TEST 8

Functions, graphs, algebra and calculus
Technology-free end-of-year examination
Total marks: 35
Suggested writing time: 50 minutes

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 1

Find an expression for $\int \frac{1}{3x+2}\,dx$.

Using the formula $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|(ax+b)| + c$.

$\int \frac{1}{3x+2}\,dx = \frac{1}{3}\log_e|(3x+2)| + c$

1 mark

QUESTION 2

a Find an expression for $\int \sin(6x)\,dx$.

$\int \sin(6x)\,dx = -\frac{1}{6}\cos(6x) + c$

1 mark

b Hence find $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(6x)\,dx$.

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(6x)\,dx = -\frac{1}{6}[\cos(6x)]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= -\frac{1}{6}(\cos(3\pi) - \cos(2\pi))$

$= -\frac{1}{6}(-1-1)$

$= \frac{1}{3}$

2 marks
(Total: 3 marks)

QUESTION 3

Use the remainder theorem to determine if the polynomial $P(x) = 3x^4 + 2x^3 - x^2 - 2$ is divisible by $(x + 1)$.

$P(x) = 3x^4 + 2x^3 - x^2 - 2$

$P(-1) = 3(-1)^4 + 2(-1)^3 - (-1)^2 - 2$

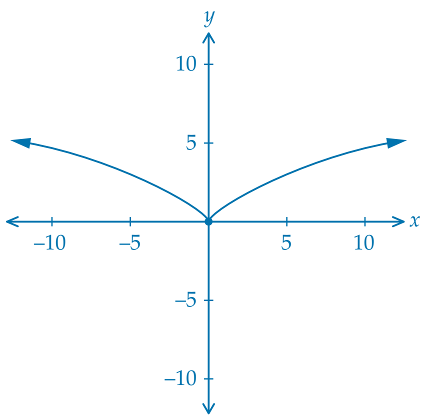
So $P(-1) = -2 \neq 0$

$\therefore P(x)$ is NOT divisible by $(x + 1)$.

2 marks

QUESTION 4

a Sketch the graph of $y = x^{\frac{2}{3}}$



2 marks

b Find the equation of the tangent to the curve $y = x^{\frac{2}{3}}$ at $x = 1$.

$y = x^{\frac{2}{3}}$

$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

At $x = 1$, $y = 1$ and $\frac{dy}{dx} = \frac{2}{3}$

Use the equation of a line $y - y_1 = m(x - x_1)$,

where m = gradient of curve.

Use point (1, 1).

$y - 1 = \frac{2}{3}(x - 1)$

The equation of the tangent is

$y = \frac{2}{3}x + \frac{1}{3}$

2 marks

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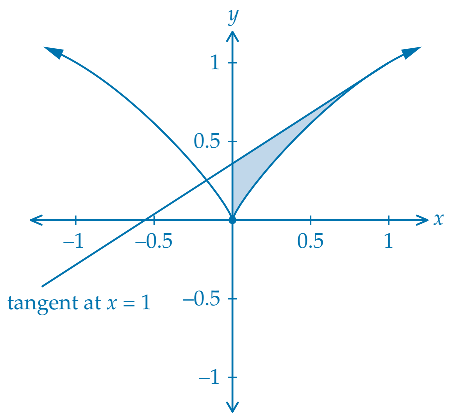
c Find the area bounded by the graph of $y = x^{\frac{2}{3}}$, the tangent found in *b*, and the line $x = 0$.



Upper curve is $y = \frac{2}{3}x + \frac{1}{3}$

Lower curve is $y = x^{\frac{2}{3}}$

Area $= \int_0^1 \left(\frac{2}{3}x + \frac{1}{3} - x^{\frac{2}{3}}\right)dx$

$= \left[\frac{x^2}{3} + \frac{x}{3} - \frac{3x^{\frac{5}{3}}}{5}\right]_0^1$

$= \frac{1}{3} + \frac{1}{3} - \frac{3}{5}$

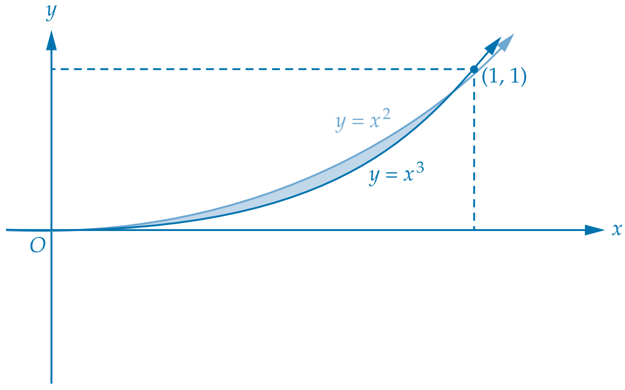
$= \frac{1}{15}$ square units

2 marks
(Total: 6 marks)

QUESTION 5

Find the area enclosed by the graphs of $y = x^2$ and $y = x^3$.

