insight

YEAR 12 *Trial Exam Paper*

2014

MATHEMATICAL METHODS (CAS)

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

SECTION 1

Question 1

Answer is D

Worked solution

The range is given by $[5-2,\ 5+2]=[3,\ 7]$.

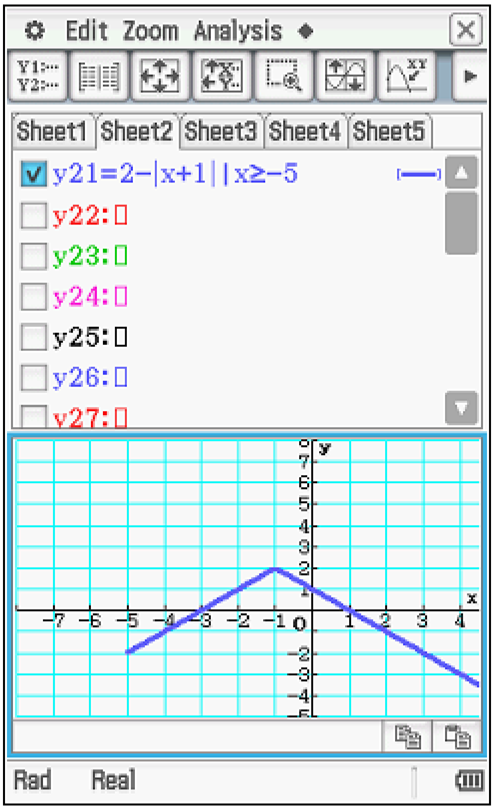
The period is given by $\frac{2\pi}{4\pi}=\frac{1}{2}$.

Question 2

Answer is D.

Worked solution

Using CAS, the range is $(-\infty, 2]$.



Tip

- *To find the range always draw the graph.*

Question 3

Answer is E.

Worked solution

For $f(x)$ to be defined, $x > k,$ so the domain is (k, ∞).
And $\sqrt{x-k} > 0$, so the range is R^{+}.

Tip

- *If it helps, an arbitrary value of* k *can be chosen and then a graph produced.*

Question 4

Answer is D.

Worked solution

Using the chain rule for $y = g\left(f(x)\right)$, $\dfrac{dy}{dx} = g'\left(f(x)\right) \times f'(x)$.

Now $f(x) = \sin(5x)$ and $f'(x) = 5\cos(5x)$, so $\dfrac{d}{dx}\left[g(\sin(5x)\right] = g'\left(\sin(5x)\right) \times 5\cos(5x)$.

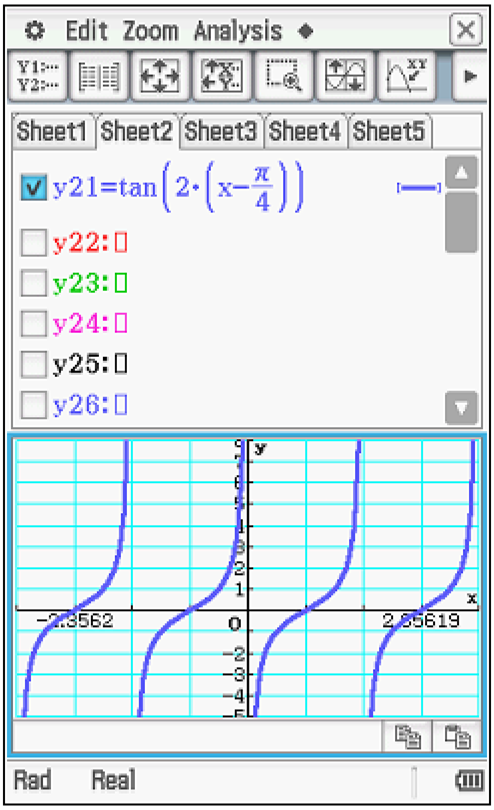
Question 5

Answer is B.

Worked solution

The graph is a tan graph with a period of $\frac{\pi}{2}$ that has been shifted to the left $\frac{\pi}{4}$ units.

Checking using CAS gives



Question 6

Answer is A.

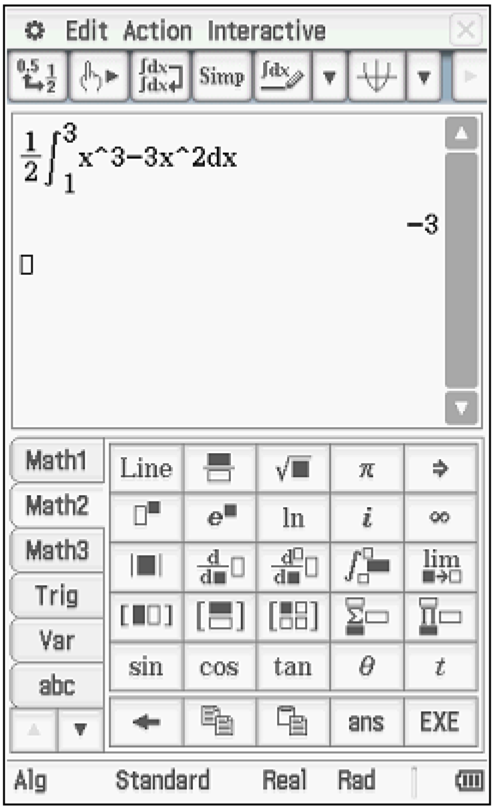
Worked solution

$f(3) = 27 - 27 = 0$

$f(1) = 1 - 3 = -2$

Average rate of change $= \dfrac{f(3) - f(1)}{3-1} = \dfrac{0-(-2)}{2} = 1$

Average value of the function is $\dfrac{1}{3-1}\int_1^3 x^3 - 3x^2\,dx = -3$ (found using CAS).



