STUDENT NUMBER | LETTER

Figures								
Words								

MATHEMATICAL METHODS

Written examination 1

Reading time: 15 minutes

Writing time: 1 hour

QUESTION AND ANSWER BOOK

Number of questions	Number of questions to be answered	Marks
11	11	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are NOT permitted to bring into the examination room: notes of any kind, blank sheets of paper, white out liquid/tape or a calculator of any type.

Materials supplied
- Question and answer book of 11 pages, with a detachable sheet of miscellaneous formulas.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the back of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

At the end of the examination
- You may keep this question book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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QUESTION 1

Let $f(x) = (x^2 + x)\log_e(x + 1)$. Find $f'(x)$.

$$f(x) = (x^2 + x)\log_e(x + 1)$$

$$f'(x) = \log_e(x + 1) \times (2x + 1) + (x^2 + x) \times \frac{1}{x+1}$$

$$= (2x + 1)\log_e(x + 1) + \frac{x(x+1)}{x+1}$$

$$= (2x + 1)\log_e(x + 1) + x$$

2 marks

QUESTION 2

Let $y = \dfrac{e^{x^2-1}}{\cos(2x-\pi)}$. Find $\dfrac{dy}{dx}$ at $x = \pi$

$$y = \frac{e^{x^2-1}}{\cos(2x-\pi)}$$

$$\frac{dy}{dx} = \frac{\cos(2x-\pi)\times 2xe^{x^2-1} + 2\sin(2x-\pi)\times e^{x^2-1}}{\cos^2(2x-\pi)}$$

At $x = \pi$

$$\frac{dy}{dx} = \frac{\cos(\pi)\times 2\pi e^{\pi^2-1} + 2\sin(\pi)\times e^{\pi^2-1}}{\cos^2(\pi)}$$

$$= \frac{-1\times 2\pi e^{\pi^2-1} - 0}{1}$$

$$= -2\pi e^{\pi^2-1}$$

2 marks

QUESTION 3

Let $\Pr(A) = 0.1$, $\Pr(B) = 0.3$

a If A and B are independent events, find $\Pr(A \cup B)$.

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

And $\Pr(A \cap B) = \Pr(A) \times \Pr(B) = 0.1 \times 0.3 = 0.03$

$\Pr(A \cup B) = 0.1 + 0.3 - 0.03$

$= 0.37$

1 mark

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b A is the event of winning a particularly difficult football game. B is the event of it being wet on the day of the game. If A and B are no longer independent events, the probability of winning the game on a wet day reduces to 0.05. Find the probability that, if the game is won, it is a wet day.

$\Pr(A \mid B) = 0.05$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\text{So } 0.05 = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$\Rightarrow \Pr(A \cap B) = 0.05 \times 0.3 = 0.015$

$$\text{Now } \Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

$$= \frac{0.015}{0.1}$$

$$= 0.15$$

The probability that, if the game is won, it is a wet day is 0.15.

3 marks
(Total: 4 marks)

QUESTION 4

A probability density function is defined below.

$$f(x) = \begin{cases} \dfrac{x}{8} + k, & -1 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } k \in \boldsymbol{R}$$

a Find k.

$$f(x) = \begin{cases} \dfrac{x}{8} + k, & -1 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } k \in \boldsymbol{R}$$

$$\text{For a PDF, } \int_{-1}^{1} \left(\frac{x}{8} + k\right) dx = 1$$

$$\left[\frac{x^2}{16} + kx\right]_{-1}^{1} = 1$$

$$\left(\frac{1}{16} + k\right) - \left(\frac{1}{16} - k\right) = 1$$

$2k = 1$

$$k = \frac{1}{2}$$

2 marks

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b Find the mean of the probability density function.

