TEST 3

Technology active end-of-year examination
Functions and graphs, Algebra
Section A: 16 marks
Section B: 30 marks
Suggested writing time: 70 minutes

Section A: Multiple-choice questions

Specific instructions to students

- A correct answer scores 1 mark, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one letter is shaded on the answer sheet.
- Choose the alternative that most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.

- Use pencil only.
- Working space is provided under those questions that require working out.

QUESTION 1

Simplified,

$\log_{10}(100) + \log_{10}\left(\frac{1}{100}\right) - 3\log_{10}\left(\sqrt{x}\right) - 2\log_{10}(2)$

can be expressed as

A $-\frac{3}{2}\log_{10}(4x)$

B $\log_{10}\left(4x^{\frac{3}{2}}\right)$

C $\log_{10}(4) + x^{\frac{3}{2}}$

D $\log_{10}\left(-4x^{\frac{3}{2}}\right)$

E $-\log_{10}\left(4x^{\frac{3}{2}}\right)$

$\log_{10}(100) + \log_{10}\left(\frac{1}{100}\right) - 3\log_{10}\left(\sqrt{x}\right) - 2\log_{10}(2)$

$= \log_{10}(10^2) + \log_{10}(10^{-2}) - 3\log_{10}\left(\sqrt{x}\right) - 2\log_{10}(2)$

$= 2\log_{10}(10) - 2\log_{10}(10) - \log_{10}\left(x^{\frac{3}{2}}\right) - \log_{10}(4)$

$= 2 - 2 - \log_{10}\left(x^{\frac{3}{2}}\right) - \log_{10}(4)$

$= -\log_{10}\left(4x^{\frac{3}{2}}\right)$

QUESTION 2

Solve for x in the equation $2^x - 3 \times 2^{-x} = 2$

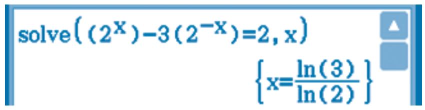
A $x = \frac{\log_2(3)}{\log_2(2)}$

B $x = \frac{\log_e(2)}{\log_e(3)}$

C $x = \log_3(2)$

D $x = 2$

E $x = 3$



$2^x - 3 \times 2^{-x} = 2$

$2^x - 3 \times \frac{1}{2^x} = 2$

$2^{2x} - 3 = 2 \times 2^x$

$2^{2x} - 2 \times 2^x - 3 = 0$

$(2^x - 3)(2^x + 1) = 0$

$2^x = 3$ and $2^x = -1$ (no solution)

$x = \log_2(3)$

Using change of base, $\log_2(3) = \frac{\log_e(3)}{\log_e(2)} = \frac{\log_2(3)}{\log_2(2)}$

QUESTION 3

Consider $f(x) = 2 + \log_e(3 - x)$ and $g(x) = 3 - e^{x-2}$. The rule and domain for the function $g(f(x))$, respectively, are

A $y = 2 + \log_e(3 - x), x \in (-\infty, 3)$

B $y = 3 - e^{x-2}, x \in (-\infty, 3)$

C $y = 3 - e^{x-2}, x \in \boldsymbol{R}$

D $y = x, x \in \boldsymbol{R}$

E $y = x, x \in (-\infty, 3)$

$f(x) = 2 + \log_e(3 - x)$ and $g(x) = 3 - e^{x-2}$.

Test $g(f(x))$.

Test ran $(f) \subseteq$ dom (g).

$\boldsymbol{R} \subseteq \boldsymbol{R}$

$f(x) = 2 + \log_e(3 - x)$ and $g(x) = 3 - e^{x-2}$ are inverses of each other so $g(f(x)) = x$.

dom $g(f(x))$ = dom $f(x) = (-\infty, 3)$

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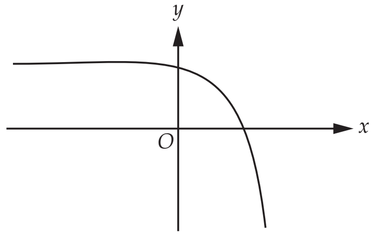
QUESTION 4

If $f(x) = 2 + \log_e(3 - x)$ and $g(x) = 3 - e^{x-2}$, then the graph for the function $g(f(x))$ looks like

