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YEAR 12 *Trial Exam Paper*

2015

MATHEMATICAL METHODS (CAS)

Written examination 2

STUDENT NAME:

Reading time: 15 minutes

Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
			Total 80

- Students are permitted to bring the following items into the examination: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer-based CAS, their full functionality may be used.
- Students are NOT permitted to bring the following items into the examination: blank sheets of paper and/or white out liquid/tape.

Materials provided

- The question and answer book of 31 pages, with a separate sheet of miscellaneous formulas.
- An answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the box provided and on the answer sheet for multiple-choice questions.
- Remove the formula sheet during reading time.
- You must answer the questions in English.

At the end of the exam

- Place the answer sheet for multiple-choice questions inside the front cover of this question book.

Students are NOT permitted to bring mobile phones or any other electronic devices into the examination.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the multiple-choice answer sheet.
Select the response that is **correct** for the question.
A correct answer scores 1 mark, whereas an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
If more than one answer is selected, no marks will be awarded.

Question 1

The maximal domain of the function with the rule $y = 1 + \sqrt{a - x}$, $a > 0$ is

A. R^+

B. $(-\infty, 1)$

C. $(-\infty, 1]$

D. $(-\infty, a)$

E. $(-\infty, a]$

Question 2

For the function $f(x) = ax^3 + bx^2 + cx + d$,

$$f(1) = -2$$
$$f(-2) = 10$$
$$f'(1) = -3$$
and $f'(3) = -36$.

A matrix equation $AX = B$ is to be used to find the values of *a*, *b*, *c* and *d* where

$$X = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ 10 \\ -3 \\ -36 \end{bmatrix}.$$

The matrix *A* is

A. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -8 & 4 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -8 & 4 & -2 & 1 \\ 3 & 2 & 1 & 1 \\ 81 & 6 & 1 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -8 & 4 & -2 & 1 \\ 3 & 2 & 1 & 0 \\ 27 & 6 & 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 1 & -2 & 1 & 3 \\ -2 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix}$

Question 3

Let $f: R \to R,\ f(x) = x(x+a)^2$, for $a < 0$.

It is true to say that the graph of $y = f(x)$ has

A. $f'(x) > 0$, for $x > 0$.

B. a local minimum located at a point where $x < 0$.

C. a local maximum located at a point where $x < 0$.

D. a stationary point of inflection.

E. $f'(x) > 0$, for $x < 0$.

Question 4

Let $f: R \to R,\ f(x) = \left|x^2 + ax\right|$ for $a > 0$.

The gradient function f' is positive for

A. $x \in R^+$

B. $x \in \left(-\dfrac{a}{2}, \infty\right)$

C. $x \in [-a, \infty)$

D. $x \in \left(-a, -\dfrac{a}{2}\right) \cup (0, \infty)$

E. $x \in \left[-a, -\dfrac{a}{2}\right] \cup [0, \infty)$

Question 5

If $\int_2^4 f(x)\,dx = 3$, then $\int_2^4 \left(2+5f(x)\right)\,dx$ is equal to

A. 17

B. 19

C. 27

D. 5

E. 25

Question 6

If a random variable, *X*, has the probability density function given by

$$f(x) = \begin{cases} \dfrac{2x^2}{k} & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

then the value of *k* is

A. $\frac{2}{3}$

B. $\frac{1}{3}$

C. 1

D. $\frac{1}{2}$

E. 3

The information below relates to Questions 7 and 8.

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases}\frac{-3}{4}(x-2)(x-4) & \text{when } 2\le x\le 4\\ 0 & \text{elsewhere}\end{cases}$$

Question 7

The mode of X is

A. 1

B. 0.75

C. 3

D. 1.5

E. 2

Question 8

$\Pr(X<2.5 \mid X<3)$, correct to 4 decimal places, is equal to

A. 0.1563

B. 0.0781

C. 0.3125

D. 0.1582

E. 0.3164

Question 9

A transformation, $T: R^2 \to R^2$, is defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \pi \\ 2 \end{bmatrix}$.

