insight

YEAR 12 *Trial Exam Paper*

2014

MATHEMATICAL METHODS (CAS)

Written examination 2

STUDENT NAME:

Reading time: 15 minutes

Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring the following items into the examination: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer-based CAS, their full functionality may be used.
- Students are NOT permitted to bring the following items into the examination: blank sheets of paper and/or white-out liquid/tape.

Materials provided

- The question and answer book of 23 pages, with a separate sheet of miscellaneous formulas.
- An answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the box provided and on the answer sheet for multiple-choice questions.
- Remove the formula sheet during reading time.
- You must answer the questions in English.

At the end of the exam

- Place the answer sheet for multiple-choice questions inside the front cover of this question book.

Students are NOT permitted to bring mobile phones or any other electronic devices into the examination.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the multiple-choice answer sheet.
Select the response that is **correct** for the question.
A correct answer scores 1 mark, whereas an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
If more than one answer is selected, no marks will be awarded.

Question 1

For $f: R \to R,\ f(x) = 5 - 2\cos(4\pi x),$ the range and period of *f*, respectively, are

A. $[3,7]$ and 4π

B. $[-2,2]$ and 2

C. $[3, 7]$ and 2π

D. $[3, 7]$ and $\frac{1}{2}$

E. $[-2, 2]$ and $\frac{\pi}{2}$

Question 2

For $f:[-5,\infty) \to R,\ f(x) = 2 - |x+1|,$ the range of f is

A. $[-2,2]$

B. R

C. $[-2,\infty)$

D. $(-\infty,2]$

E. $[2,\infty)$

Question 3

The maximal domain and the range of the function $f(x) = \frac{5}{\sqrt{x-k}}$, where *k* is a positive real number, are given, respectively, by

A. $R \setminus \{0\}$ and $(-k,\infty)$

B. $R \setminus \{k\}$ and $R \setminus \{0\}$

C. $R \setminus \{0\}$ and (k,∞)

D. (k,∞) and $R \setminus \{k\}$

E. (k,∞) and R^+

Question 4

The function g is differentiable for all real values of x. The derivative of the function $g\left(\sin(5x)\right)$ is given by

A. $g'\left(\cos(5x)\right)$

B. $g'\left(-5\sin(5x)\right)$

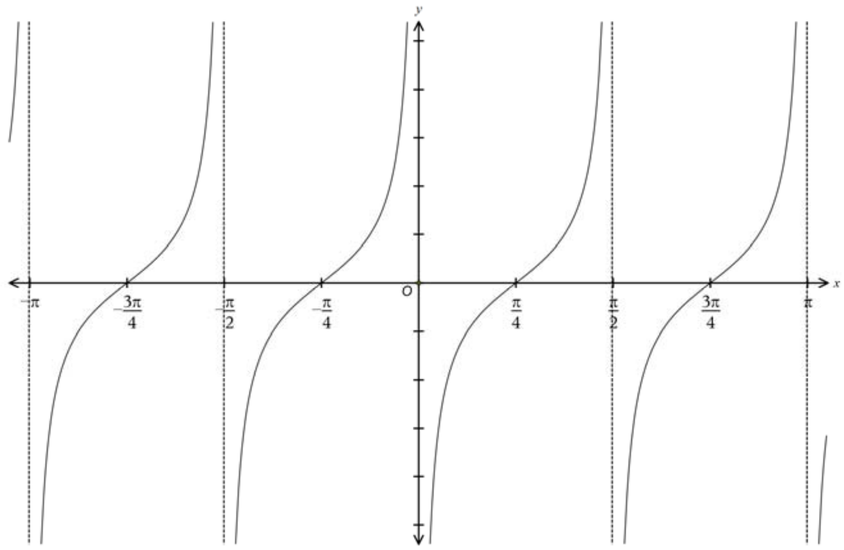
C. $5g'\left(\cos(5x)\right)\sin(5x)$

D. $5\cos(5x)\,g'\left(\sin(5x)\right)$

E. $5\sin(5x)\,g\left(\cos(5x)\right)$

Question 5

A section of the graph of f is shown below.