A



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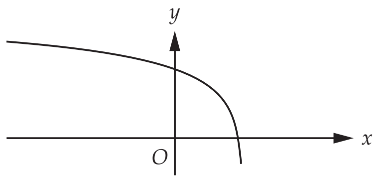

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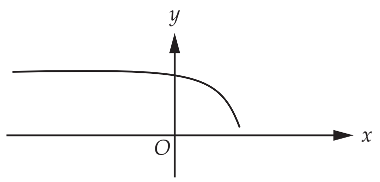
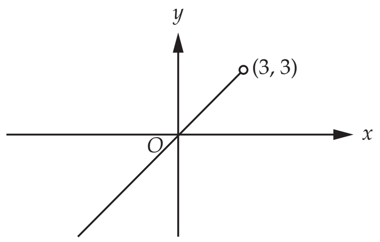
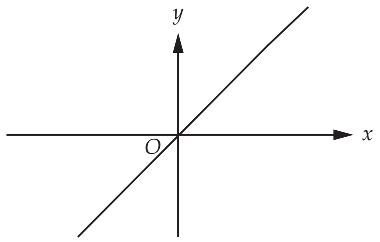


D



E

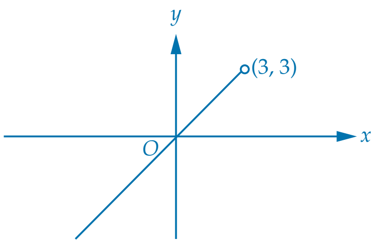


$f(x) = 2 + \log_e(3 - x)$ and $g(x) = 3 - e^{x-2}$.

Rule $g(f(x)) = x$.

dom $g(f(x))$ = dom $f(x) = (-\infty, 3)$

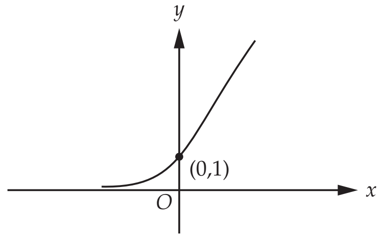
$g(f(x))$ looks like



QUESTION 5

For $f(x) = \begin{cases} x+1, & x \geq 0 \\ e^x, & x < 0 \end{cases}$ the graph looks like

A



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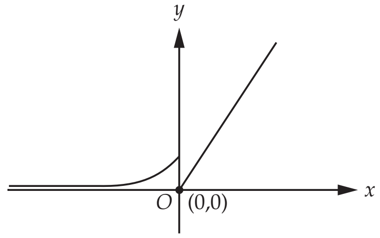

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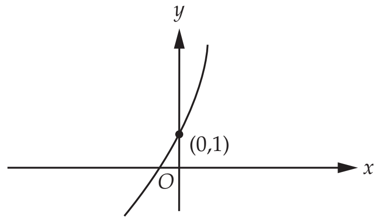

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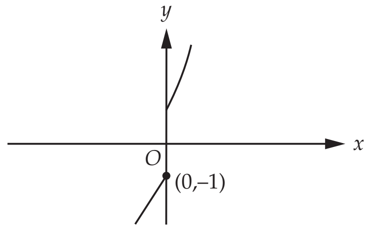
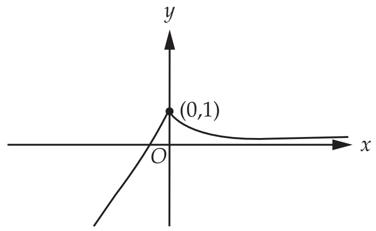
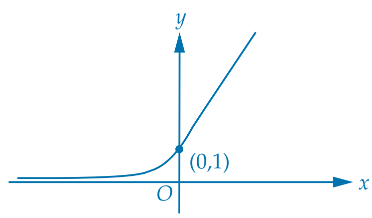


E



$f(x) = \begin{cases} x+1, & x \geq 0 \\ e^x, & x < 0 \end{cases}$ looks like



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QUESTION 6

The maximal domain and range, respectively, of the graph of $y = x^{\frac{5}{2}} + 1$ is