The equation of the image of the curve $y = \sin(3x)$ under this transformation is

A. $y = 2\sin\big(-(x+\pi)\big) - 2$

B. $y = 2\sin\big(-(x+\pi)\big) + 2$

C. $y = \frac{1}{2}\sin(\pi - x) - 2$

D. $y = \frac{1}{2}\sin\big(6(x-\pi)\big) + 2$

E. $y = 2\sin\big(-(x+3\pi)\big) + 2$

Question 10

At the point (2, 7), the graph of the curve $y = f(x)$ has a tangent with equation $y = 5x - 3$.
At the point (0, 8), the graph of the curve $y = 1 + f(x+2)$ has a tangent with equation

A. $y = 3x - 4$

B. $y = 5x - 12$

C. $y = 5x - 14$

D. $y = 5x + 8$

E. $y = 5x - 13$

Question 11

If $g(x)=\dfrac{f(x)}{e^x}$, then $g'(x)$ is equal to

A. $\dfrac{f'(x)}{e^x}$

B. $\dfrac{f'(x)}{e^{2x}}$

C. $\dfrac{f'(x)-f(x)}{e^{2x}}$

D. $\dfrac{f(x)-f'(x)}{e^x}$

E. $\dfrac{f'(x)-f(x)}{e^x}$

Question 12

A die has been altered so that the probability of throwing a 6 is 0.3.
Ellen rolls the die ten times. The probability that she obtains a 6 more than twice is closest to

A. 0.2668

B. 0.2335

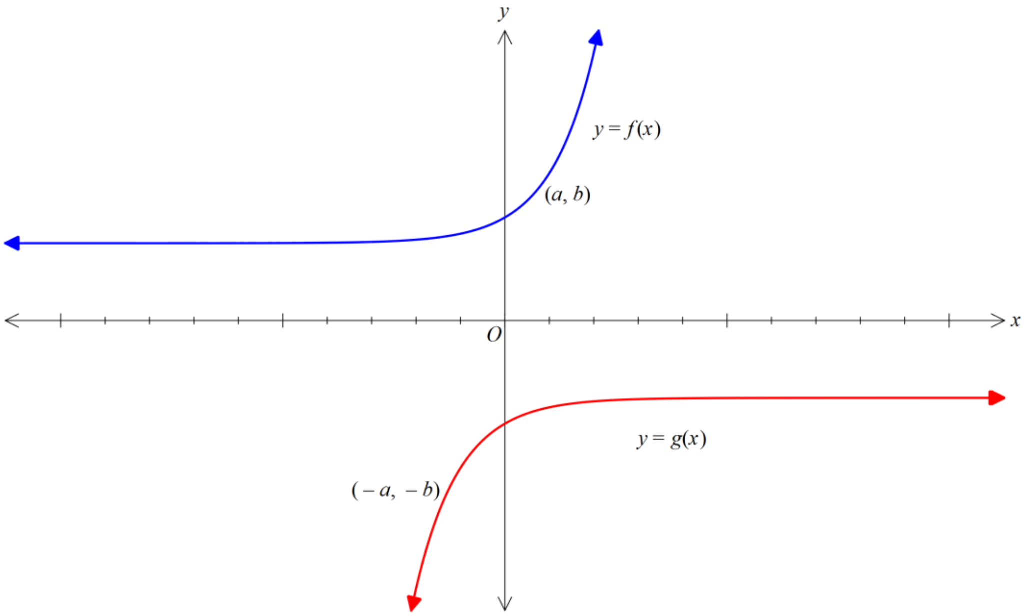
C. 0.3828

D. 0.8507

E. 0.6172

Question 13

The diagram below shows the graphs of $y = f(x)$ and $y = g(x)$. The function f has undergone a transformation to become the function g.



A transformation, $T : R^2 \to R^2$, that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$ is given by

A. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

B. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

C. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

D. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

E. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Question 14

Let $f(x) = x^2 + 3$ and $g(x) = \log_e(x)$.

Both functions have a maximal domain.

The range of the function $g(f(x))$ is

A. $(0, \infty)$

B. $[0, \infty)$

C. $(\log_e(3), \infty)$

D. $[\log_e(3), \infty)$

E. $R \setminus \{\pm 1\}$

Question 15

For the function $f(x) = |2x - 1| - 2,$ the rate of change of *y* with respect to *x* is

A. not defined for $x = 0$

B. zero at $x = \frac{3}{2}$

C. never equal to zero

D. -2 for $x \in (-\infty, \frac{1}{2}]$

E. -2 for $x \in (-\infty, -\frac{1}{2}) \cup (\frac{3}{2}, \infty)$

Question 16

Let $f:[0,a]\to R,\ f(x)=\cos(3x-\pi)$.