$$\text{Mean} = E(X) = \int_{-1}^{1} x\left(\frac{x}{8}+k\right)d. \qquad \text{where } k = \frac{1}{2}$$

$$= \int_{-1}^{1}\left(\frac{x^2}{8}+\frac{x}{2}\right)dx$$

$$= \left[\frac{x^3}{24}+\frac{x^2}{4}\right]_{-1}^{1}$$

$$= \left(\frac{1}{24}+\frac{1}{4}\right)-\left(-\frac{1}{24}+\frac{1}{4}\right)$$

$$= \frac{1}{12}$$

2 marks

c Find the median of the probability density function.

$$\int_{-1}^{m}\left(\frac{x}{8}+\frac{1}{2}\right)dx = \frac{1}{2} \text{ where } m \text{ is the median}$$

$$\left[\frac{x^2}{16}+\frac{x}{2}\right]_{-1}^{m} = \frac{1}{2}$$

$$\left(\frac{m^2}{16}+\frac{m}{2}\right)-\left(\frac{1}{16}-\frac{1}{2}\right) = \frac{1}{2}$$

$$\frac{m^2}{16}+\frac{m}{2}+\frac{7}{16} = \frac{1}{2}$$

$$m^2 + 8m + 7 = 8$$

$$m^2 + 8m - 1 = 0$$

$$m = \frac{-8 \pm \sqrt{64+4}}{2} = \frac{-8 \pm 2\sqrt{17}}{2} = -4 \pm \sqrt{17}$$

Select the m value that is within the domain.

$$m = -4 + \sqrt{17}$$

2 marks
(Total: 6 marks)

QUESTION 5

a Find $\frac{d}{dx}\left(x^2 \log_e(x)\right)$.

$$\frac{d}{dx}\left(x^2 \log_e(x)\right) = 2x \log_e(x) + x^2 \times \frac{1}{x}$$

$$= 2x \log_e(x) + x$$

1 mark

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b Hence find $\int_1^2 4x\log_e(x)dx$.

$$\frac{d}{dx}\left(x^2\log_e(x)\right) = 2x\log_e(x) + x$$

$$\Rightarrow \int\left(2x\log_e(x)+x\right)dx = x^2\log_e(x)\ (+c)$$

$$\Rightarrow \int_1^2\left(2x\log_e(x)+x\right)dx = [x^2\log_e(x)]_1^2$$

$$\int_1^2\left(2x\log_e(x)\right)dx + \int_1^2 x\,dx = [x^2\log_e(x)]_1^2$$

$$\int_1^2\left(2x\log_e(x)\right)dx = [x^2\log_e(x)]_1^2 - \int_1^2 x\,dx$$

$$\int_1^2\left(4x\log_e(x)\right)dx = 2[x^2\log_e(x)]_1^2 - 2\int_1^2 x\,dx$$

$$= 2(4\log_e(2)) - 2\left[\frac{x^2}{2}\right]_1^2$$

$$= 8\log_e(2) - (4-1)$$

$$\int_1^2\left(4x\log_e(x)\right)dx = 8\log_e(2) - 3$$

3 marks
(Total: 4 marks)

QUESTION 6

a Show that $P(x) = 3x^3 - x^2 - 2$ has only one real factor.

$P(x) = 3x^3 - x^2 - 2$

$P(1) = 3 - 1 - 2 = 0$, so $x - 1$ is a factor.

By synthetic division

	3	–1	0	–2
1		3	2	2
	3	2	2	0

The quadratic factor is $(3x^2 + 2x + 2)$.

$\Delta = 2^2 - 4 \times 3 \times 2 = -20$

So there are no real factors for the quadratic.

$\therefore (x - 1)$ is the only real factor.

2 marks

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b Find the equation of the tangent to the curve $y = 3x^3 - x^2 - 2$ at $x = 1$.

$y = 3x^3 - x^2 - 2$

So $\frac{dy}{dx} = 9x^2 - 2x$

Gradient at $x = 1$ is 7.

Use the equation of a line $y - y_1 = m(x - x_1)$, where m = gradient of curve.

