin)

e. Area = $\int_0^2 \left(\frac{\sqrt{2}}{4}x\right)dx + \int_2^4 \left(\frac{\sqrt{2}}{4}x - \sqrt{x-2}\right)dx$ **1A**

$\int_0^2 \left(\frac{\sqrt{2}}{4}x\right)dx$ **1A**

$\int_2^4 \left(\frac{\sqrt{2}}{4}x - \sqrt{x-2}\right)dx$ **1A**

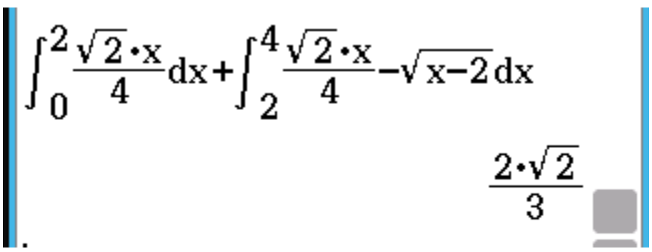
OR

Area= $\int_0^4 \left(\frac{\sqrt{2}}{4}x\right)dx - \int_2^4 \left(\sqrt{x-2}\right)dx$ **1A**

$\int_0^4 \left(\frac{\sqrt{2}}{4}x\right)dx$ **1A**

$\int_2^4 \left(\sqrt{x-2}\right)dx$ **1A**

f. Area = $\frac{2\sqrt{2}}{3}$ **1A**

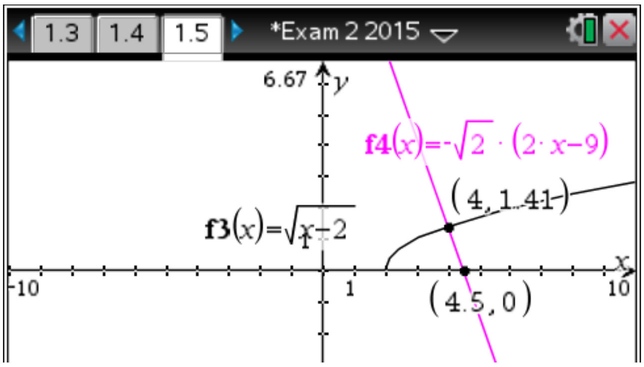


g. Equation of normal is $y = -2\sqrt{2}x + 9\sqrt{2}$. **1A**

The normal intersects the x-axis at $x = 4.5$.

Area $= \int_2^4 \left(\sqrt{x-2}\right)dx + \int_4^{4.5} \left(-2\sqrt{2}x + 9\sqrt{2}\right)dx$ **1A**

Area $= \dfrac{19\sqrt{2}}{12}$ square units **1A**



$$\text{normal}(\sqrt{x-2}, x, 4) \quad -2\cdot\sqrt{2}\cdot x + 9\cdot\sqrt{2}$$

$$\int_2^4 \sqrt{x-2}\,dx + \int_4^{4.5} -2\cdot\sqrt{2}\cdot x + 9\cdot\sqrt{2}\,dx \quad \frac{19\cdot\sqrt{2}}{12}$$

Question 2 (15 marks)