If the inverse function f^{-1} exists, then the largest value that a can take is

A. $\frac{\pi}{4}$

B. 1

C. $\frac{\pi}{2}$

D. $\frac{\pi}{3}$

E. $\frac{\pi}{6}$

Question 17

An approximation for $4.2^{\frac{3}{2}}$ can be found using the linear approximation $f(4+h)\approx f(4)+hf'(4)$ and by considering the point (4, 8), which lies on the graph of $f(x)=x^{\frac{3}{2}}$.

The approximation is equal to

A. 8.4

B. 8.9

C. 8.6

D. 4.6

E. 8.2

Question 18

Let $f:(-2\pi, 2\pi) \to R,\ f(x) = \left|\tan\left(\frac{x}{4}\right)\right|$.

The number of solutions to the equation $f(x) = 1$ is/are

A. 0

B. 1

C. 2

D. 3

E. 4

Question 19

The area enclosed by the graph of $y = 1 - e^{(x+3)}$, the *x*-axis and the *y*-axis is given by

A. $\int_{-3}^{0} (e^{(x+3)} - 1)\, dx$

B. $\int_{-3}^{0} (1 - e^{(x+3)})\, dx$

C. $\int_{-3}^{0} (e^{(x+3)} + 1)\, dx$

D. $\int_{-3}^{e^3-1} (e^{(x+3)} - 1)\, dx$

E. $\int_{-3}^{e^3-1} (1 - e^{(x+3)})\, dx$

SECTION 1 – continued

Question 20

The area enclosed by the graph $y = \log_e(x+1)$, the x-axis and the line $x = 4$ is approximated using the method of right rectangles with rectangles of width 1 unit.

The approximation, in square units, is

A. $\log_e(6)$

B. $\log_e(2)$

C. $\log_e(36)$

D. $\log_e(120)$

E. $\log_e(24)$

Question 21

The velocity, in metres per second, of a car moving in a straight line is given by

$v(t) = 4.5 - \frac{1}{2}t^2, t \geq 0.$

The distance, in metres, travelled in the first 6 seconds is

A. –9

B. 27

C. 9

D. 18

E. –6

Question 22

For $f(x) = \log_e(x)$, which one of the following is **not** correct for all positive real values of x and y?

A. $f^{-1}(x-y) = \dfrac{f^{-1}(x)}{f^{-1}(y)}$

B. $f\left(\dfrac{x}{y}\right) = f(x) - f(y)$

C. $f(x^y) = yf(x)$

D. $f(e^x) = x$

E. $f^{-1}(x) = \dfrac{1}{f(x)}$

END OF SECTION 1
TURN OVER

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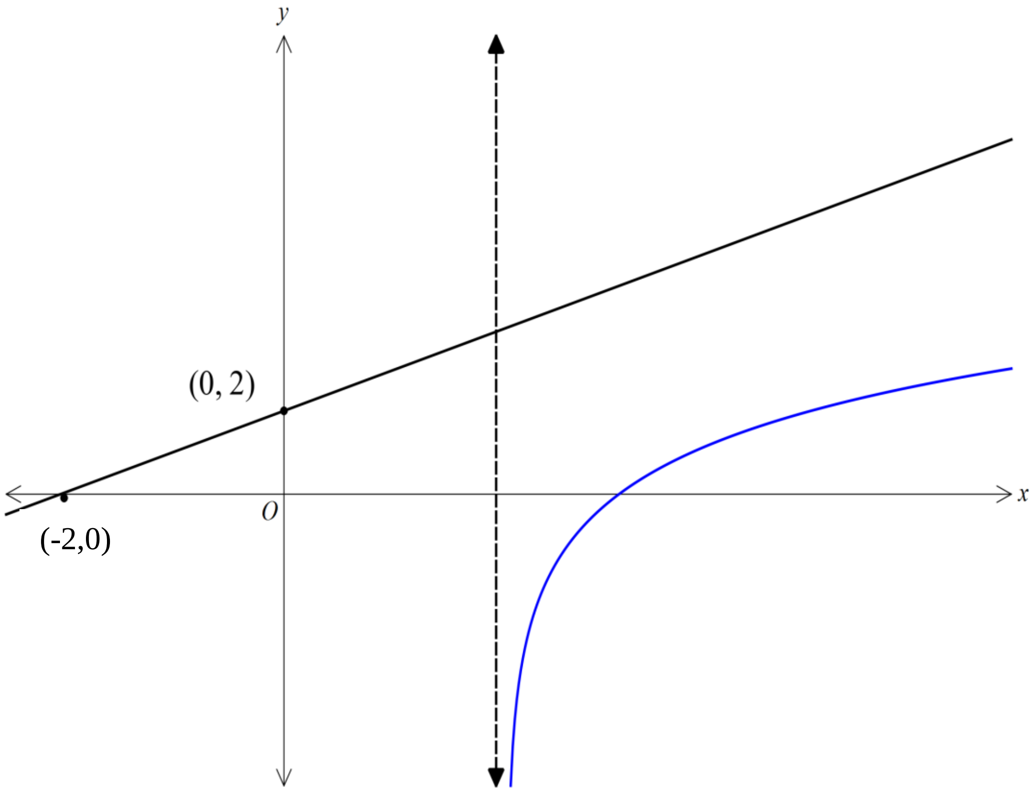
SECTION 2