The rule of f could be

A. $f(x) = \tan(x)$

B. $f(x) = \tan 2(x - \frac{\pi}{4})$

C. $f(x) = \tan(2x)$

D. $f(x) = \tan 2(x - \frac{\pi}{2})$

E. $f(x) = \tan(x - \frac{\pi}{4})$

Question 6

For the function with rule $f(x) = x^3 - 3x^2$, the average rate of change of $f(x)$ with respect to x on the interval [1, 3] is

A. 1

B. −1

C. –2

D. 2

E. –3

Question 7

The function $f : B \to R,\ f(x) = \log_e(2|x| + 3)$ will have an inverse function when

A. $B = [0, \infty)$

B. $B = (-3, \infty)$

C. $B = [-\frac{3}{2}, \infty)$

D. $B = (-\frac{3}{2}, \infty)$

E. $B = R$

Question 8

For the system of equations

$$z = 1$$
$$z - x = 2$$
$$z + y = 5$$

an equivalent matrix representation is

A. $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

SECTION 1 – continued
TURN OVER

Question 9

For $y = e^{3x}\cos(4x)$, the rate of change of y with respect to x when $x = \frac{\pi}{4}$ is

A. 0

B. -3

C. $-3e^{\frac{3\pi}{4}}$

D. $-3e^{\frac{\pi}{4}}$

E. $-4e^{\frac{3\pi}{4}}$

Question 10

Let $g: R \to R,\ g(x) = ax^2 - x^4$, where a is a positive real number.
Also, $g'(u) = 0$ and $g'(t) = 0$, where u and t are real, non-zero numbers and $u < t$.
The function g has a positive gradient for

A. $x \in (-\infty, t) \cup (0, u)$

B. $x \in (u, 0) \cup (t, \infty)$

C. $x \in (u, t)$

D. $x \in (-\infty, u) \cup (t, \infty)$

E. $x \in (-\infty, u) \cup (0, t)$

Question 11

An approximation for $9.2^{\frac{3}{2}}$ can be found using the linear approximation

$f(9+h) \approx f(9) + hf'(9)$

and by considering the point $(9, 27)$ that lies on the graph of $f(x) = x^{\frac{3}{2}}$.

The approximation is equal to

A. 27.9

B. 27.45

C. 27.6

D. 27.06

E. 27.2

Question 12

A transformation $T: R^2 \to R^2$ that maps the curve with the equation $y = \sin(x)$ onto the curve with the equation $y = 1 - 2\sin(3x + \pi)$, is given by

A. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{3} \\ 1 \end{bmatrix}$

B. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ -1 \end{bmatrix}$

C. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{3} \\ 1 \end{bmatrix}$

D. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ -1 \end{bmatrix}$

E. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ -1 \end{bmatrix}$

Question 13

Let X be a discrete random variable with a binomial distribution. The mean of X is 20 and the variance is 12.

The values of n (the number of independent trials) and p (the probability of each trial) are

A. $p = 50$ and $n = 0.4$

B. $p = 0.6$ and $n = 50$

C. $p = 0.2$ and $n = 100$

D. $p = 0.72$ and $n = 50$

E. $p = 0.4$ and $n = 50$

Question 14

The discrete random variable X has a probability distribution as shown.

x	0	1	2	3
$\Pr(X = x)$	$4a$	$2a$	$3a$	a

The mean of X is

A. 1.1

B. 1.5

C. 1

D. $\frac{11a}{3}$

E. $\frac{11a}{4}$

Question 15

The tangent at the point $(1, -3)$ on the graph of the curve $y = f(x)$ has the equation $y = 2x - 5$.

Hence, the tangent at the point $(-1, -1)$ on the curve $y = f(-x) + 2$ has the equation

A. $y = -2x + 7$

B. $y = -2x - 7$

C. $y = 2x - 5$

D. $y = -2x - 3$

E. $y = 2x - 3$

Question 16

If $\int_{-4}^{5} f(x)\,dx = 4$, then $\int_{-1}^{2} \left(f(3x-1)\right)dx$ is equal to

A. 11

B. $\frac{4}{3}$

C. 1

D. 9

E. 4

SECTION 1 – continued
TURN OVER

Question 17

Isabelle likes to drink both green tea and black tea. She drinks one cup of either each day and likes to switch from day to day. If she drinks green tea one day, then she drinks green tea the next with a probability of 0.65. If she drinks black tea one day, then she drinks green tea the next with a probability of 0.3.

