The Mathematical Association of Victoria
MATHEMATICAL METHODS (CAS)
SOLUTIONS: Trial Exam 2014

Written Examination 2

SECTION 1

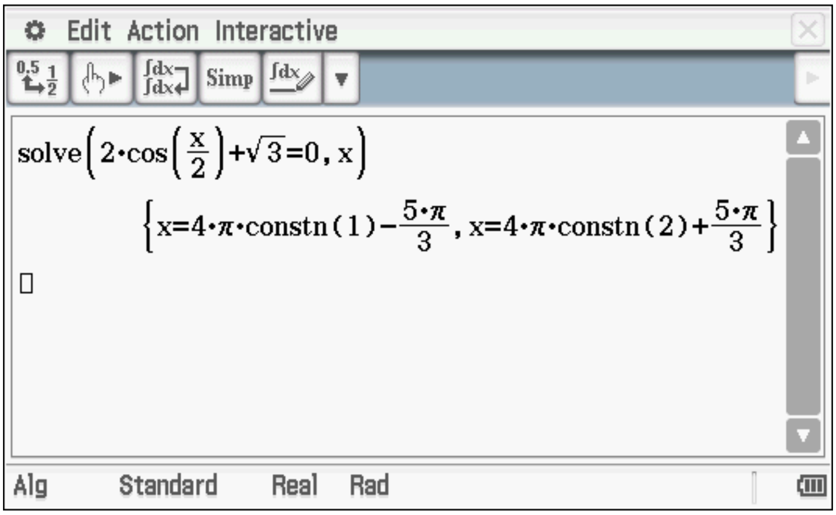
1. C 2. D 3. C 4. D 5. B 6. A 7. B 8. A 9. A 10. E 11. E

12. D 13. C 14. C 15. A 16. B 17. E 18. B 19. C 20. D 21. A 22. E

Question 1

$2\cos\left(\frac{x}{2}\right)+\sqrt{3}=0$

$x=4\pi n\pm\frac{5\pi}{3}, n\in Z$ **C**



Question 2

Range of $y=-\sin(a(x+b))$ is $[-1,1]$

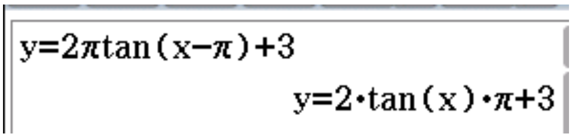
Range of $y=-4\sin(a(x+b))$ is $[-4,4]$

Range of the graph of $f(x)=-4\sin(a(x+b))-b$ is $[-4-b,4-b]$ **D**

Question 3

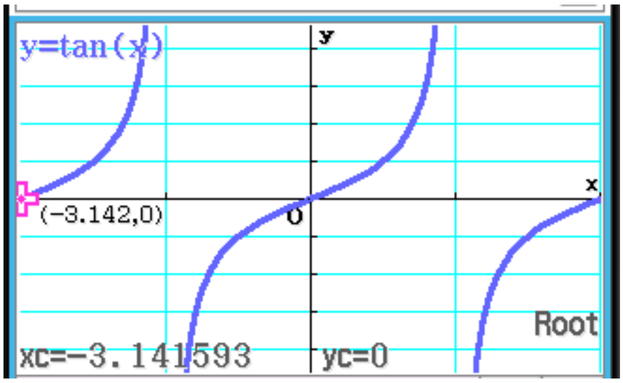
Consider $f:[-\pi, 2\pi] \to R$, where $f(x) = 2\pi \tan(x-\pi) + 3$.

The horizontal translation in a positive direction of π will make no difference to the vertical asymptotes of the graph of $y = 2\pi \tan(x) + 3$.



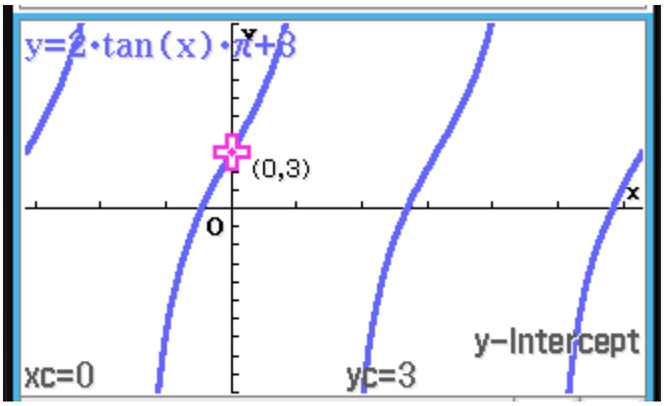
where the constants $2\pi, 3$ do not alter asymptotes

The graph of $y = \tan(x)$ from $-\pi$ to π



shows expected asymptotes at $x = \pm\frac{\pi}{2}$.

For domain $[-\pi, 2\pi]$ we add the asymptote $x = \frac{3\pi}{2}$



The equations of asymptotes are $x = \frac{3\pi}{2}, x = \pm\frac{\pi}{2}$ **C**

Question 4

$y = \sin(x)$

Then $y = \sin\left(x + \frac{\pi}{2}\right)$

Then $y = \sin\left(x + \frac{\pi}{2}\right) + \frac{1}{2}$

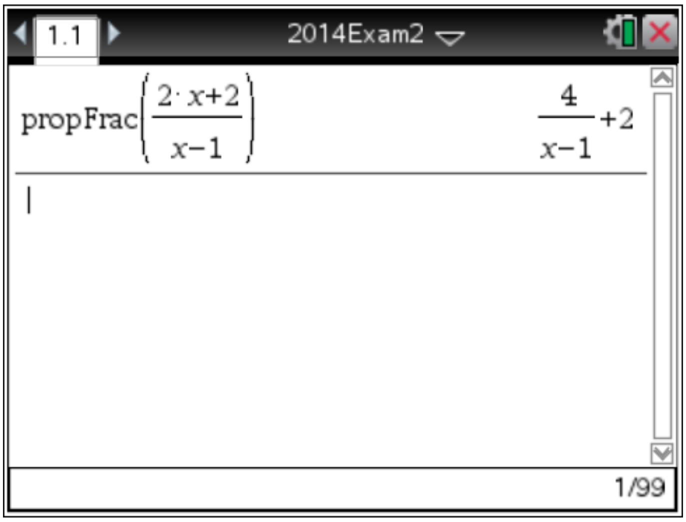
Then $y = \sin\left(\frac{x}{2} + \frac{\pi}{2}\right) + \frac{1}{2}$

Using complementary functions this now becomes $y = \cos\left(\frac{x}{2}\right) + \frac{1}{2}$ **D**

Question 5

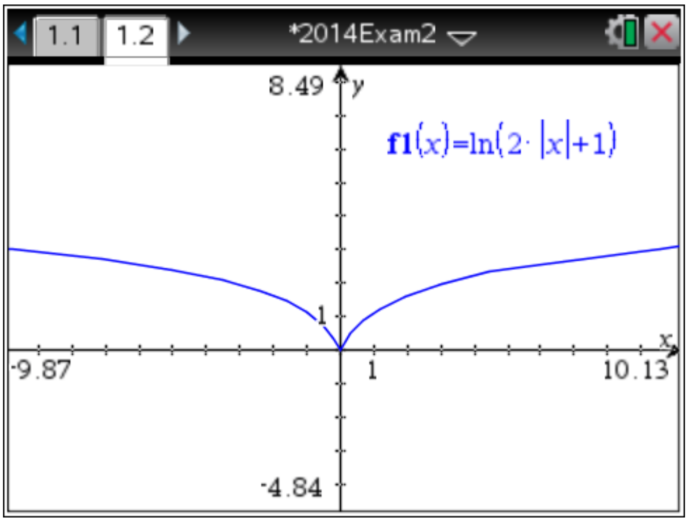
$f(x) = \frac{2x+2}{x-1} = 2 + \frac{4}{x-1}$