Instructions for Section 2
Answer **all** questions in the spaces provided.
A decimal answer will not be accepted if an **exact** answer is required to a question.
For questions where more than one mark is available, appropriate working must be shown.
Unless otherwise stated, diagrams are not drawn to scale.

Question 1 (8 marks)

The graphs of the functions

$f:(2,\infty)\to R,\ f(x)=\frac{1}{2}\log_e(x-2)$ and $g:R\to R,\ g(x)=x+2$ are shown below.



a. Find $\left\{x:\frac{1}{2}\log_e(x-2)+x+2=0\right\}$, correct to 4 decimal places.

1 mark

b. On the axes in part **a.**, sketch the graph of $y = f(x) + g(x)$.

2 marks

Consider the function h, where $h(x) = g(x) - f(x)$.

c. State the domain of h.

1 mark

d. **i.** Find $h'(x)$.

1 mark

ii. Hence, find the value of x for which $h(x)$ is a minimum.
(There is no need to justify that the stationary point is a minimum.)

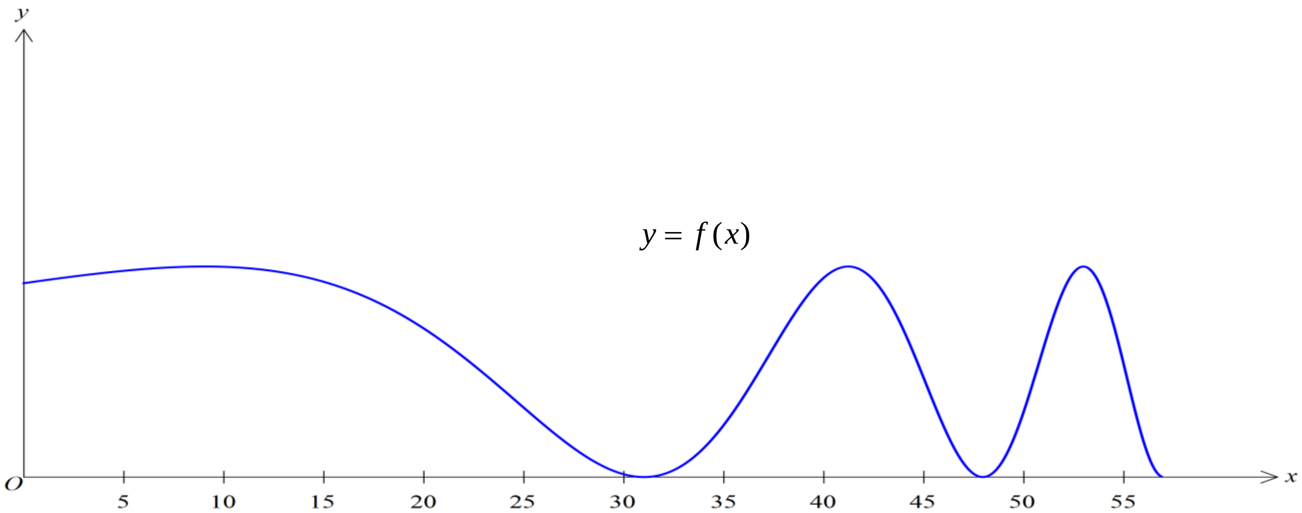
2 marks

e. Hence, find the minimum vertical distance between the graphs of $y = g(x)$ and $y = f(x)$. Express your answer as an exact value.