Using point (1, 0),

$y - 0 = 7(x - 1)$

The equation of the tangent is

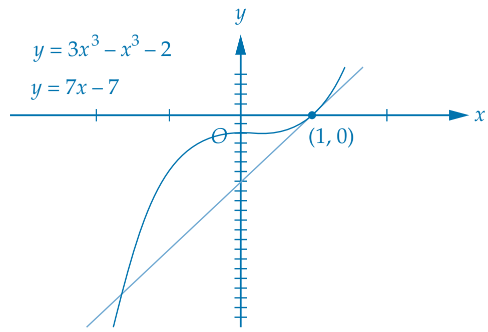
$y = 7x - 7$

2 marks

c Hence, find the coordinates of the point where the tangent to the curve $y = 3x^3 - x^2 - 2$ at $x = 1$ intersects again with the curve $y = 3x^3 - x^2 - 2$.

Upper curve: $y = 3x^3 - x^2 - 2$

Lower curve: $y = 7x - 7$



Point of intersection

Equate $3x^3 - x^2 - 2 = 7x - 7$

$3x^3 - x^2 - 7x + 5 = 0$

We already know that one point of intersection is (1, 0).

By synthetic division

	3	–1	–7	5
1		3	2	–5
	3	2	–5	0

The quadratic factor is $(3x^2 + 2x - 5)$.

So $y = (x - 1)(x - 1)(3x + 5)$

2nd point of intersection is $\left(-\frac{5}{3}, -\frac{56}{3}\right)$.

Alternate solution

We know that $x = 1$ meets at the tangent to the curve, so we expect a repeated factor.

So $y = (x - 1)^2$(linear factor)

$\quad = (x^2 - 2x + 1)$(linear factor)

By observation of $y = 3x^3 - x^2 - 7x + 5$, the linear factor must be $(3x + 5)$.

The 2nd point of intersection is $\left(-\frac{5}{3}, -\frac{56}{3}\right)$.

2 marks
(Total: 6 marks)

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QUESTION 7

Consider the graph of $y = 2\sin\left(x - \frac{\pi}{2}\right) + 1$.

a Find the x-intercepts of the graph for the domain $x \in [0, 2\pi]$.

Solve $2\sin\left(x - \frac{\pi}{2}\right) + 1 = 0$ for $x \in [0, 2\pi]$.

$2\sin\left(x - \frac{\pi}{2}\right) = -1 \Rightarrow \sin\left(x - \frac{\pi}{2}\right) = -\frac{1}{2}$

Reference angle $= \frac{\pi}{6}$

$\left(x - \frac{\pi}{2}\right) = -\frac{\pi}{6}, \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$= -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$x = -\frac{\pi}{6} + \frac{\pi}{2}, \frac{7\pi}{6} + \frac{\pi}{2}, \frac{11\pi}{6} + \frac{\pi}{2}$

$= \frac{2\pi}{6}, \frac{10\pi}{6}, \frac{14\pi}{6}$

$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$ where $\frac{7\pi}{3}$ is now out of the domain $[0, 2\pi]$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

2 marks

b Hence find the x-intercepts of the graph of $y = 2\sin\left(x - \frac{\pi}{2}\right) + 1$ after it is translated $\frac{\pi}{4}$ units in the positive direction of the x-axis and then reflected over the x-axis.

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

Translated $\frac{\pi}{4}$ units in the positive direction of the x-axis:

$x = \frac{\pi}{3} + \frac{\pi}{4}, \frac{5\pi}{3} + \frac{\pi}{4}$

$= \frac{7\pi}{12}, \frac{23\pi}{12}$

Reflected over the x-axis, intercepts remain the same.

$x = \frac{7\pi}{12}, \frac{23\pi}{12}$

2 marks
(Total: 4 marks)

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QUESTION 8

Sam finds that, on average, he solves an equation correctly 7 out of 10 times. In a particular examination, Sam is faced with 4 such equations. What is the probability that Sam will solve at least one of these equations correctly?