The transition matrix *T* that represents the situation is

A. $\begin{bmatrix} 0.65 & 0.7 \\ 0.35 & 0.3 \end{bmatrix}$

B. $\begin{bmatrix} 0.35 & 0.3 \\ 0.65 & 0.7 \end{bmatrix}$

C. $\begin{bmatrix} 0.65 & 0.3 \\ 0.35 & 0.7 \end{bmatrix}$

D. $\begin{bmatrix} 0.65 & 0.35 \\ 0.3 & 0.7 \end{bmatrix}$

E. $\begin{bmatrix} 0.65 \\ 0.3 \end{bmatrix}$

Question 18

Let $f : R \to R,\ f(x) = e^x + e^{-x}$. Then, $f(2u)$ is equal to

A. $2f(u)$

B. $f(u^2)$

C. $[f(u)]^2 - 2$

D. $2f(u) - 2$

E. $f(u^2) - 2$

Question 19

The random variable *X* has a normal distribution with a mean of 50 and a variance of 49.

If the random variable *Z* has a standard normal distribution, then the probability that *X* is greater than 71 is

A. $1 - \Pr(Z < -3)$

B. $1 - \Pr(Z > 3)$

C. $\Pr(-3 < Z < 3)$

D. $\Pr(Z < 3)$

E. $\Pr(Z < -3)$

Question 20

For the function *f*, $f\left(\frac{3\pi}{2}+\theta\right)=-f\left(\frac{3\pi}{2}-\theta\right)$ and $f\left(-\frac{\pi}{2}+\theta\right)=f\left(-\frac{3\pi}{2}+\theta\right)$ for all values of θ.

The rule for *f* could be

A. $f(x)=\cos(x)$

B. $f(x)=-\cos(x)$

C. $f(x)=\sin(x)$

D. $f(x)=-\tan(x)$

E. $f(x)=\tan(x)$

Question 21

The tangent to the graph of $y=-\log_e(x)$ at the point $(a,-\log_e(a))$ crosses the *x*-axis at the point (*b*, 0), where $b>0$.

Which of the following is **false**?

A. $a>0$

B. The gradient of the tangent is negative.

C. $1<a<e$

D. The gradient of the tangent is $-\frac{1}{a}$.

E. $a>e$

Question 22

Let $f(x)=a-b\cos(x)$, where a and b are real numbers and $a<0$, $b>0$.

Then, $f(x)<0$ for all real values of x when

A. $a>b$

B. $b>a$

C. $a>-b$

D. $b>0$

E. $a<-b$

END OF SECTION 1
TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal answer will not be accepted if an **exact** answer is required to a question.
For questions where more than one mark is available, appropriate working must be shown.
Unless otherwise stated, diagrams are not drawn to scale.

Question 1 (14 marks)

A special electronic switch is prone to failure. The time, in years, to failure is given by the random variable X, with the probability density function

$$f(x)=\begin{cases}0 & \text{for } x\le 1\\ \frac{1}{4}e^{-\frac{(x-1)}{4}} & \text{for } x>1\end{cases}$$

a. Find the exact value of the mean time to failure for one of these special electronic switches.

2 marks

b. Find the median time, in years, to failure for one of these special electronic switches. Give your answer correct to 2 decimal places.

2 marks

SECTION 2 – Question 1 – continued

c. Find the variance of the time to failure for one of these special electronic switches.

2 marks

d. Find the exact probability that one of these special switches will last for at least 5 years.

2 marks

Four of these switches are installed in a particular brand of television. The switches operate independently of one another. These televisions can continue operating if at least two of the switches have not failed.

e. What is the expected number of switches in the television that will last for at least 5 years?