1 mark

Question 2 (13 marks)

An oceanologist is studying the ocean floor off the coast of Victoria. The ocean floor in this area is uneven and has deep and shallow sections. A cross-section of the sandy floor is shown on the graph below, where the height, h, in metres above the base level is given by the function $f:\left[0, 20\log_e\left(\frac{11\pi}{2}\right)\right] \to R,\quad f(x) = \sin(e^{\frac{x}{20}}) + 1.$



The x-axis represents the base level and x, in metres, represents the distance from the edge of a lighthouse.

The function f is a composite function, where $f(x) = h(g(x))$.

a. **i.** Write down the rule for $h(x)$ and for $g(x)$.

2 marks

ii. Explain whether or not the composite function $g(h(x))$ exists. Give reasons.

1 mark

SECTION 2 – Question 2 – continued

b. Find $f'(x)$.

1 mark

c. Find the coordinates of the point on the sandy floor when the sandy floor first drops to base level.

1 mark

d. **i.** Find the maximum height of the sandy floor above base level.

1 mark

ii. Find the distance(s) from the lighthouse where this maximum height occurs. Express your answer(s) as an exact value.

2 marks

For scientific purposes it is relevant to record an average sand floor height for this section of the ocean floor. This measurement represents the surface of the sand floor as if it were smoothed to a horizontal cross-section.

e. What is the average sand floor height recorded for this section of the ocean floor for $x \in \left[0, 20\log_e\left(\frac{11\pi}{2}\right)\right]$?

Express your answer in metres, correct to 2 decimal places.

2 marks

A straight metal pole has to be inserted into the ocean floor at right angles to the slope of the sand. The pole is pushed through until its end reaches the base level so that a sample of the sand floor can be recorded.

The pole is inserted at a point for which $x \in [30, 40]$ and the gradient of the pole is –10 when it has been inserted.

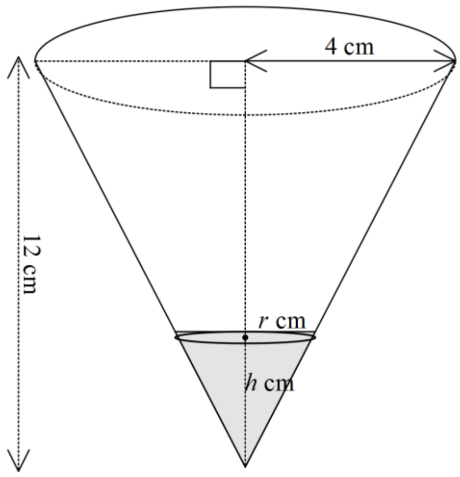
f. Find the horizontal distance, in metres of the bottom of the pole from the edge of the lighthouse. Express your answer correct to 3 decimal places.

3 marks

Question 3 (11 marks)

Chocolate mousse is manufactured in a factory production line. The liquid mousse is poured into an inverted right circular cone with base radius 4 cm and height 12 cm.

As the mousse is being poured into the container, an automated arm above the machine lowers a decorative chocolate curl towards the top of the container.



The container is initially empty. Liquid mousse is poured simultaneously as the chocolate curl is being lowered.

At time t minutes after 9 a.m., liquid mousse is poured into the container so that the height of the mousse is h cm and the radius of the surface of the mousse in the container is r cm.

a. **i.** Show that $h = 3r$.

1 mark

ii. Hence, find an expression for the volume of mousse in the container at any time t minutes, in terms of h.

1 mark

The liquid mousse is poured into the container at a rate of $\frac{4\pi}{9}$ cm^3/min.

b. Show that $\frac{dh}{dt}$, i.e. the rate at which the height of the mousse in the container is increasing, is $\frac{4}{h^2}$ cm/min.

2 marks

c. **i.** Find the rate at which the height is increasing when the height is 2 cm.

1 mark

ii. Find the height of the mousse when the rate at which the height is increasing is half of that in part **c. i.**

1 mark

d. **i.** Write an expression for $\frac{dt}{dh}$ in terms of h.

1 mark

ii. Write an equation for the height, h, in terms of time, t.