a. $5 - 2x > 0, x < \dfrac{5}{2}$

$a = \dfrac{5}{2}$ **1A**

b. Solve $g(x) = -\log_e(5-2x) + 1 = 0$

$x = \dfrac{5-e}{2}$ **1A**

c. $g(0) = 1 - \log_e(5)$ **1A**

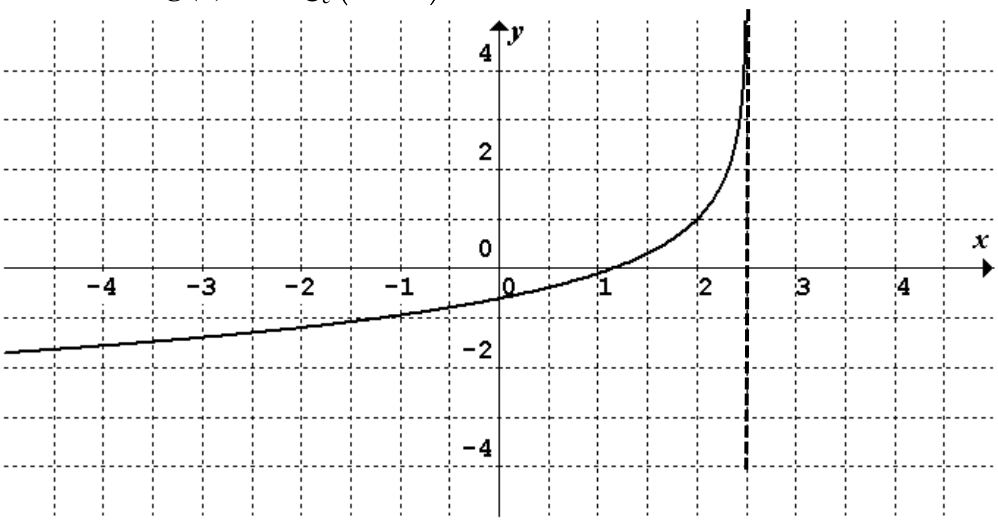
```
define g(x)=-ln(5-2x)+1
                                   done
solve(g(x)=0,x)
                          {x=-e/2+5/2}
g(0)
                              -ln(5)+1
```

These are now the axial intercepts.

d. Shape **1A**
Asymptote **1A**

Axial Intercepts $\left(0, 1-\log_e(5)\right)$, $\left(\frac{5-e}{2}, 0\right)$ **1A**

Graph of $g(x) = -\log_e\left(5-2x\right)+1$



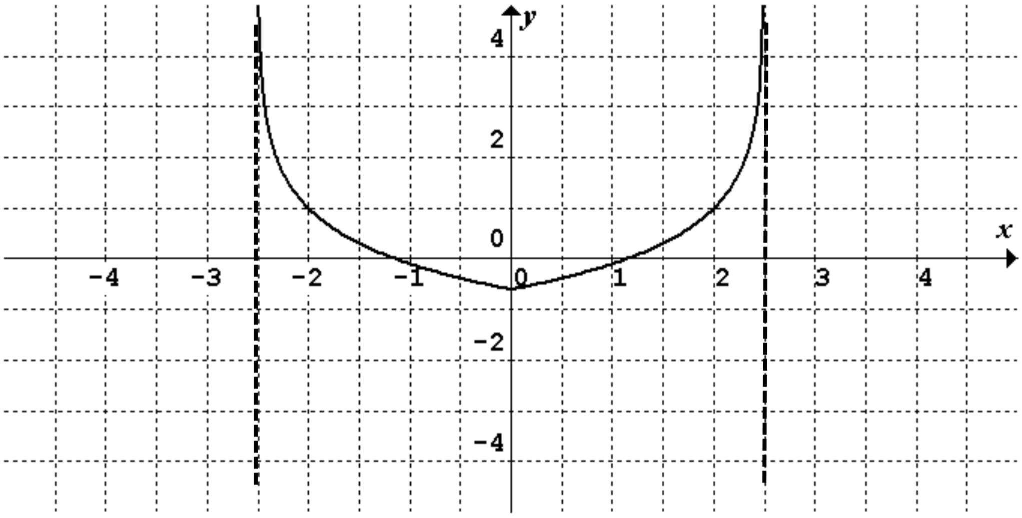
$x = 2.5$

e. Shape (cusp) **1A**

Asymptotes $x = \pm\frac{5}{2}$ **1A**

Axial intercepts $\left(0, 1-\log_e(5)\right)$ cusp, $\left(\frac{5-e}{2}, 0\right)$, $\left(-\frac{5-e}{2}, 0\right)$ **1A**