A $\boldsymbol{R}, \boldsymbol{R}$

B $\boldsymbol{R}, (0, \infty)$

C $[0, \infty), [-1, \infty)$

D $[0, \infty), [1, \infty)$

E $(0, \infty), [0, \infty)$

> $y = x^{\frac{5}{2}} + 1$
> domain: $[0, \infty)$, range: $[1, \infty)$

QUESTION 7

$P(x) = ax^3 + bx^2 + x - 10$ is a cubic polynomial. Factorised into the product of linear factors, $P(x)$ can be expressed as $P(x) = (2x + 5)(x - 1)(x + 2)$. The values of a and b are

A $a = 5, b = -2$

B $a = 1, b = -10$

C $a = 2, b = 7$

D $a = 7, b = -2$

E $a = 2, b = -7$

> $P(x) = ax^3 + bx^2 + x - 10 = (2x + 5)(x - 1)(x + 2)$.
> Expanded, $(2x + 5)(x - 1)(x + 2) = 2x^3 + 7x^2 + x - 10$
> $\therefore a = 2, b = 7$

QUESTION 8

A graph is reflected over the y-axis, then dilated by 2 units parallel to the y-axis. The transformation matrix, T, which describes this is

A $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

B $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

C $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

D $\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$

E $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

> A graph is reflected over the y-axis, then dilated by 2 units parallel to the y-axis.
> $T = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

QUESTION 9

The graph of $y = \frac{1}{(x-1)^2}$ is reflected over the y-axis, then dilated by 2 units parallel to the y-axis. The image after this transformation is

A $y = \frac{-1}{(x-1)^2}$

B $y = \frac{1}{(x-1)^2}$

C $y = \frac{2}{(x+1)^2}$

D $y = \frac{1}{(x+1)^2}$

E $y = \frac{-2}{(x-1)^2}$

> The graph of $y = \frac{1}{(x-1)^2}$ is reflected over the y-axis, then dilated by 2 units parallel to the y-axis. Image is
> $y = \frac{2}{(-x-1)^2} = \frac{2}{(x+1)^2}$

QUESTION 10

If $T: \boldsymbol{R}^2 \to \boldsymbol{R}^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is applied to the function $y = \sqrt{x}$, the image function is

A $y + 4 = \sqrt{x+2}$

B $y = \sqrt{x-2} + 4$

C $y = \sqrt{x-2} - 4$

D $y = 4\sqrt{x-2}$

E $y = 4\sqrt{x+2}$

> If $T: \boldsymbol{R}^2 \to \boldsymbol{R}^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is applied to the function $y = \sqrt{x}$,
> $y = f(x - b)$ means that $y = f(x)$ has been translated 'b' units to the right.
> $y = f(x) + c$ means that $y = f(x)$ has been translated 'c' units up.
> Translation of b units right and c units up
> $\begin{bmatrix} x+2 \\ y+4 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ is applied to the function $y = \sqrt{x}$,
> so $x = x_1 - 2$ and $y = y_1 - 4$
> So $y = \sqrt{x}$ becomes $y_1 - 4 = \sqrt{x_1 - 2}$
> $\therefore y = \sqrt{x-2} + 4$

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QUESTION 11

The transformations required to change $y = \frac{1}{\sqrt{x}}$ to $y = 3 - \frac{2}{\sqrt{x+1}}$ are:

A dilation by a factor of 2 from the x-axis, translation of 1 unit to the right, reflection over the y-axis, translation of 3 units down

B dilation by a factor of 2 from the y-axis, translation of 1 unit to the right, translation of 3 units up

C dilation by a factor of 2 from the x-axis, translation of 1 unit to the left, reflection over the x-axis and translation of 3 units up

D dilation by a factor of 2 from the x-axis, reflection over the y-axis, translation of 1 unit to the left, translation of 3 units up

E dilation by a factor of $\frac{1}{2}$ from the y-axis, translation of 1 unit to the right, translation of 3 units up

> $y = \frac{1}{\sqrt{x}}$ to $y = 3 - \frac{2}{\sqrt{x+1}}$
> - dilation by a factor of 2 from the x-axis
> - translation of 1 unit to the left
> - reflection over the x-axis
> - translation of 3 units up

QUESTION 12

If $f(x) = -\sqrt{-(x+1)} - 2$, then the domain of the inverse function f^{-1} is

A $(-\infty, -1)$

B $(-\infty, -2)$

C $(-2, \infty)$

D $[2, \infty)$

E $(-\infty, -2]$

> $f(x) = -\sqrt{-(x+1)} - 2$
> Domain of f^{-1} = range $f(x) = (-\infty, -2]$.