Domain $R \setminus \{1\}$, Range $R \setminus \{2\}$ **B**



Question 6

The graph of $f(x) = \log_e(2|x|+1)$ has no asymptotes. **A**



Question 7

$(x-1)(x^2+ax+b)=0$

$x = 1$ is one solution

$x=\dfrac{-a\pm\sqrt{a^2-4b}}{2}$ will give another solution if

$a^2-4b=0,\ a^2=4b$

$x=-\dfrac{a}{2}$ is the solution

$-\dfrac{a}{2}\neq 1,\ a\neq -2$ **B**

Question 8

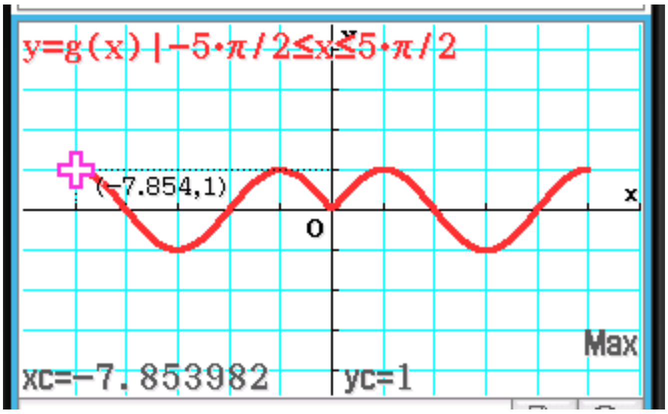
$h(x)=g\left(f(x)\right)=\left(\sqrt{x+2}\right)^2+5=x+7$

Dom(h) = Dom(f), $[-2,\infty)$ **A**

Question 9

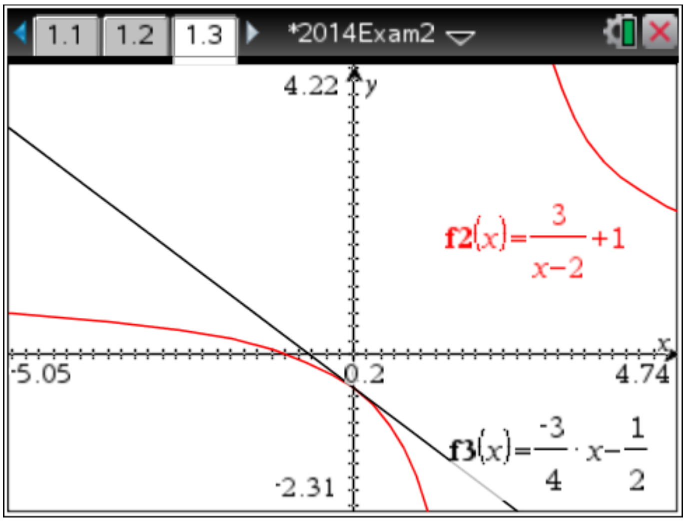
$g:\left[-\dfrac{5\pi}{2},\dfrac{5\pi}{2}\right]\to R$, where $g(x)=\sin|x|$, is **not** differentiable at cusp and endpoints.

At $x=\pm\dfrac{5\pi}{2}$, $x=0$ **A**



Question 10

The tangent line will be above the graph of $f(x) = \dfrac{3}{x-2} + 1$ when $x < 2$. **E**



Question 11

$g: R \setminus \{-1\} \to R,\ g(x) = \dfrac{1}{x+1}$

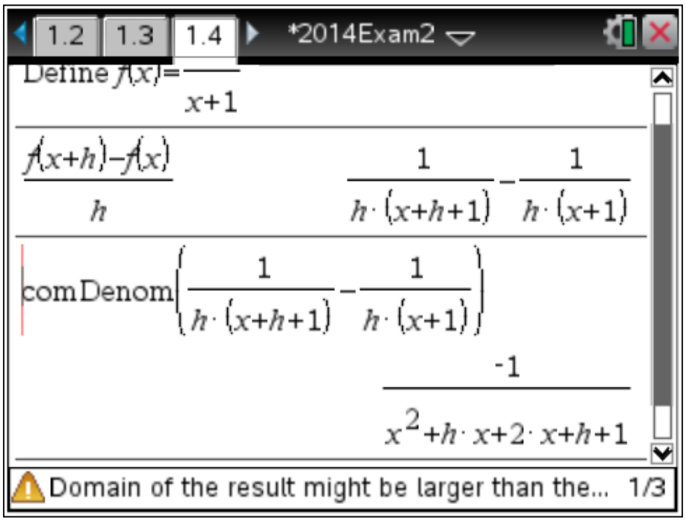
Using First Principles

$$\lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{x+1-(x+h+1)}{h(x+1)(x+h+1)}\right)$$

$$= \lim_{h \to 0}\left(\frac{-1}{x^2+xh+2x+h+1}\right)$$ **E**



Question 12

f is the graph of $h(x)=(x+1)^3(x-1)$ translated 3 units up.

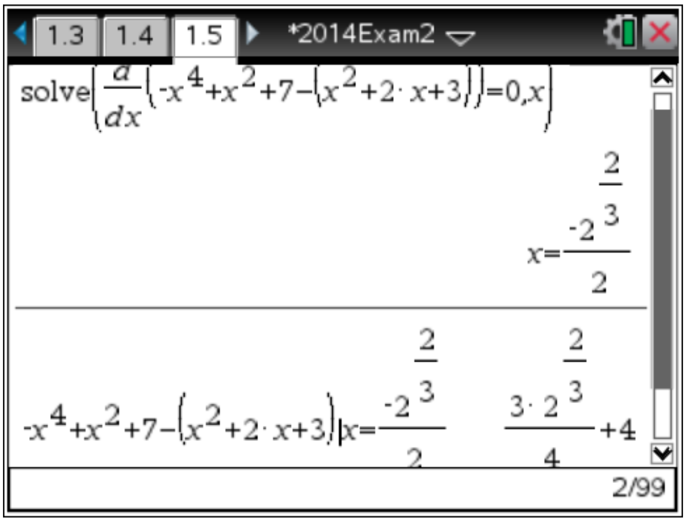
$f(x)=(x+1)^3(x-1)+3$ **D**

Question 13

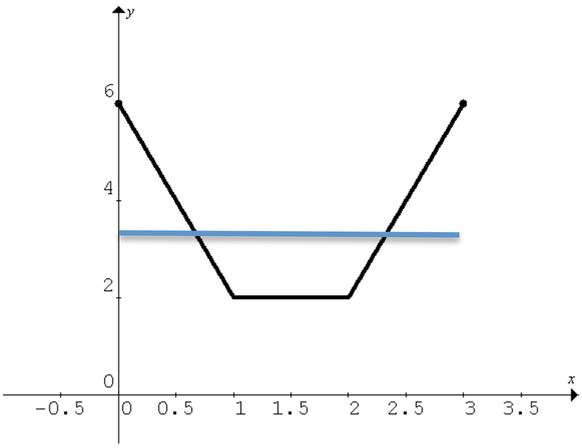
$g-f=-x^4+x^2+7-\left(x^2+2x+3\right)=-x^4+x^2-2x+4$

Solve $\frac{d}{dx}\left(-x^4+x^2-2x+4\right)=0$ for x

Maximum value is $\frac{3}{2^{\frac{4}{3}}}+4$ **C**



Question 14



The area above the line $y = \frac{10}{3}$ equals the area below the line.