Graph of $g\left(|x|\right) = 1-\log_e\left(5-2|x|\right)$



$x = -2.5$ $x = 2.5$

f. differentiable for domain $x \in \left(-\frac{5}{2}, 0\right) \cup \left(0, \frac{5}{2}\right)$ **1A**

g. For $x \in \left(0, \frac{5}{2}\right)$, $g(|x|) = 1 - \log_e(5 - 2x)$

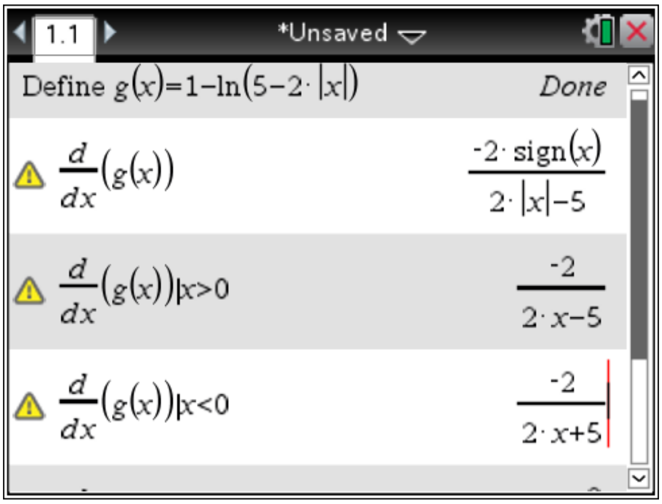
$g'(|x|) = \frac{2}{5 - 2x}$ **1A**

For $x \in \left(-\frac{5}{2}, 0\right)$, $g(|x|) = 1 - \log_e(5 + 2x)$

$g'(|x|) = \frac{-2}{5 + 2x}$ **1A**

Correct domain **1A**

$$\therefore g'(|x|) = \begin{cases} \frac{-2}{5+2x}, x \in \left(-\frac{5}{2}, 0\right) \\ \frac{2}{5-2x}, x \in \left(0, \frac{5}{2}\right) \end{cases}$$



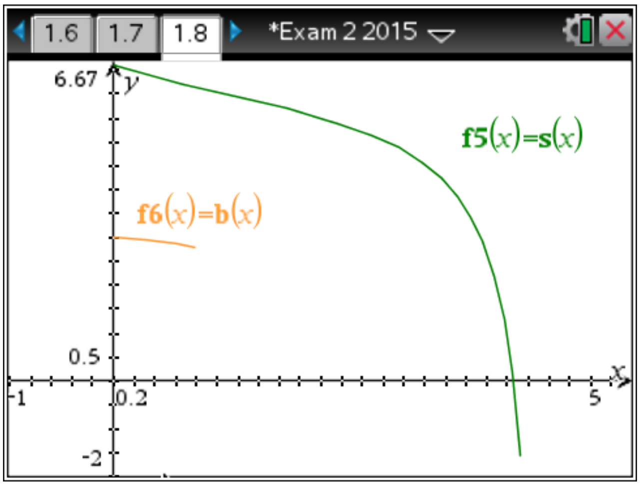
h. 3 correct transformations in a correct order **1A**

The remaining correct and in a correct order **1A**

There are other possibilities.

- Translation of 1 unit in the negative direction of the y-axis
- Translation of 2.5 units in the negative direction of the x-axis
- Dilation by a factor of 2 units from the y-axis
- Reflection over the y-axis
- Reflection over the x-axis

Question 3



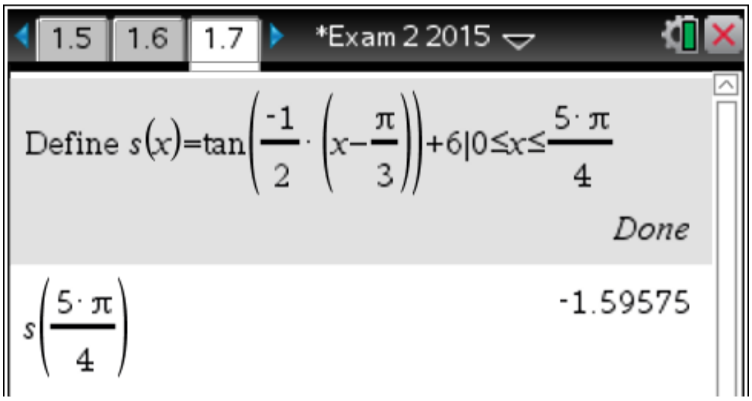
a. This is at the point of inflection.