QUESTION 13

The functionality equation $f(x) \times f(y) = f(x + y)$ is satisfied by

A $f(x) = e^x$

B $f(x) = \frac{1}{x}$

C $f(x) = \log_e(x)$

D $f(x) = x$

E $f(x) = 2x^2$

> $f(x) \times f(y) = f(x + y)$ looks like the rule $e^x \times e^y = e^{x+y}$ and is satisfied by $f(x) = e^x$.

QUESTION 14

Which of the following is **not** a polynomial?

A $f(x) = 1 + 2x$

B $f(x) = -3 + 4x + 2x^2$

C $f(x) = 4x + 2x^3$

D $f(x) = x^6 - 1$

E $f(x) = x^2 + \frac{1}{x}$

> $f(x) = x^2 + \frac{1}{x}$ is not a polynomial because of the $\frac{1}{x}$ term, which is the power of -1.

QUESTION 15

The sum of the solutions to the equation $\tan(2x) = -1$ for $-\pi \le x \le 0$ is

A $\frac{3\pi}{4}$

B $-\frac{3\pi}{4}$

C $-\frac{\pi}{2}$

D $-\frac{\pi}{8}$ and $-\frac{5\pi}{8}$

E $\frac{5\pi^2}{64}$

> $\tan(2x) = -1$ for $-\pi \le x \le 0$
>
> solve(tan(2·x)=−1|−π≤x≤0, x)
> $\left\{x=\frac{-5\cdot\pi}{8}, x=\frac{-\pi}{8}\right\}$
>
> Reference angle = $\frac{\pi}{4}$
>
> $2x = -\frac{\pi}{4}, -\pi - \frac{\pi}{4}$
>
> $x = -\frac{\pi}{8}, -\frac{5\pi}{8}$
>
> Sum $= -\frac{\pi}{8} - \frac{5\pi}{8} = -\frac{6\pi}{8} = -\frac{3\pi}{4}$

QUESTION 16

$\sin\left(\frac{11\pi}{6}\right)$ is equivalent to

A $\cos\left(\frac{11\pi}{6}\right)$

B $\sin\left(\frac{\pi}{6}\right)$

C $\cos\left(\frac{16\pi}{3}\right)$

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D $\cos\left(\dfrac{11\pi}{3}\right)$

E $\tan\left(\dfrac{11\pi}{3}\right)$

$\sin\left(\dfrac{11\pi}{6}\right) = -\dfrac{1}{2}$ and $\cos\left(\dfrac{16\pi}{3}\right) = -\dfrac{1}{2}$

ONE ANSWER PER LINE **USE PENCIL ONLY**

1	A	B	C	D	■	9	A	B	■	D	E
2	■	B	C	D	E	10	A	■	C	D	E
3	A	B	C	D	■	11	A	B	■	D	E
4	A	B	C	■	E	12	A	B	C	D	■
5	■	B	C	D	E	13	■	B	C	D	E
6	A	B	C	■	E	14	A	B	C	D	■
7	A	B	■	D	E	15	A	■	C	D	E
8	■	B	C	D	E	16	A	B	■	D	E

Section B: Extended-response questions

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 17

a If $3x^3 + ax^2 + 5x = 3(x + b)^3 + c$, find all possible values of a, b and c.