2 marks

f. What is the probability that the television will still be operational after 5 years? Give your answer correct to 4 decimal places.

2 marks

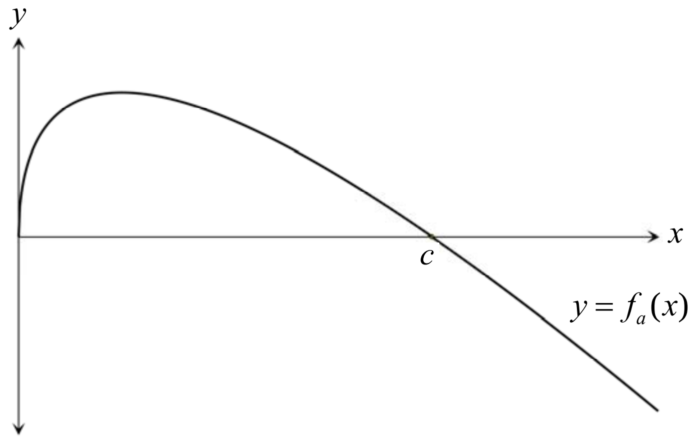
g. A random sample of 12 of these televisions is selected from the population and inspected. What is the probability that seven of these are operational after 5 years? Give your answer correct to 4 decimal places.

2 marks

Question 2 (15 marks)

Consider the family of functions $f_a : [0, \infty) \to R$, which is defined by $f_a(x) = 4a\sqrt{x} - x$, where a is a real number and $a > 0$.

Part of the graph of f_a is shown below.



a. Find c in terms of a, where $f_a(c) = 0$ and c is not zero.

2 marks

b. Determine the interval over which f_a is strictly decreasing.

2 marks

c. Show that the equation of the tangent to the graph of f_a at the point (c, 0) is $y=-\frac{1}{2}(x-16a^2)$.

3 marks

The function $f_a(x)$ is transformed to form $g_a(x)$, where $g_a(x)$ is defined as $g_a(x)=f_a(x)-b$.

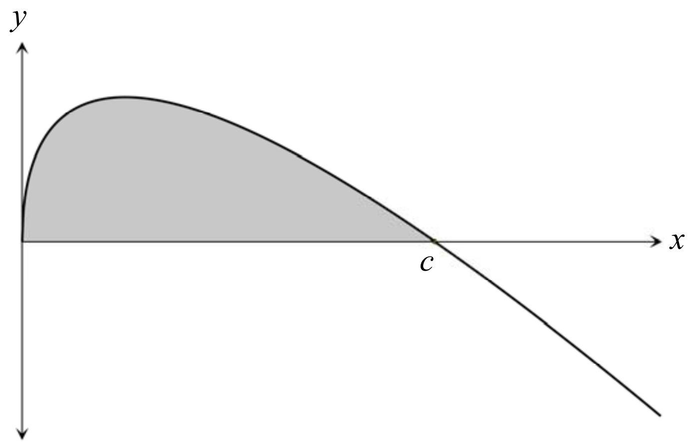
d. Find the value of b, in terms of a, such that the tangent drawn to the curve of $g_a(x)$ at $x=c$ passes through the origin.

2 marks

e. Let $h_a(x) = f_a\left(\frac{x}{2}\right)$.

i. Find the exact value of the area, in terms of a, that is bounded by the x-axis and the graph of $y = f_a(x)$, as shaded in the diagram below.

2 marks



ii. Hence, find the area, in terms of a, that is bounded by the x-axis and the graph of $y = h_a(x)$.

2 marks

f. Find the value of d, in terms of a, such that the average value of the function $y = h_a(x)$ over the interval $[0, d]$ is equal to zero.

2 marks

Question 3 (14 marks)

The brightness (or luminosity, which is measured in candela with the units cd) of an exploding firework can be altered by the addition of a catalyst chemical.