The average value is $\frac{10}{3}$. **C**

Question 15

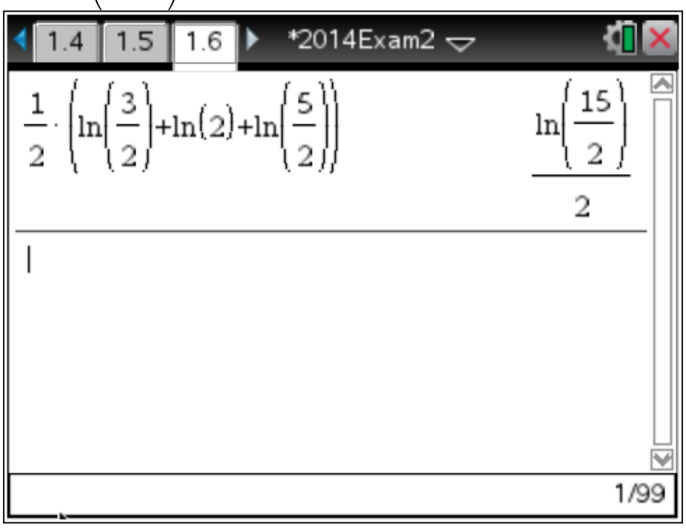
$y = \log_e(x-1)$

Area = $\frac{1}{2}\left(\log_e(1.5) + \log_e(2) + \log_e(2.5)\right)$

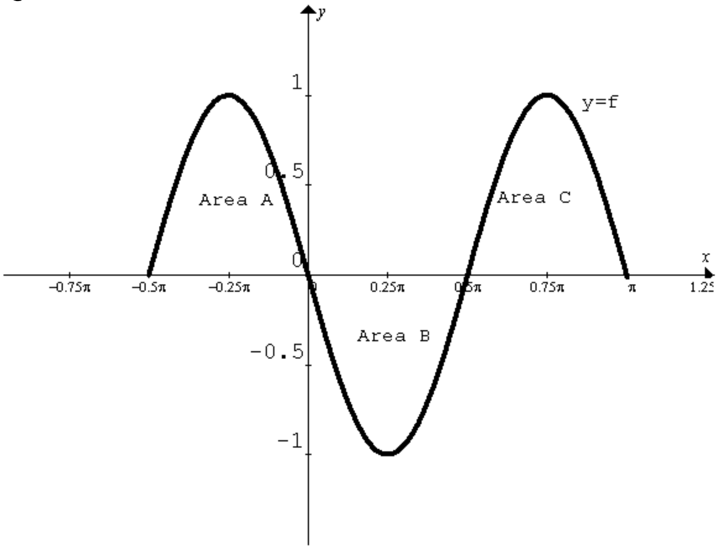
$= \frac{1}{2}\log_e\left(\frac{3}{2} \times 2 \times \frac{5}{2}\right)$

$= \frac{1}{2}\log_e\left(\frac{15}{2}\right)$

$= \log_e\left(\sqrt{\frac{15}{2}}\right)$ **A**

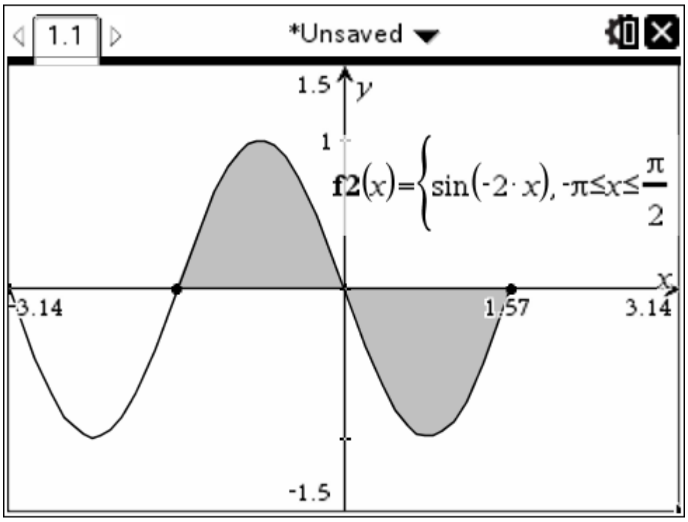


Question 16

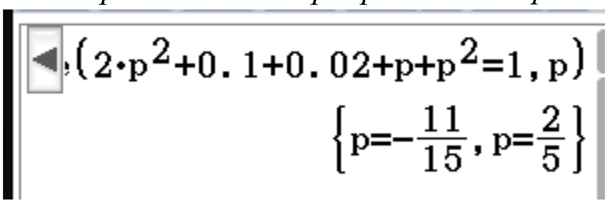


The graph of f has been reflected in the x and y axes.
Area A = Area B

Hence $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-f(-x)\right)dx = 0$ **B**



Question 17
To find p.
Solve $2p^2 + 0.1 + 0.02 + p + p^2 = 1$ where $p > 0$

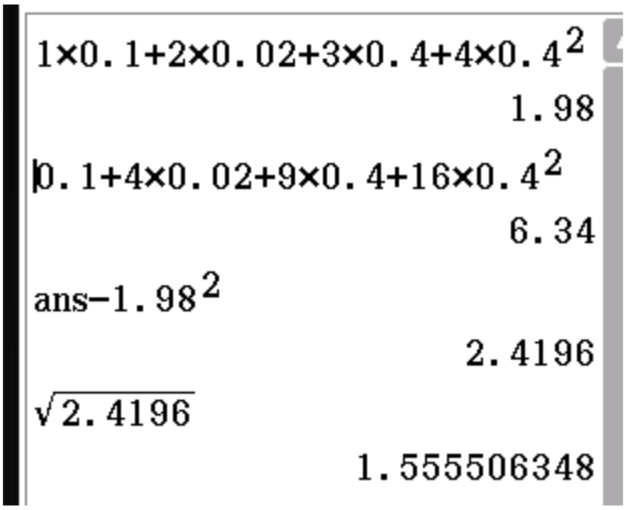


gives $p = \frac{2}{5} = 0.4$

$E(X) = 1.98$

$Var(X) = 2.4196$

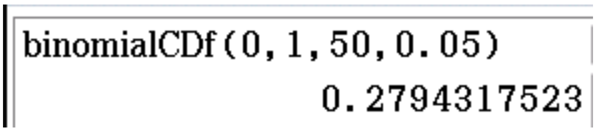
The standard deviation of X is closest to 1.5555 **E**



Question 18

Bi(n, p) = Bi(50, 0.05)

$\Pr(X < 2) = \Pr(X \leq 1) = 0.2794$ **B**

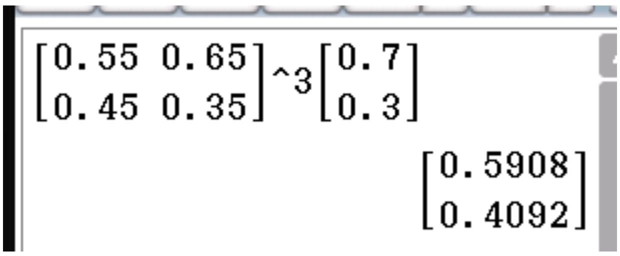


Question 19

$$S_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \begin{matrix} \leftarrow W \\ \leftarrow L \end{matrix}$$

$$T = \begin{bmatrix} 0.55 & 0.65 \\ 0.45 & 0.35 \end{bmatrix}$$

Need $S_3 = T^3 S_0$



$$S_3 = \begin{bmatrix} 0.5908 \\ 0.4092 \end{bmatrix} \begin{matrix} \leftarrow W \\ \leftarrow L \end{matrix}$$ **C**

Question 20

P and Q mutually exclusive when $\Pr(P \cap Q) = 0$

Given $\Pr(P \cap Q) + \frac{1}{2}\Pr(P' \cap Q) = \frac{3}{10}$

So

$$0 + \frac{1}{2}\Pr(P' \cap Q) = \frac{3}{10}$$

$$\therefore \Pr(P' \cap Q) = \frac{3}{5}$$

	P	P'	
Q	0	$\frac{3}{5}$	$\frac{3}{5}$
Q'	$\Pr(P \cap Q')$	$\Pr(P' \cap Q')$	$\frac{2}{5}$
	—	—	1