1 mark

The machine making the mousse also decorates the filling mousse container. As the mousse begins to pour into the container, an automated arm lowers a chocolate curl to be placed on the top of the filling container. The chocolate curl starts at a height of 24 cm above the top of the container and is lowered towards the top of the mousse at a rate of 1.5 cm/min.

e. **i.** Write an expression for the distance of the chocolate curl, in cm, above ground level at any time t minutes. (Note: The vertex of the cone is at ground level.)

1 mark

ii. At what time would the chocolate curl first touch the chocolate mousse? Give your answer correct to the nearest minute.

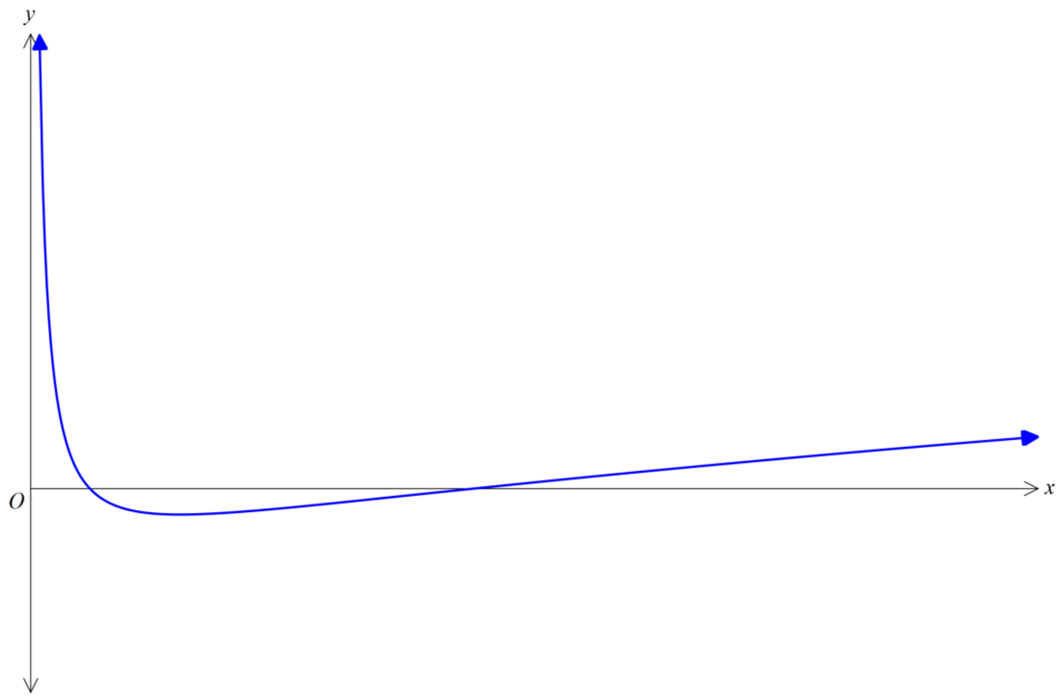
2 marks

Question 4 (13 marks)

Consider the family of functions $f_a : R^+ \to R$, which is defined by $f_a(x) = \frac{a}{x} + \sqrt{x} - 3$,

where a is a real number and $a > 0$.

Part of the graph of f_a is shown below.



a. State the interval for which the graph of f_a is strictly increasing, in terms of a.

2 marks

b. Determine the absolute minimum value of f_a, in terms of a.

2 marks

c. Show that the equation of the tangent to the graph of f_a at the point when $x = 4$ is $y = -\frac{(a-4)}{16}x + \frac{(a-4)}{2}$.

3 marks

The function $f_a(x)$ is transformed to form $g_a(x)$, where $g_a(x)$ is defined as $g_a(x) = f_a(x) + b$.