The brightness of an exploding firework at time *t* seconds after the firework is ignited may be modelled using the formula

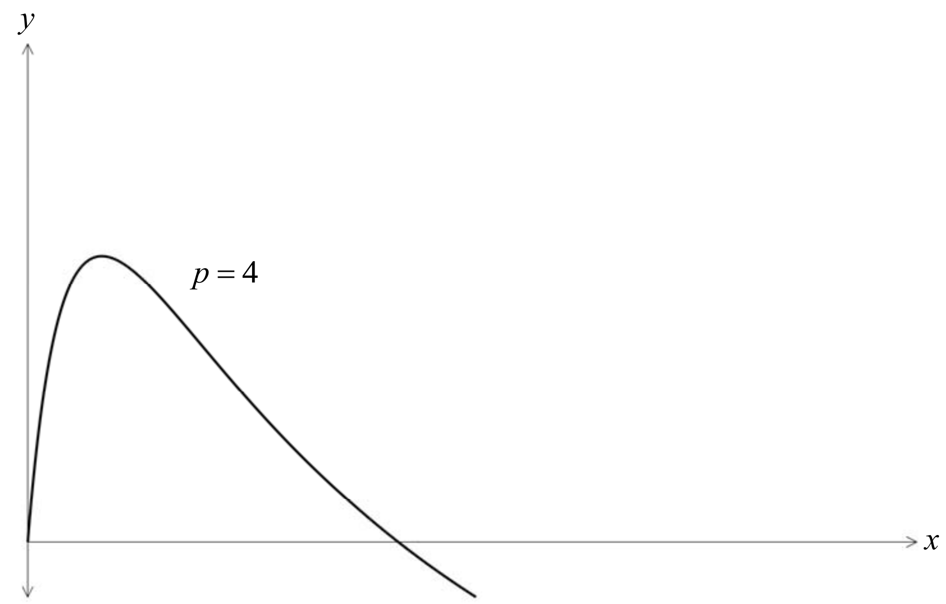
$$F = \frac{-pt(pt-24)}{(t+2)^2},$$

where $p > 0$ and *F* cd is the brightness of the firework at time *t* minutes and *p* grams is the amount of catalyst chemical added.

The firework is extinguished or finished when the brightness becomes zero.

a. The graph of *F* when $p = 4$ is shown below. Sketch graphs of *F* for $p = 2$ and $p = 6$ on the same axes. Label turning points and intercepts as coordinates, correct to 2 decimal places.

2 marks



b. Find an expression, as a function of p, for the time taken for the firework to extinguish.

2 marks

c. **i.** Find an equation in terms of p and t that, when solved for t, will give an expression for the time at which the maximum brightness is achieved as a function of p.

2 marks

ii. Solve the equation derived in **part c.i.** to give this time as a function of p.

2 marks

d. **Hence**, find an expression in terms of p for the maximum brightness.

2 marks

e. For aesthetic reasons, the brightness must always be less than 11 cd. Find the greatest value of K such that if $p \leq K$, then $F \leq 11$.

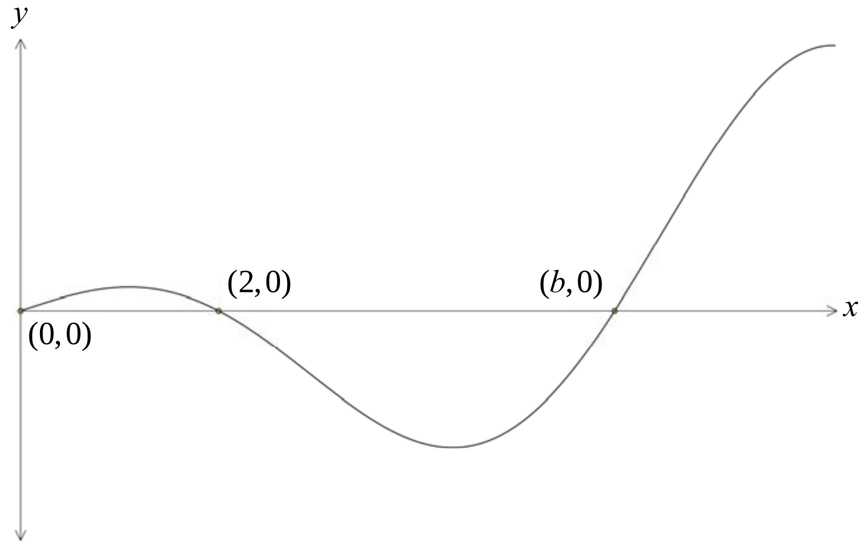
2 marks

f. Additionally, the firework must have reached maximum brightness within 1.5 seconds of ignition. Find the interval of values for p that satisfies these conditions.