$\Pr(P \cap Q') + \Pr(P' \cap Q') = \frac{2}{5}$ **D**

Question 21

$N(\mu, \sigma^2)$

$\Pr(X > 80) = \frac{85}{1000}$ and $\Pr(X < 10) = 0.01$

$\Pr(Z > 1.37220) = \frac{85}{1000}$ and $\Pr(Z < -2.32635) = 0.01$

```
invNormCDf("R", 85/1000, 1, 0)
                    1.372203809
invNormCDf("L", 0.01, 1, 0)
                   -2.326347874
```

Solve using $z = \frac{x - \mu}{\sigma}$

$$\text{solve}\left(\begin{cases} 1.37220=\dfrac{80-m}{s} \\ -2.32635=\dfrac{10-m}{s} \end{cases}\right|_{m,s}$$

{m=54.02928175, s=18.9263

Giving μ = 54.029 **A**

Question 22

$$f(x)=\begin{cases} |\sin(x)| & \dfrac{3\pi}{2}\le x\le 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

Variance $=\int_{\frac{3\pi}{2}}^{2\pi} x^2 f(x)dx-\left(E(X)\right)^2$

Variance $=\int_{\frac{3\pi}{2}}^{2\pi} x^2 f(x)dx-\left(\int_{\frac{3\pi}{2}}^{2\pi} xf(x)dx\right)^2=0.14159$ **E**

$$f(x)=\begin{cases} |\sin(x)|, & \dfrac{3\cdot\pi}{2}\le x\le 2\cdot\pi \\ 0, & \dfrac{3\cdot\pi}{2}\ge x\ge 2\cdot\pi \end{cases}$$

done

$$\int_{\frac{3\cdot\pi}{2}}^{2\cdot\pi} x^2\cdot f(x)dx$$

28.05363964

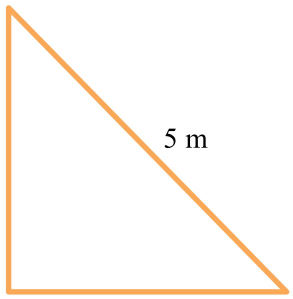
$$\text{ans}-\left(\int_{\frac{3\cdot\pi}{2}}^{2\cdot\pi} x\cdot f(x)dx\right)^2$$

0.1415926519

SECTION 2

Question 1 (10 marks)

a. i.



$h^2 + b^2 = 25$

$h = \sqrt{25 - b^2}\,,\ h > 0$

$\frac{dh}{db} = \frac{-2b}{2\sqrt{25-b^2}} = \frac{-b}{\sqrt{25-b^2}}$ **1M**

$\frac{db}{dt} = 2$ m/s

$\frac{dh}{dt} = \frac{dh}{db} \times \frac{db}{dt}$ **1M**

$\frac{dh}{dt} = \frac{-b}{\sqrt{25-b^2}} \times 2$

$b = 3$ m

$\frac{dh}{dt} = \frac{-3}{\sqrt{25-9}} \times 2 = \frac{-6}{\sqrt{16}} = \frac{-3}{2}$ m/s

The ladder is moving $\frac{3}{2}$ m/s down the wall. **1A**

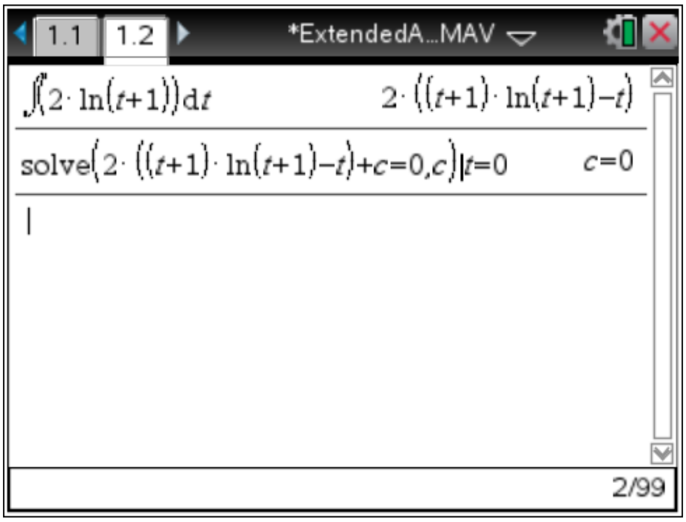
ii. $h = 2\sqrt{6}$ m, $h^2 = 24,\ b = 1$ m, $\frac{db}{dt} = 2$ m/s, $t = 2$ s **1A**

b. i. $A = \int \left(2\log_e(t+1)\right) dt$

$A = 2(t+1)\log_e(t+1) - 2t + c$ **1M**

$A(0) = c = 0$

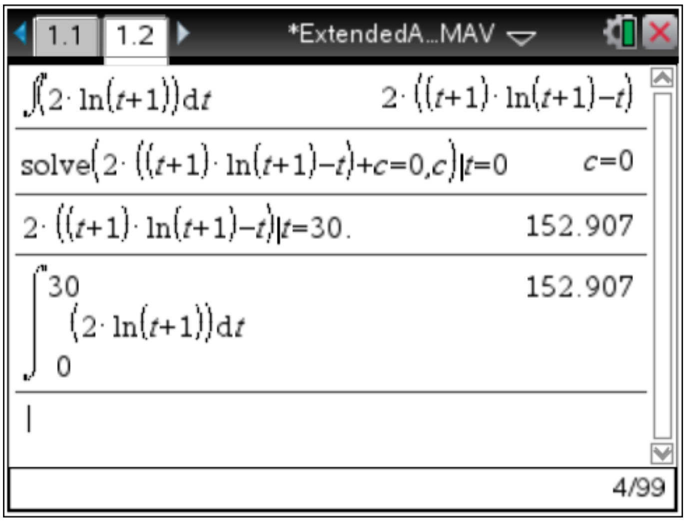
$A = 2(t+1)\log_e(t+1) - 2t$ **1A**



ii. $A(30) = 152.9\ \text{m}^2$ correct to one decimal place **1M 1A**

OR

$A = \int_0^{30} \left(2\log_e(t+1)\right)dt = 152.9\ \text{m}^2$ correct to one decimal place **1M 1A**