Points of intersection are at (0, 0) and (1, 1).



Upper curve is $y = x^2$

Lower curve is $y = x^3$

Area $= \int_0^1 (x^2 - x^3)\,dx$

$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$

$= \frac{1}{3} - \frac{1}{4}$

$= \frac{1}{12}$ square units

3 marks

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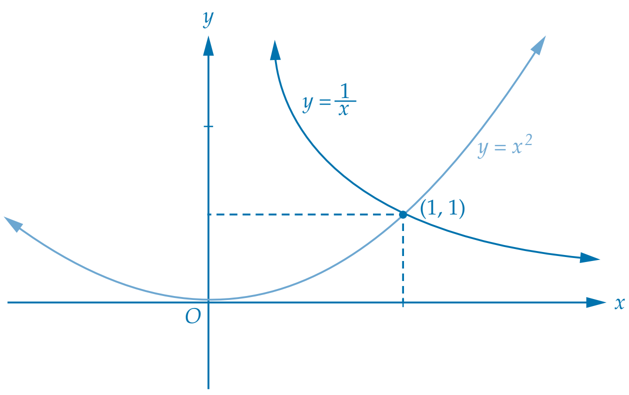
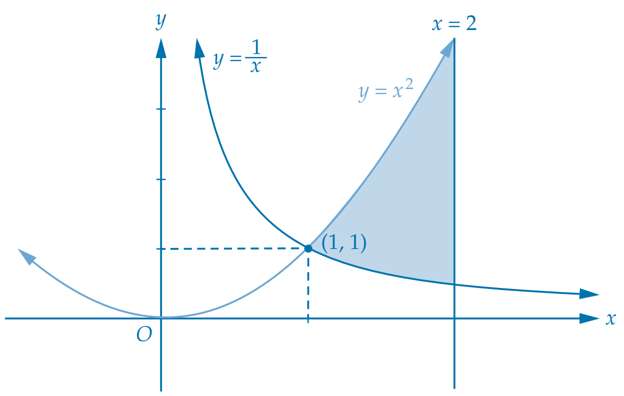
QUESTION 6

a Sketch the graphs of $y = x^2$ and $y = \frac{1}{x}$, labelling any points of intersection.



2 marks

b Find the area enclosed by the graphs of $y = x^2$ and $y = \frac{1}{x}$ and the line $x = 2$.



Upper curve is $y = x^2$

Lower curve is $y = \frac{1}{x}$

$$\text{Area} = \int_1^2 \left(x^2 - \frac{1}{x} \right) dx$$

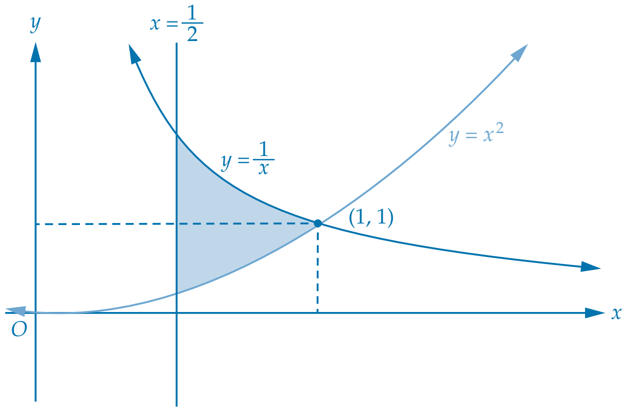
$$= \left[\frac{x^3}{3} - \log_e(x) \right]_1^2$$

$$= \left(\frac{8}{3} - \log_e(2) \right) - \left(\frac{1}{3} - \log_e(1) \right)$$

$$= \frac{7}{3} - \log_e(2) \text{ square units}$$

2 marks

c Find the area enclosed by the graphs of $y = x^2$ and $y = \frac{1}{x}$ from $x = \frac{1}{2}$ to $x = 1$.