$x = \frac{\pi}{3}$ **1A**

b. $x = \frac{5\pi}{4}$

$S\left(\frac{5\pi}{4}\right) = -160$ cm to the nearest cm **1A**

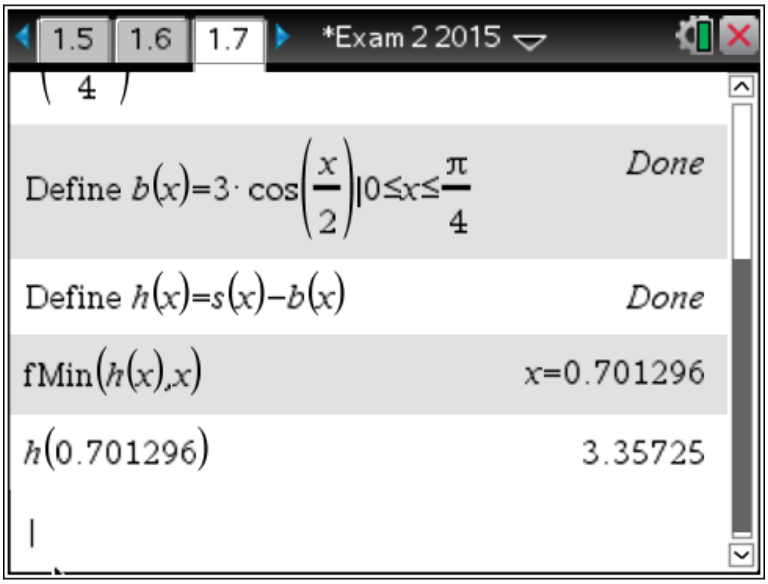
S_1 is 160 cm below the water **1A**



c. $h:\left[0,\frac{\pi}{4}\right] \to R, h(x) = \tan\left(-\frac{1}{2}\left(x-\frac{\pi}{3}\right)\right)+6-3\cos\left(\frac{x}{2}\right)$ **1A** Rule **1A** Domain

d. The minimum value occurs when $x = 0.701\ldots$ **1M**

$h(x) = 336$ cm **1A**



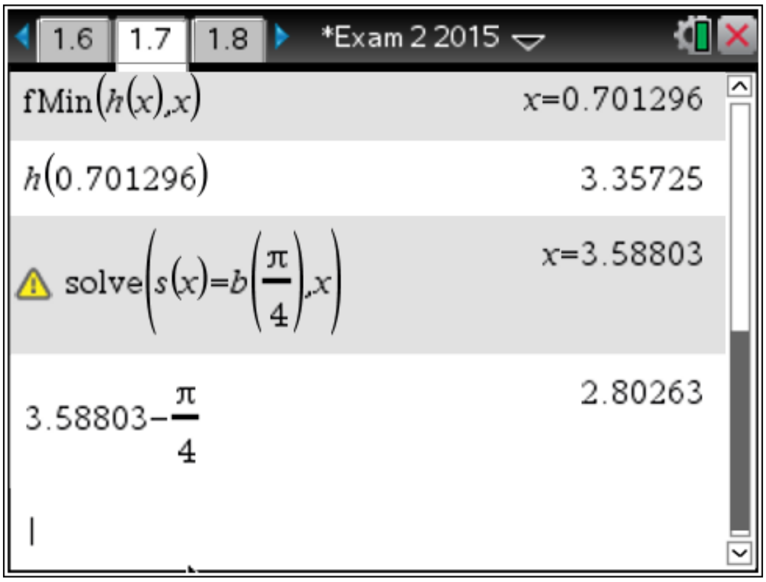
e. $x = \frac{\pi}{4}$ when B_1 ends

$B\left(\frac{\pi}{4}\right) = 2.7716...$

Solve $S(x) = 2.7716...$ for x

$x = 3.588...$ **1A**

Horizontal distance is $3.588... - \frac{\pi}{4} = 280$ cm to the nearest cm **1A**