iii. Solve $A = 2(t+1)\log_e(t+1) - 2t = 202.907...$ for t **1M**

$t = 37.0...$ min

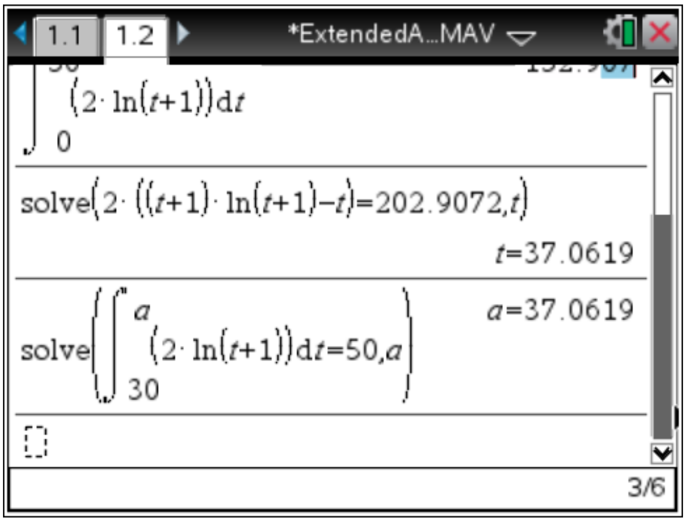
It takes an extra 7 minutes **1A**

OR

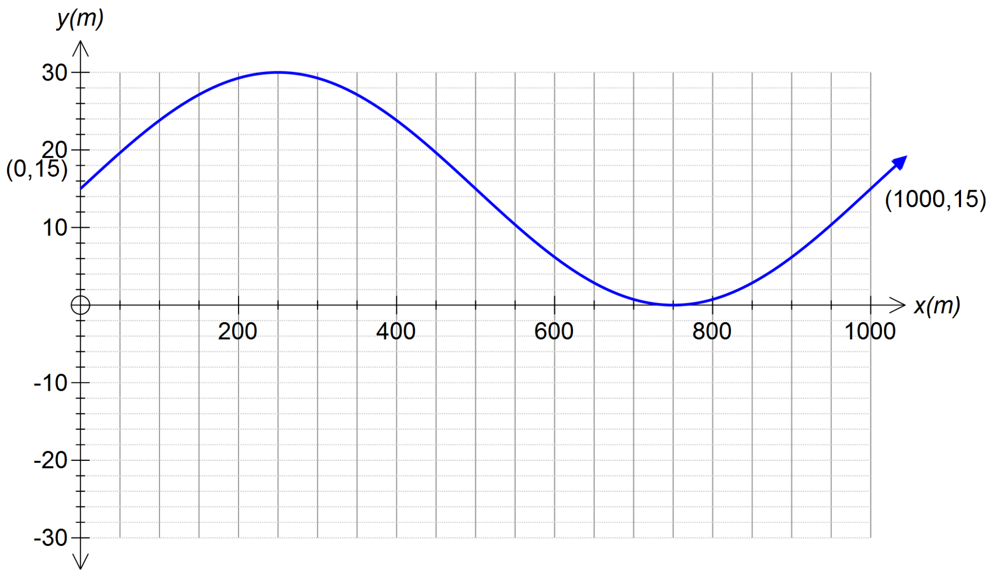
Solve $\int_{30}^{a} \left(2\log_e(t+1)\right)dt = 50$ for a **1M**

$t = 37.0...$ min

It takes an extra 7 minutes **1A**



Question 2 (17 marks)



a. **i.** a is amplitude so $a = 15$ **1A**

ii. c is vertical translation so $c = 15$ **1A**

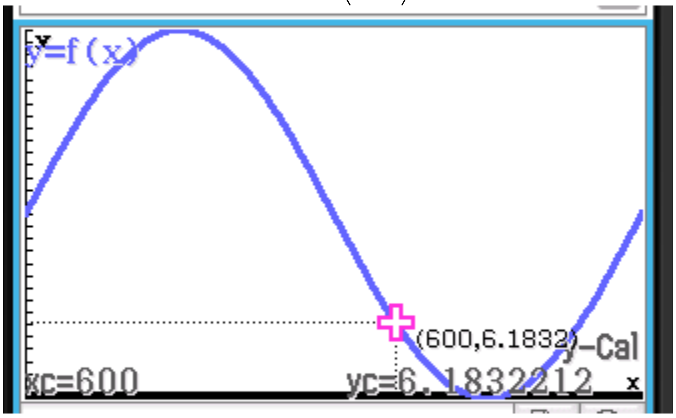
iii. period $= 1000 = \dfrac{2\pi}{\pi/b}$

giving $1000 = 2b$

$b = 500$. **1M**

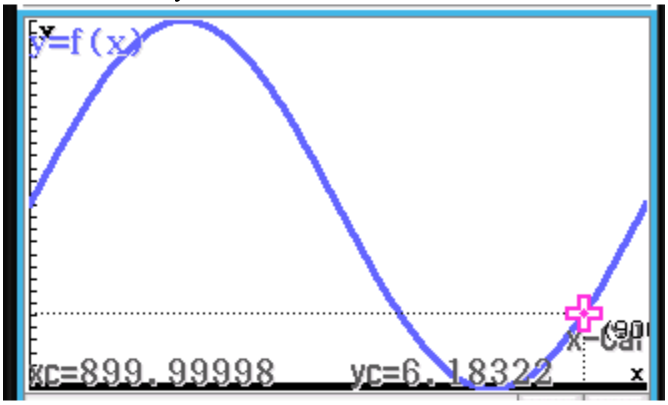
b. $y = 15 + 15\sin\left(\dfrac{\pi x}{500}\right)$ **1A**

c. **i.** $y(600) = 15 + 15\sin\left(\dfrac{6\pi}{5}\right)$



Depth of the lake 600 metres from the café is 6.18 metres (or 618 **centimetres)** **1A**

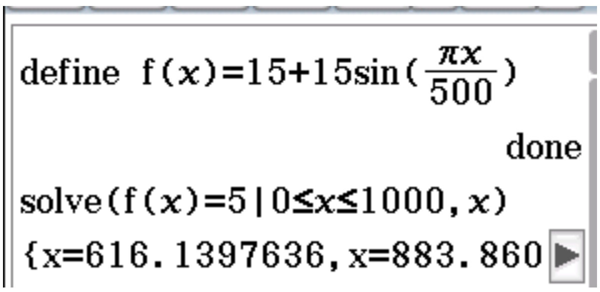
ii. Solve y = 6.18322 for the domain $0 \le x \le 1000$

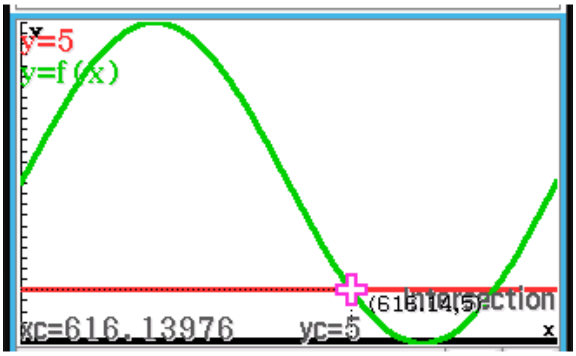


Depth of water found at x = 600 and x = 900 to nearest **metre.** **1A**

Compete when $y \ge 5$

d. **i.** Solve $y \ge 5$ **1M**

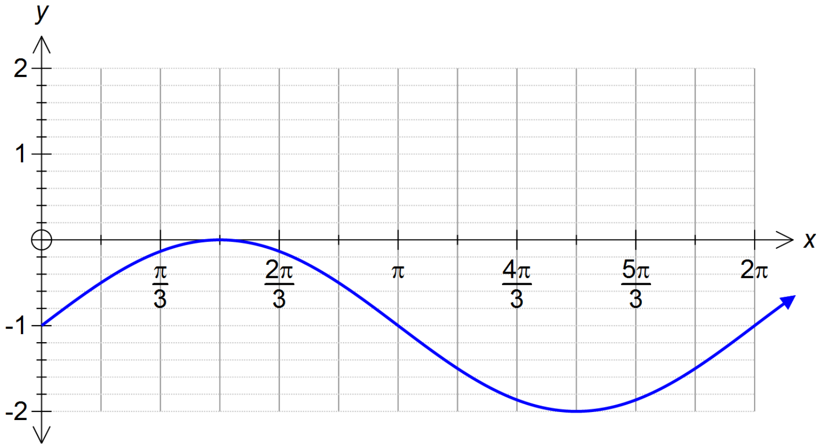




Depth of water suitable for $x \in [0, 616] \cup [884, 1000]$ to the nearest **metre** **1A**

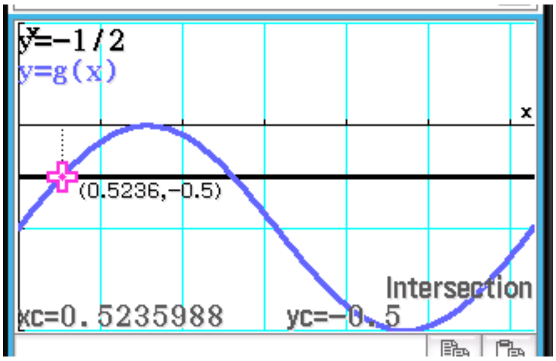
ii. Length of race, to nearest **metre,** before it is interrupted by shallow water is 616 metres. **1A**