Upper curve is $y = \frac{1}{x}$

Lower curve is $y = x^2$

$$\text{Area} = \int_{\frac{1}{2}}^{1} \left(\frac{1}{x} - x^2 \right) dx$$

$$= \left[\log_e(x) - \frac{x^3}{3} \right]_{\frac{1}{2}}^{1}$$

$$= \left(\log_e(1) - \frac{1}{3} \right) - \left(\log_e\left(\frac{1}{2}\right) - \frac{1}{24} \right)$$

$$= \log_e(2) - \frac{7}{24} \text{ square units}$$

2 marks
(Total: 6 marks)

QUESTION 7

a Find $\frac{d}{dx}(x\log_e(x))$.

$$\frac{d}{dx}(x\log_e(x)) = \log_e(x) \times 1 + x \times \frac{1}{x}$$
$$= \log_e(x) + 1$$

2 marks

b Hence find $\int_1^2 -2\log_e(x)\,dx$.

Statement: $\int (\log_e(x)+1)\,dx = x\log_e(x)\ (+\,c)$

$\therefore \int_1^2 (\log_e(x)+1)\,dx = [x\log_e(x)]_1^2$

$\Rightarrow \int_1^2 (\log_e(x))\,dx = [x\log_e(x)]_1^2 - \int_1^2 1\,dx$

We require $-2\int_1^2 (\log_e(x))\,dx = -2[x\log_e(x)]_1^2 + 2\int_1^2 1\,dx$

Hence

$$\int_1^2 -2\log_e(x)\,dx = -2 \times 2\log_e(2) + 2 \times 1$$
$$= -4\log_e(2) + 2$$

2 marks
(Total: 4 marks)

QUESTION 8

A function is of the form $f(x) = ax + \frac{b}{x}$.

a Find an expression for $f'(x)$.

$f(x) = ax + \frac{b}{x} = ax + bx^{-1}$

$f'(x) = a - bx^{-2} = a - \frac{b}{x^2}$

1 mark

b Hence find x such that $f'(x) = 0$

$f'(x) = 0$ gives

$a - \frac{b}{x^2} = 0 \Rightarrow a = \frac{b}{x^2}$

$\Rightarrow x = \pm\sqrt{\frac{b}{a}}$

1 mark

c It is known that $f'(2) = 0$ and $f(2) = 4$. Find a and b.

$f'(2) = 0 \Rightarrow a - \frac{b}{4} = 0$

$f(2) = 4 \Rightarrow 2a + \frac{b}{2} = 4$

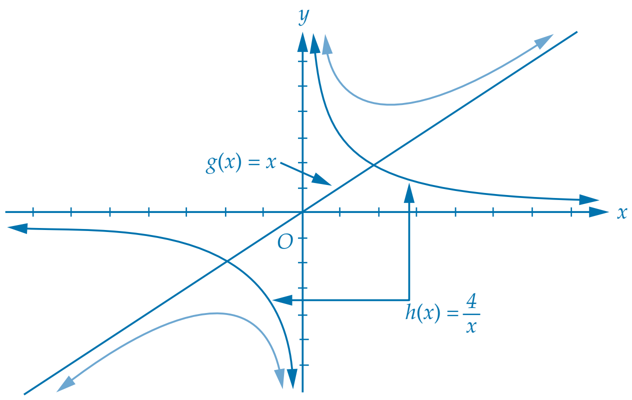
Solve the above equations simultaneously to get

$a = 1, b = 4$

2 marks

d Hence, use addition of ordinates to sketch the graph of $f(x) = ax + \frac{b}{x}$.

$f(x) = ax + \frac{b}{x} = x + \frac{4}{x}$



2 marks
(Total: 6 marks)

QUESTION 9

Find the minimum value of the function $f(x) = x - \log_e(x)$.

$f(x) = x - \log_e(x)$

$f'(x) = 1 - \frac{1}{x} = 0$ for max/min

$\Rightarrow \frac{1}{x} = 1$

So $x = 1$, and due to the shape of the graph formed from the addition of ordinates, this is the minimum.

At $x = 1, f(1) = 1$.

Minimum value = 1

2 marks

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QUESTION 10

Consider the graph of $y = x^2 + a$. If the line $y = 6x - 1$ is a tangent to this graph, find the value of a.

$y = x^2 + a$

$\frac{dy}{dx} = 2x$

At $x = x_1$, $\frac{dy}{dx} = 2x_1$

Gradient = 6, so $2x_1 = 6 \Rightarrow x_1 = 3$

Use the equation of a line $y - y_1 = m(x - x_1)$

where m = gradient of curve.

Use point $(3, 9 + a)$.

$y - (9 + a) = 6\,(x - 3)$

The equation of the tangent is

$y = 6x - 18 + (9 + a)$

$\quad = 6x - 9 + a$

Compare with $y = 6x - 1$

This gives $-9 + a = -1$

$\therefore a = 8$

2 marks

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