f. Period $= \frac{2\pi}{\left(\frac{2\pi}{3}\right)} = 6$ **1A**

$\frac{300}{6} = 50$ waves **1A**

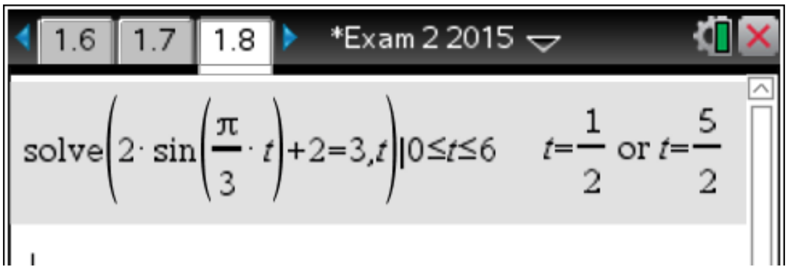
g. $w(t) > 3$ for the slide to be completely under water. **1M**

Solve $w(t) = 3$ for t

$t = \frac{1}{2}, \frac{5}{2}$

B_1 will be under water for 2 seconds per cycle (6 seconds)

$\frac{1}{3}$ of the time **1A**



Question 4 (17 marks)

Let P = pass and F = fail

a. $\Pr(PPP) = 0.3 \times 0.6 \times 0.6$

$= 0.108$ or $\frac{27}{250}$ **1A**

b. $\Pr(PFF) + \Pr(FPF) + \Pr(FFP)$ **1M**

$= 0.3 \times 0.4 \times 0.7 + 0.7 \times 0.3 \times 0.4 + 0.7 \times 0.7 \times 0.3$

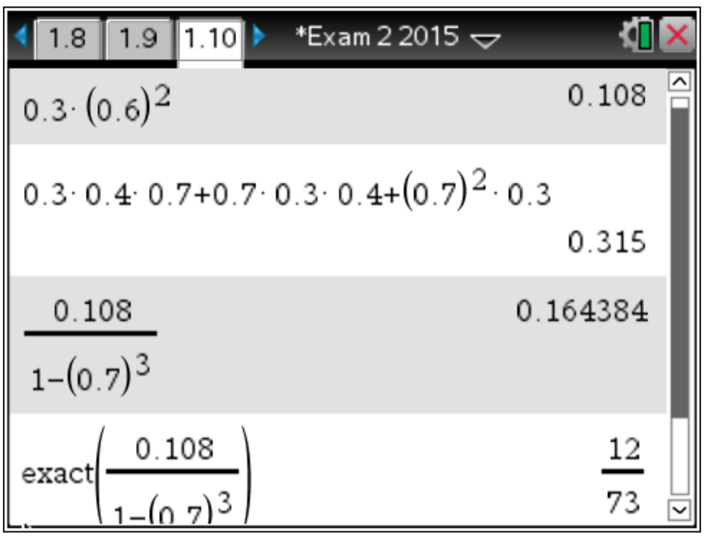
$= 0.315$ **1A**

c. $\Pr(P = 3 \mid P \geq 1) = \frac{\Pr(P = 3 \cap P \geq 1)}{\Pr(P \geq 1)}$ **1M**

$= \frac{\Pr(P = 3)}{1 - \Pr(F = 3)}$

$= \frac{0.3 \times 0.6 \times 0.6}{1 - 0.7^3}$

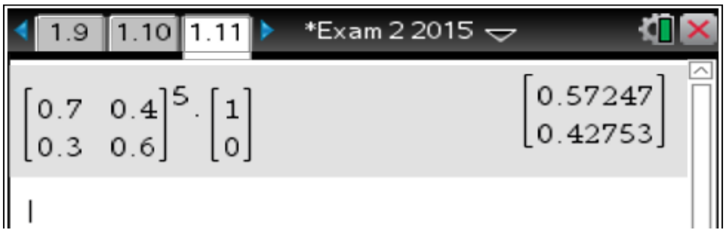
$= \frac{12}{73}$ **1A**



d. $\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}^5 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **1M**