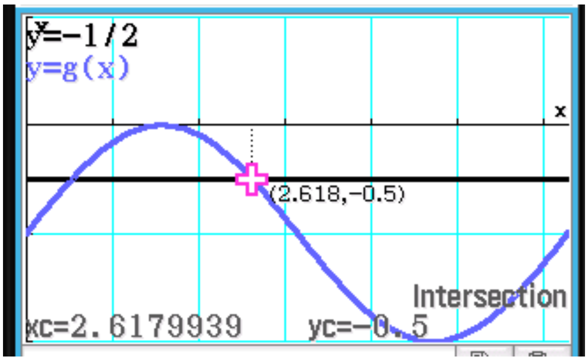
$f(x) = \sin(x) - 1$ for $0 \le x \le 2\pi$



e. zero depth at $\left(\frac{\pi}{2}, 0\right)$ **1A**

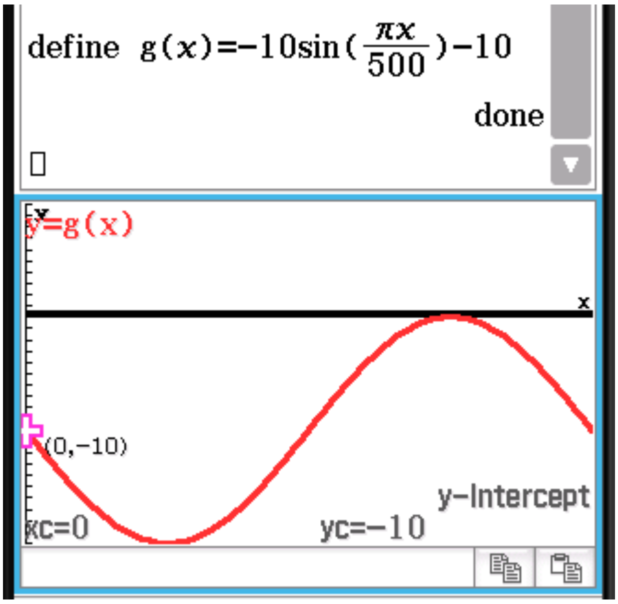
Stand in water when $y > -0.5$ **1M**





f. Tourists can walk at distance from the café, to nearest **centimetre,** for $x \in (0.52, 2.62)$ **1A**

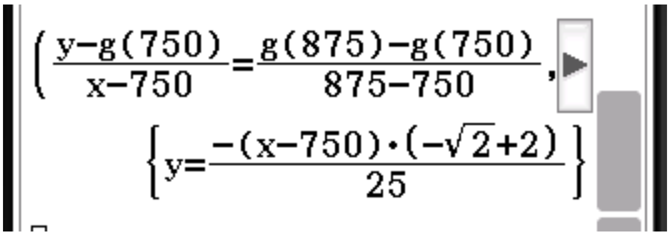
$$g(x) = -10\sin\left(\frac{\pi x}{500}\right) - 10 \text{ for } 0 \le x \le 1000$$



g. Straight line, between $x = 750$ and $x = 875$.

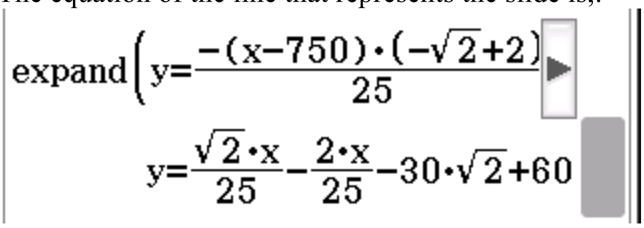
Using the equation of a straight line $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

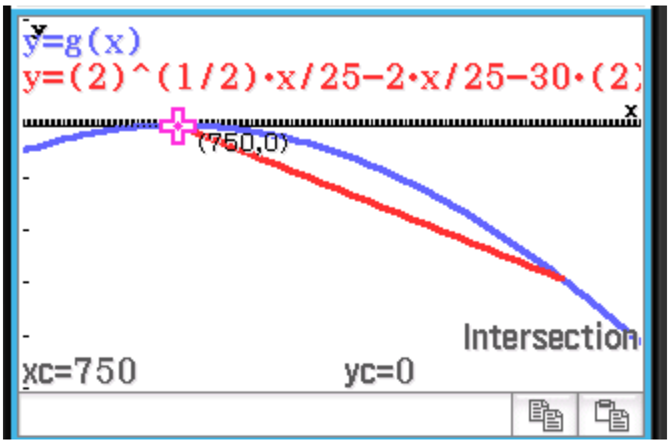
where $\dfrac{y - y(750)}{x - 750} = \dfrac{y(875) - y(750)}{875 - 750}$ **1M**



1M

The equation of the line that represents the slide is,:





graph of the water slide

Simplified equation of water slide $y = -\frac{x}{25}\left(2-\sqrt{2}\right) - 30\sqrt{2} + 60$ **1A**

h. Transform $y = 15 + 15\sin\left(\frac{\pi x}{500}\right)$ to the image graph $g(x) = -10\sin\left(\frac{\pi x}{500}\right) - 10$

Step 1: Dilate from the x-axis by a factor of $\frac{2}{3}$ units **1A**

Step 2: Reflect in the x-axis **1A**

Question 3 (16 marks)
Rowing Ergo (R), Cycle (C), Step (S).
60 staff members, all take part in at least one class.
Pr (R) = 0.25
Pr (C) = 0.55
Pr (S) = 0.75
R, C and S are independent.

a. **i.** Independent $\therefore \Pr(R \cap S) = \Pr(R) \times \Pr(S)$

$\Pr(R \cap S) = 0.25 \times 0.75 = 0.1875$ or $\frac{3}{16}$ **1A**

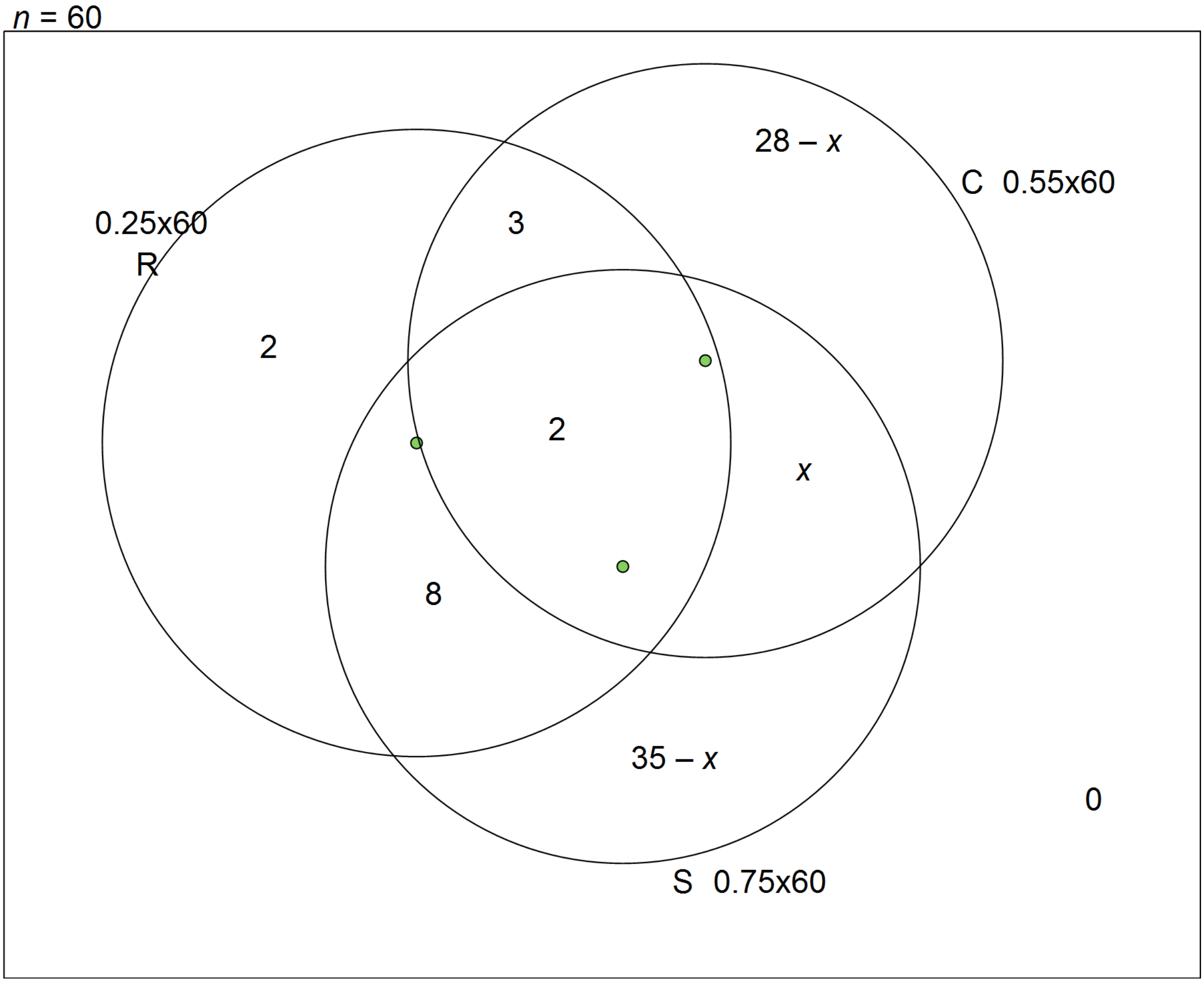
ii. $\Pr(R \mid S) = \frac{\Pr(R \cap S)}{\Pr(S)}$