$3x^3 + ax^2 + 5x = 3(x + b)^3 + c = 3(x^3 + 3x^2b + 3xb^2 + b^3) + c$

So $3x^3 + ax^2 + 5x = 3x^3 + 9x^2b + 9xb^2 + 3b^3 + c$

Equating coefficients

$a = 9b$ (coefficients of x^2)

$5 = 9b^2$ (coefficients of x^1)

$0 = 3b^3 + c$ (coefficient of x^0)

expand(3·(x+b)³+c)

$3 \cdot x^2 + 3 \cdot b^2 + c^2 + 6 \cdot b \cdot x$

$\begin{cases} a=9b \\ 5=9b^2 \\ 0=c+3b^3 \end{cases} \Big|\, a, b, c$

$\left\{\left\{a=-3\cdot\sqrt{5}, b=\dfrac{-\sqrt{5}}{3}, c=\dfrac{5\cdot\sqrt{5}}{9}\right\}, \left\{a=3\cdot\sqrt{5}, b=\dfrac{\sqrt{5}}{3}, c=\dfrac{-5}{\ldots}\right.\right.$

Hence $a = \pm 3\sqrt{5}$, $b = \mp\dfrac{\sqrt{5}}{3}$, $c = \pm\dfrac{5\sqrt{5}}{9}$

$\therefore 3x^3 + ax^2 + 5x = 3\left(x - \dfrac{\sqrt{5}}{3}\right)^3 + \dfrac{5\sqrt{5}}{9}$

Or

$3x^3 + ax^2 + 5x = 3\left(x + \dfrac{\sqrt{5}}{3}\right)^3 - \dfrac{5\sqrt{5}}{9}$

4 marks

b Letting $f_1(x) = 3(x + b)^3 + c$, and using values from part **a**, where $b < 0$ and $c > 0$, find the x-coordinates of the point(s) where $f(x) = 0$.

$b < 0$ and $c > 0$

$f(x) = 3\left(x - \dfrac{\sqrt{5}}{3}\right)^3 + \dfrac{5\sqrt{5}}{9}$

$f(x) = 0$ gives

$\left(x - \dfrac{\sqrt{5}}{3}\right)^3 = -\dfrac{5\sqrt{5}}{27}$

$x = -\dfrac{\sqrt{5}}{3} + \dfrac{\sqrt{5}}{3}$

$x = 0$

1 mark

c Letting $f_2(x) = 3(x + b)^3 + c$, and using values from part **a**, where $b > 0$ and $c < 0$, find the x-coordinates of the point(s) where $f(x) = 0$.

$b > 0$ and $c < 0$

$f(x) = 3\left(x + \dfrac{\sqrt{5}}{3}\right)^3 - \dfrac{5\sqrt{5}}{9}$

$f(x) = 0$ gives

$\left(x + \dfrac{\sqrt{5}}{3}\right)^3 = \dfrac{5\sqrt{5}}{27}$

$x = \dfrac{\sqrt{5}}{3} - \dfrac{\sqrt{5}}{3}$

$x = 0$

1 mark

d Hence, find where $f_1(x) = f_2(x)$.

Point of intersection at (0, 0).

1 mark

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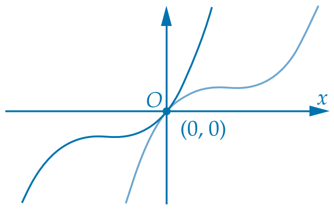
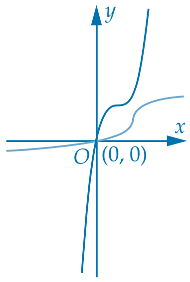
e Sketch the graphs of $f_1(x)$ and $f_2(x)$ on the same axes, labelling point(s) of intersection.



3 marks

f Sketch the inverse of graph $f_1(x)$ on the same axes, labelling point(s) of intersection.



2 marks
(Total: 12 marks)

QUESTION 18

a Find the general solution to the equation $2\cos(2x) = 1$.

$\cos(2x) = \frac{1}{2}$

$2x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$

$2x = 2n\pi \pm \frac{\pi}{3}$

$x = n\pi \pm \frac{\pi}{6}, n \in \mathbf{Z}$

2 marks

b Hence, give the first four positive solutions to $\cos(2x) = \frac{1}{2}$.

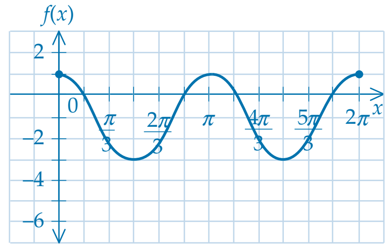
N	0	1	2
$\frac{\pi}{6} + n\pi$	$\frac{\pi}{6}$	$\frac{7\pi}{6}$	~~$\frac{13\pi}{6}$~~
$-\frac{\pi}{6} + n\pi$	~~$\frac{\pi}{6}$~~	$\frac{5\pi}{6}$	$\frac{11\pi}{6}$

Answer: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

4 marks

c Sketch the graph of $f(x) = 2\cos(2x) - 1$ for the domain $x \in [0, 2\pi]$, showing the coordinates of the axial intercepts.