2 marks

Question 4 (15 marks)

Part of the graph of $y = \frac{\pi x}{4}\cos(nx)$ is shown below.



a. Show that the value of n is $\frac{\pi}{4}$.

2 marks

b. State the value of b.

1 mark

The pollution level, y units, along a straight road between two factories, A and B, which are 4.3 km apart, is given by

$$y = 4 - \frac{\pi x}{4}\cos(px), \text{ where } 0 \le x \le 4.3$$

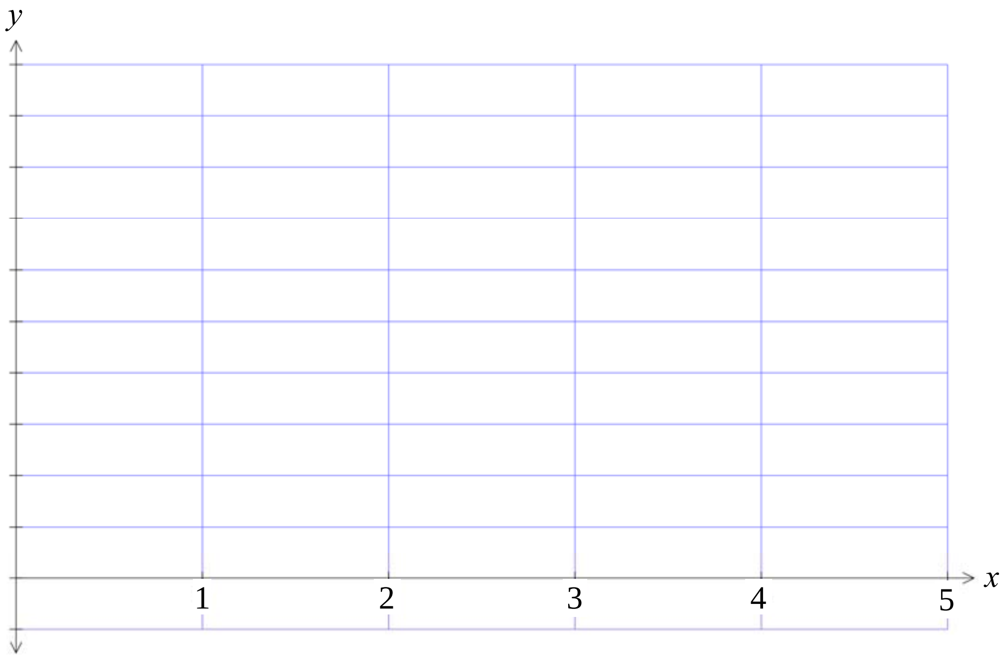
where x km is the distance from factory A and p is a positive constant.

On a particular day, the value of p is measured to be $p = \frac{\pi x}{4}$.

c. **i.** Sketch the graph of $y = 4 - \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)$, where $0 \le x \le 4.3$.

Label endpoints and turning points as coordinates, correct to 2 decimal places.

3 marks



ii. Find the exact pollution level when $x = 1$.

1 mark

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____

iii. Ben travels from factory A to factory B. For what length of his journey (in kilometres, correct to 3 decimal places) is the pollution level at least 6 units?

2 marks

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On another day, the value of p is unknown and the pollution level, y units, is given by

$$y = 4 - \frac{\pi x}{4}\cos(px), \text{ where } 0 \le x \le 4.3$$

where x km is the distance from factory A and p is a positive constant.

d. **i.** For what value of p, correct to 4 decimal places, does the minumum value of y occur when $x = 1$?

2 marks

ii. For what values of p, correct to 4 decimal places, does the maximum value of y occur when $x = 4.3$.

4 marks

END OF QUESTION AND ANSWER BOOK