$\Pr(R \mid S) = \frac{0.1875}{0.75} = 0.25$ or $\frac{1}{4}$ **1A**

b. **i.** $\Pr(R \cap S \cap C') = 0.25 \times 0.75 \times 0.45$ **1M**

$\Pr(R \cap S \cap C') = 0.084375$ or $\frac{27}{320}$ **1A**

ii. Venn diagram.



1M
From Venn diagram we get 2 + 3 + 2 + 8 + 28 –x + x + 35-x = 60
$x = 18$
10 Staff members take C only **1A**

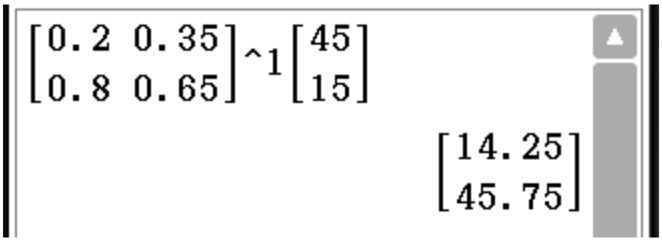
60 staff members
If a staff member takes Step there is a probability of 20% of them taking Step the next day.
If a staff member takes Cycle there is a probability 35% of them taking Step Class the next day.
No R classes are offered.

On Monday 15 take Cycle Class.

$$T = \begin{bmatrix} 0.2 & 0.35 \\ 0.8 & 0.65 \end{bmatrix}$$

$$S_0 = \begin{bmatrix} 45 \\ 15 \end{bmatrix}$$

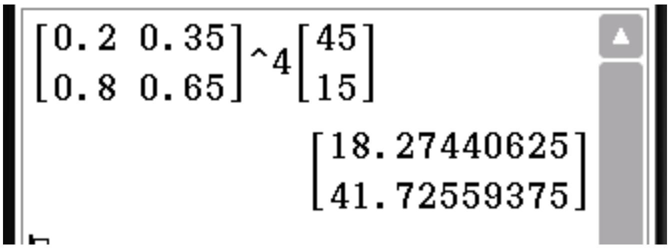
c. **i.** Cycle class on Tuesday using $S_1 = T^1 S_0$



$$\begin{bmatrix} 0.2 & 0.35 \\ 0.8 & 0.65 \end{bmatrix} \begin{bmatrix} 45 \\ 15 \end{bmatrix} = \begin{bmatrix} 14.25 \\ 45.75 \end{bmatrix}$$ **1M**

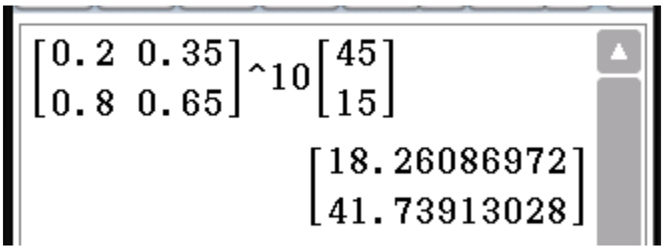
46 staff take Cycle class on Tuesday **1A**

ii. Step class on Friday using $S_4 = T^4 S_0$



18 staff take Step class on Friday **1A**

iii. Cycle and Step using $S_\infty = T^\infty S_0$



18 staff take Step class and 42 take Cycle class long term **2A**

10 Mathematics staff members. Independent events

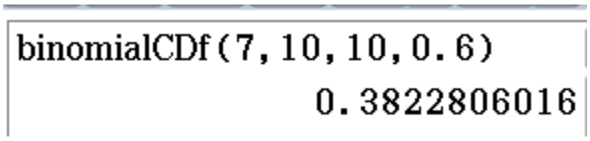
Pr (C) = 0.2

Pr(S) = 0.6

d. $\Pr(S \cap S \cap C \cap C \cap C) = 0.6^2 \times 0.2^3$

$\Pr(S \cap S \cap C \cap C \cap C) = 0.00288$ or $\dfrac{9}{3125}$ **1A**

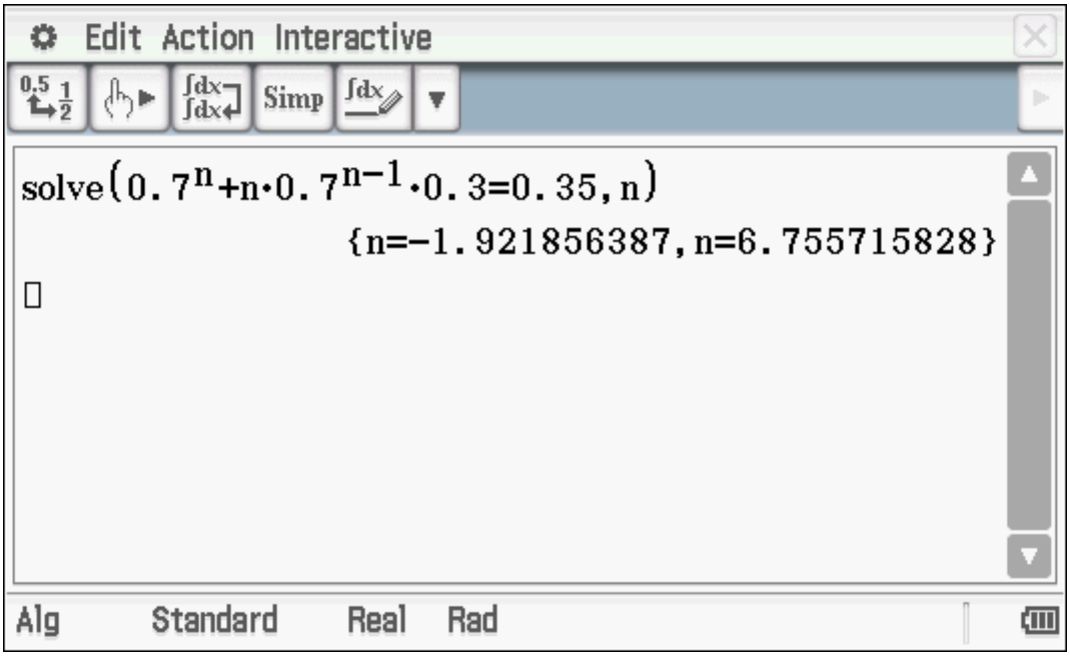
e. $\text{Bi}(n, p) = \text{Bi}(10, 0.6)$ **1M**



$\Pr(X > 6) = \Pr(X \geq 7) = 0.3823$ **1A**

f. Now Pr (C) = 0.3

$\text{Bi}(n, p) = \text{Bi}(n, 0.3)$
$\Pr(X \geq 2) > 0.65$
Let $\Pr(X \geq 2) = 0.65$
$1 - \left[\Pr(X = 0) + \Pr(X = 1)\right] = 0.65$
$\therefore \Pr(X = 0) + \Pr(X = 1) = 0.35$ **1M**
${}^{n}C_0 q^n p^0 + {}^{n}C_1 q^{n-1} p^1 = 0.35$
Where q = 0.7, p = 0.3.
${}^{n}C_0 0.7^n 0.3^0 + {}^{n}C_1 0.7^{n-1} 0.3^1 = 0.35$
Giving
$0.7^n + n \times 0.7^{n-1} \times 0.3^1 = 0.35$. Solve for *n*.