$= \begin{bmatrix} 0.572 \\ 0.428 \end{bmatrix}$

The probability he will pass the 6th test is 0.428 correct to 3 decimal places **1A**



e. Steady states $\frac{0.3}{0.3+0.4} = \frac{3}{7}$ fail, $\frac{0.4}{0.3+0.4} = \frac{4}{7}$ pass

Jeremy is more likely to fail a test in the long term, **1A**

as the probability of Jeremy passing in the long term is $\frac{3}{7}$ and failing is $\frac{4}{7}$. **1A**

f. $X : N(\mu, \sigma^2)$

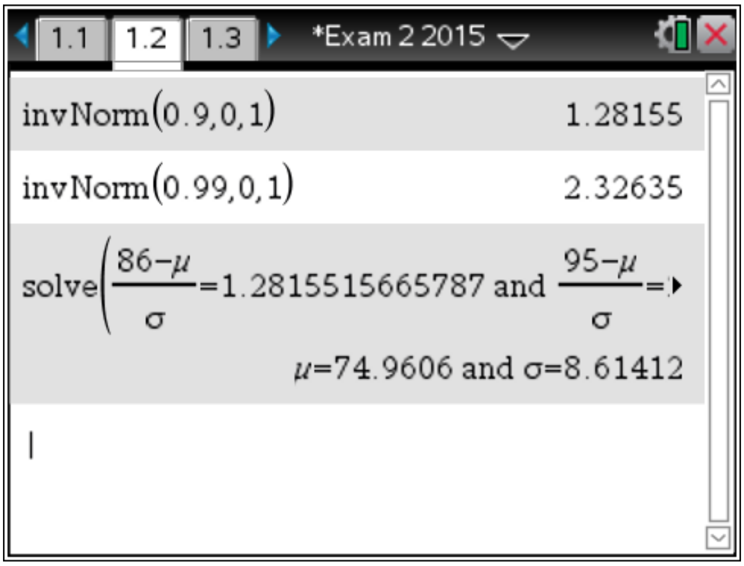
$\Pr(X > 86) = 0.1, \ \Pr(X > 95) = 0.01$

$\frac{86-\mu}{\sigma} = 1.281...$ (1) **1A**

$\frac{95-\mu}{\sigma} = 2.326...$ (2) **1A**

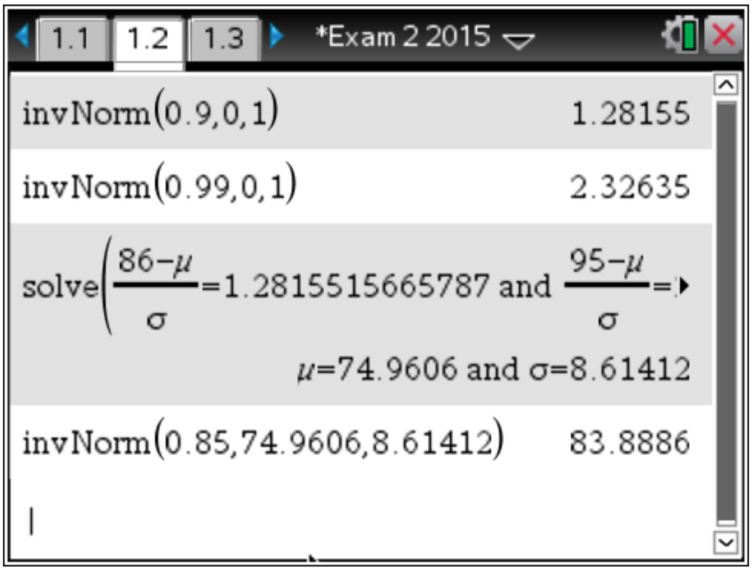
$\mu = 74.96$ **1A**

$\sigma = 8.61$ **1A**



g. $\Pr(X < a) = 0.85$

$a = 83.89$ **1A**



h. $p^3 + p^2 + p = 1$ **1A**

$p = 0.543...$

$\Pr(S \cap R) = \Pr(S) \times \Pr(R)$ **1M**

Let $a = \Pr(S \cap R)$

Solve $a = (a + p) \times p^3$ for a

$a = 0.10$ **1A**