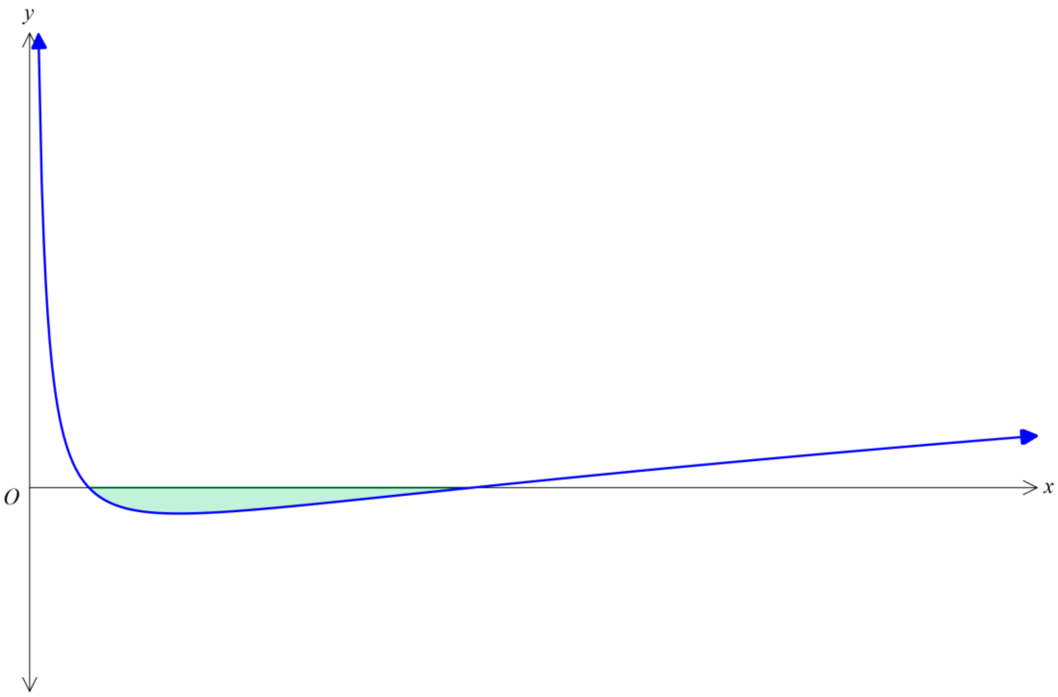
d. Find the value of *b*, in terms of *a*, such that the tangent drawn to the curve of $g_a(x)$ at $x = 4$ passes through the origin.

2 marks

Consider the graph of $f_a(x)$ for $a = 2$.
Let $h_a(x) = f_a(2x)$.

e. **i.** Find the value of the area that is bounded by the x-axis and the graph of $y = f_a(x)$ for $a = 2$, as shaded in the diagram below, correct to 3 decimal places.

2 marks



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ii. Hence, find the area that is bounded by the x-axis and the graph of $y = h_a(x)$ for $a = 2$, correct to 3 decimal places.

2 marks

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Question 5 (13 marks)

Thomas Lindsay is a star full forward for the local football team. He has found from past experience that his chance of scoring a goal is 0.7 and is independent of any attempt he makes. At training he is practising going for goal and makes ten attempts.

a. **i.** What is the probability, correct to 4 decimal places, that Thomas scores goals with his first four attempts?

1 mark

ii. What is the probability that Thomas scores exactly four goals out of his ten attempts, correct to 4 decimal places?

1 mark

iii. What is the probability that the last six of Thomas' ten attempts miss, given that exactly four of his ten attempts are goals? Give your answer correct to 4 decimal places.

2 marks

The club statistician records data relating to the distance Thomas stands from the boundary line when he scores a goal. The data is recorded according to two categories: home games and away games.

For each category, the distance from the boundary line, in metres, is normally distributed with a mean and standard deviation that is either given or to be determined. See the table below.

Goal distance	Mean	Standard deviation
Home	22.5	5.5
Away	μ	σ

b. Calculate the probability that Thomas scores a goal at a home game when he is more than 30 metres from the boundary line. Give your answer correct to 4 decimal places.

1 mark

Of the away games, 8% of Thomas' goals are scored more than 35 metres from the boundary line, whereas 4% are scored when he is less than 5 metres from the boundary line.

c. **i.** Show that the mean and the standard deviation of the away game goals can be found by solving the pair of equations $\mu + 1.4051\sigma = 35$ and $\mu - 1.7507\sigma = 5$.

2 marks

ii. Hence, evaluate the mean and standard deviation for the away games, expressing your answer correct to 3 decimal places.

2 marks

d. Find the probability that of the 20 away goals Thomas scores, at least seven of them will be scored from more than 35 metres from the boundary line. Give your answer correct to 5 decimal places.

2 marks

In preparing a video highlights package of Thomas' goals, it is found that 40% of his goals are scored at away games. Video footage of one of the goals is chosen to be used by the league to promote the game. The footage shows Thomas scoring a goal from more than 28 metres from the boundary line.

e. What is the probability that it was a goal from a home game? Give your answer correct to 4 decimal places.

2 marks

END OF QUESTION AND ANSWER BOOK