We need 7 Science staff members. **1A**

Question 4 (15 marks)

a. $A = -1$

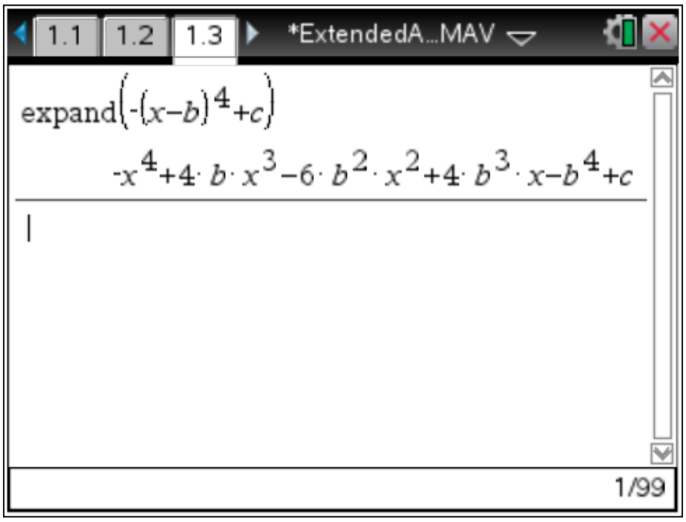
Expand $-(x-B)^4 + C = -x^4 + 4Bx^3 - 6B^2x^2 + 4B^3x - B^4 + C$ **1M**

Equate coefficients

$4B = 8,\ B = 2$

$-B^4 + C = -15,\ -16 + C = -15,\ C = 1$

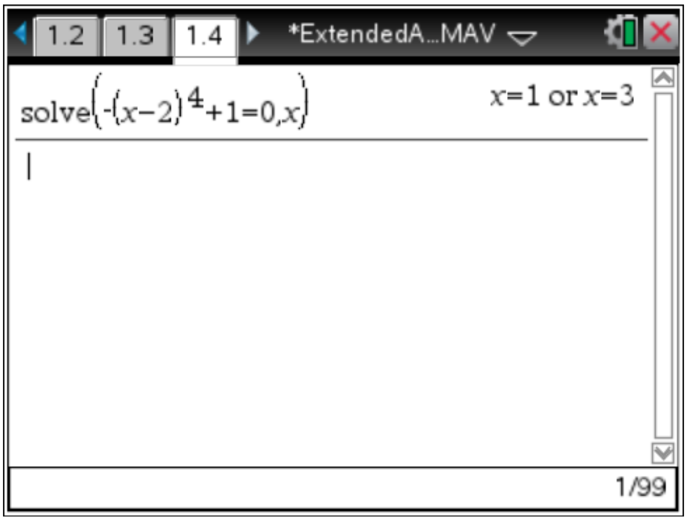
$f(x) = -(x-2)^4 + 1$ **1A**



b. $f(1) = -(1-2)^4 + 1 = 0$ **1M**

Could use the turning point (2, 1)

Solve $f(x) = -(x-2)^4 + 1 = 0$ for x, $x = 3,\ b = 3$ **1A**



c. **i.** $[1,\ 3]$ **1A**

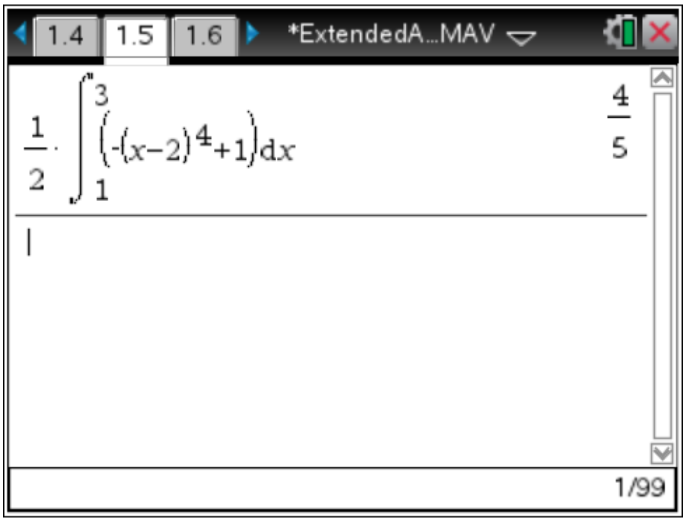
ii $(1,\ 3)$ **1A**

d. $c = 2$ **1A**

e. g is not differentiable at $x = 3$ and therefore is not differentiable for all values of x. **1A**

f. i. $k = \frac{1}{2}\int_1^3 (f(x))dx$ **1M**

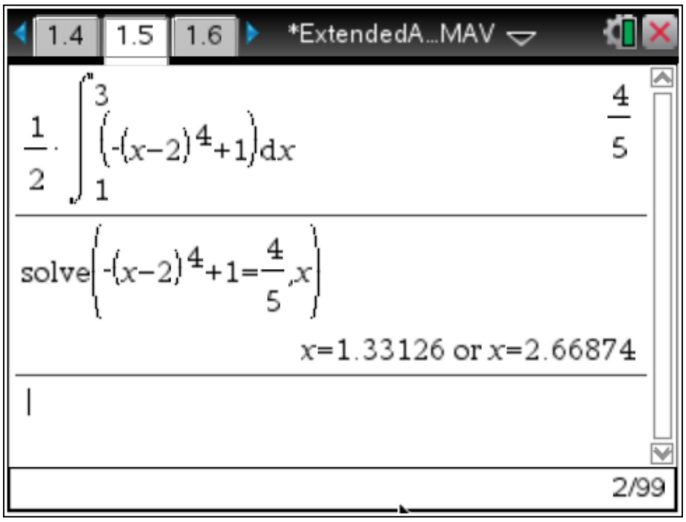
$= \frac{4}{5}$ **1A**



ii. $\int_m^n (f(x) - k)\,dx = 2\int_1^m (k - f(x))dx$

Solve $f(x) = \frac{4}{5}$ for x **1M**

$m = 1.3313$ and $n = 2.6687$ correct to four decimal places **1A**



g. $f_1 : [1,3] \rightarrow R, f_1(x) = -x^4 + 8x^3 + ax^2 + bx + d$

Find a and b in terms of d.

Solve $f_1(1) = 0$ and $f_1(3) = 0$ for a and b

$a = \dfrac{d-57}{3}$ and $b = -\dfrac{4(d-9)}{3}$ **1M**

Solve $\dfrac{d}{dx}\left(-x^4 + 8x^3 + \dfrac{d-57}{3}x^2 - \dfrac{4(d-9)}{3}x + d\right) = 0$ for x **1M**

$x = 2$ or $x = \dfrac{12 \pm \sqrt{6d+90}}{6}$

For three solutions $6d + 90 > 0$, $d > -15$ and $\dfrac{\sqrt{6d+90}}{6} < 1$, $d < -9$ as $1 < x < 3$

$-15 < d < -9$ **1A**