$\text{Bi}(n, p) = \text{Bi}(4, 0.7)$

$\Pr(X \geq 1) = 1 - \Pr(X = 0)$

$$\begin{aligned} 1 - \Pr(X = 0) &= 1 - {}^4C_0(0.3)^4(0.7)^0 \\ &= 1 - (0.3)^4 \\ &= 1 - 0.0081 \\ &= 0.9919 \end{aligned}$$

2 marks

QUESTION 9

The curve $y = 2x^2 + 3$ for $x \geq 0$ is added to the curve $y = px^2 - 6$ for $x < 2$ to create the function $f(x) = 2x^2 + 3 + px^2 - 6$ where p is a real constant.

a State the domain for which $f(x)$ exists.

Intersection of domains $x \geq 0$ and $x < 2$.

$x \in [0, 2)$

1 mark

b Find the value(s) of p for which the inverse, $f^{-1}(x)$, exists.

$f(x) = 2x^2 + 3 + px^2 - 6$

$y = (2 + p)x^2 - 3$

Interchange x and y to find the inverse.

$x = (2 + p)y^2 - 3$

$y^2 = \sqrt{\dfrac{x+3}{p+2}}$

$y = \pm\sqrt{\dfrac{x+3}{p+2}}$

Select the +ve branch because of the domain of $f(x)$.

$f^{-1}(x) = \sqrt{\dfrac{x+3}{p+2}}$

$f^{-1}(x)$ exists for $p + 2 > 0$.

$\therefore p > -2$

2 marks
(Total: 3 marks)

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QUESTION 10

Consider the functions with maximal domains: $f(x) = x^2$ and $g(x) = \log_e(x)$.

a State, with a reason, if $f(g(x))$ exists.

For $f(g(x))$, test ran (inner) $\subseteq$ dom (outer).

$\boldsymbol{R} \subseteq \boldsymbol{R}$

$\therefore f(g(x))$ exists.

1 mark

b State, with a reason, if $g(f(x))$ exists.

For $g(f(x))$, test ran (inner) $\subseteq$ dom (outer).

$[0, \infty) \not\subset (0, \infty)$

$\therefore g(f(x))$ does not exist.

1 mark

c Define $h'(x)$ if $h(x) = f(g(x))$

$h(x) = f(g(x))$, which exists

$h(x) = (\log_e(x))^2$

$$h'(x) = 2\log_e(x) \times \frac{1}{x}$$

$$= \frac{2}{x}\log_e(x)$$

Domain $f(g(x))$ = domain $g(x) = (0, \infty)$.

Domain $h'(x) = (0, \infty)$

$$h': (0, \infty) \to \boldsymbol{R},\ h'(x) = \frac{2}{x}\log_e(x)$$

2 marks

(Total: 4 marks)

QUESTION 11

From a sample of 60 Year 12 students, 45 said they like chocolate. Estimate the probability of Year 12 students liking chocolate and the variance of the sampling distribution.

$$p \approx \hat{p} = \frac{45}{60} = \frac{3}{4}$$

$$\text{Var}(\hat{p}) \approx \frac{\hat{p}(1-\hat{p})}{n} = \frac{\frac{3}{4}\left(1-\frac{3}{4}\right)}{60} = \frac{\frac{3}{16}}{60}$$

$$\text{Var}(\hat{p}) = \frac{1}{320}$$

The probability of Year 12 students liking chocolate is about 0.75, with a variance of $\frac{1}{320}$.

3 marks

Mathematical Methods Formulas

Mensuration

area of trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\int e^{ax}\, dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\int \frac{1}{x}dx = \log_e |x| + c$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$$

$$\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$$

$$\int \cos(ax) dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

variance: $Var(X) = \sigma^2 = E\left((X-\mu)^2\right) = E(X^2) - \mu^2$

mean: $\mu = E(X)$

Probability distribution		Mean	Variance
Discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2 p(x)$
Continuous	$\Pr(a < X < b) = \int_a^b f(x)\, dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\ f(x)\, dx$

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