$\left(\frac{\pi}{6}, 0\right), \left(\frac{5\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$



3 marks
(Total: 9 marks)

QUESTION 19

The temperature of a cup of tea cools according to the rule $T = T_0 \times 2^{-kt}$, where T is the temperature in degrees Celsius and t is time in hours. The original temperature of the cup of tea is 90°C.

a What is T_0?

$T_0 = 90°C$

1 mark

It takes 30 minutes for the temperature to halve.

b What fraction of the original temperature is the temperature of the cup of tea after 60 minutes?

$T = T_0 \times 2^{-kt}$

It takes 30 minutes for the temperature to halve, so $t = \frac{1}{2}$ and $T = \frac{1}{2}T_0$.

$\Rightarrow \frac{1}{2}T_0 = T_0 \times 2^{-0.5k}$

$\Rightarrow \frac{1}{2} = 2^{-0.5k}$

$2^{-1} = 2^{-0.5k}$

This gives $\frac{1}{2}k = 1$

$k = 2$

After 60 minutes, i.e. at $t = 1$

$T = T_0 \times 2^{-2 \times 1}$

$T = T_0 \times \frac{1}{4}$

The fraction of the original temperature after 60 minutes $= \frac{1}{4}$

3 marks

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c What is the temperature of the cup of tea after 60 minutes?

Using the result from part **b**,

$T = 90 \times \frac{1}{4} = 22.5°C$

1 mark

d A different drink's temperature follows the formula $T = T_0 \times 2^{-kt} + 20$, where T is the temperature in degrees Celsius and t is time in hours. It takes 20 minutes for the temperature to halve and the original temperature of the cup of tea is 90°C. Find the value of k in this case, giving your answer correct to 2 decimal places.

$T = T_0 \times 2^{-kt} + 20$

$t = 0$, gives $T = T_0 + 20$

Original temperature of the cup of tea is 90°C.

$90 = T_0 + 20$, so $T_0 = 70$

It takes 20 minutes for the temperature to halve,

so $t = \frac{1}{3}$.

$\Rightarrow \frac{1}{2} \times 90 = 70 \times 2^{-k\frac{1}{3}} + 20$

$45 = 70 \times 2^{-k\frac{1}{3}} + 20$

This gives $k \approx 4.46$

$\text{solve}\left(45=70\cdot 2^{-\frac{k}{3}}+20, k\right)$

$\{k=4.456280482\}$

2 marks

The formula is changed to suit another drink.
The graph of $T = T_0 \times 2^{-kt}$ has a sequence of transformations applied to it, in the order given.

- The graph is translated 15 units in the negative direction of the t-axis.
- It is then translated 7 units in the negative direction of the T-axis
- It is then dilated by a factor of 5 from the t-axis
- It is then dilated by a factor of $\frac{1}{2}$ from the T-axis
- The graph is then reflected in the T-axis

e After all these transformations are applied, what is the new rule for $T(t)$?

$T = T_0 \times 2^{-kt}$	
Step 1: translated 15 units in the negative direction of the t-axis	$T_1 = T_0 \times 2^{-k(t+15)}$
Step 2: translated 7 units in the negative direction of the T-axis	$T_2 = T_0 \times 2^{-k(t+15)} - 7$
Step 3: dilated by a factor of 5 from the t-axis	$T_3 = 5T_0 \times 2^{-k(t+15)} - 35$
Step 4: dilated by a factor of $\frac{1}{2}$ from the T-axis	$T_4 = 5T_0 \times 2^{-k(2t+15)} - 35$
Step 5: reflected in the T-axis	$T_5 = 5T_0 \times 2^{-k(-2t+15)} - 35$
New rule for $T(t)$	$T_5 = 5T_0 \times 2^{k(2t-15)} - 35$

2 marks
(Total: 9 marks)

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