Tip

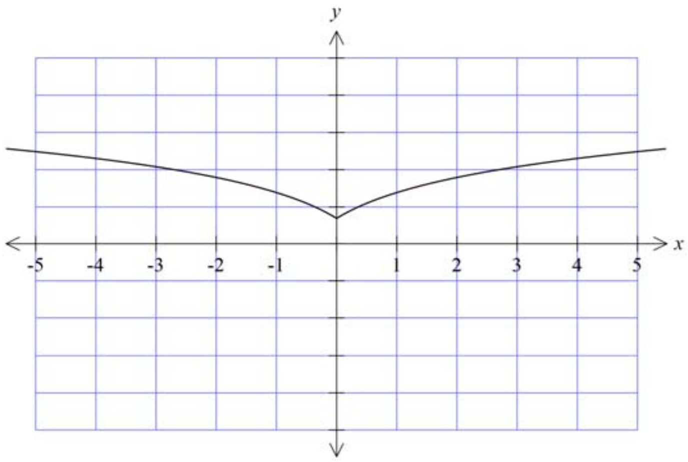
- *Be careful! Option E is the answer to the* ***average value*** *of the function—this is easily and readily confused.*

Question 7

Answer is A.

Worked solution

For an inverse function to exist, the function $f(x)$ must be one-to-one; i.e. for each x value there is exactly one y value.
The graph of $f(x)$ is



To be one-to-one $x \in [0, \infty)$.

Tip

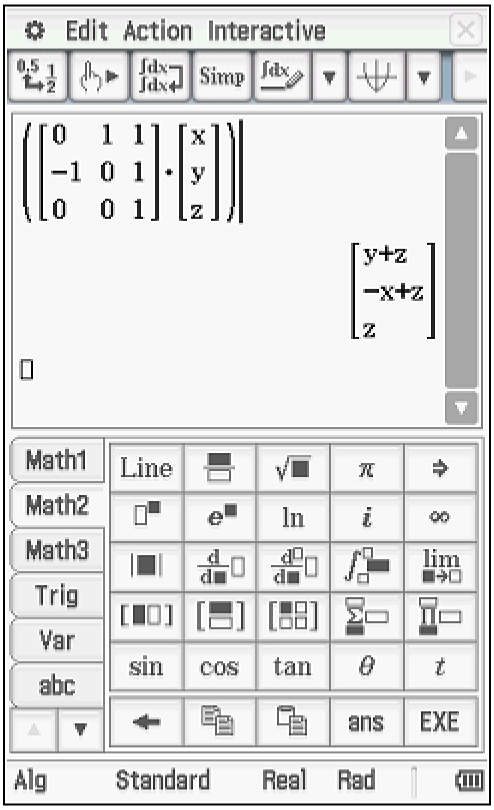
- *For the graph to be one-to-one it must pass the horizontal line test.*

Question 8

Answer is D.

Worked solution

Expanding the matrix gives



Tip

- *Ensure the equations are first re-written in the form*

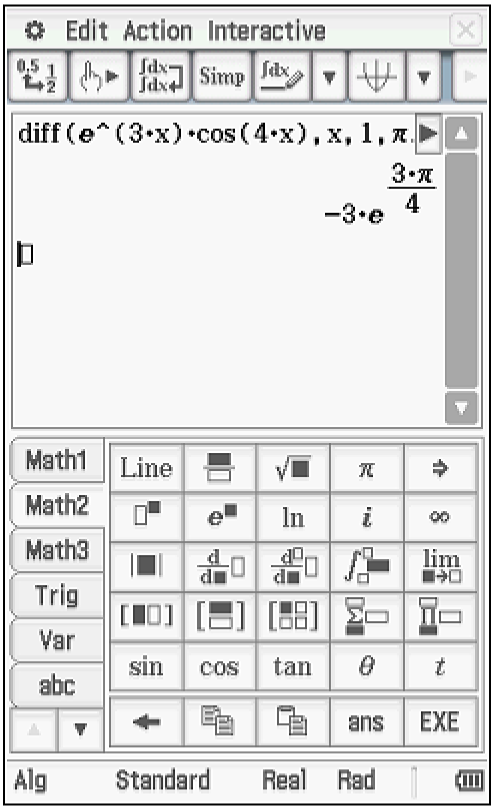
$$0x + 0y + z = 1$$
$$-1x + 0y + z = 2$$
$$0x + y + z = 5$$

Question 9

Answer is C.

Worked solution

Using CAS gives

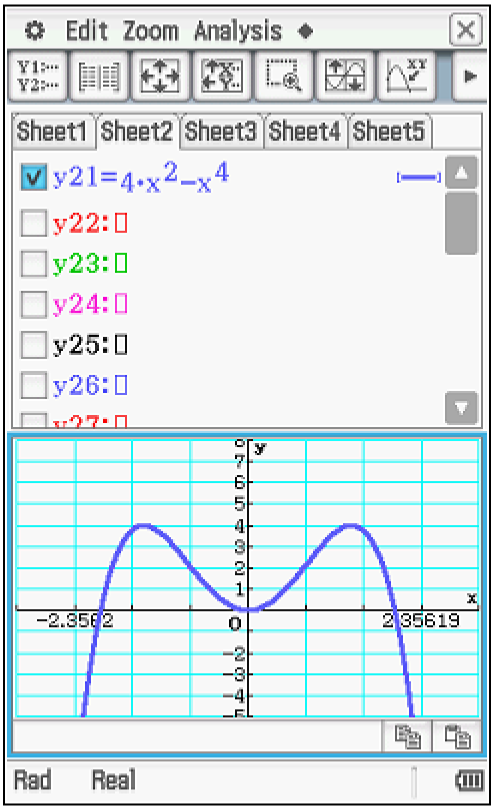


Question 10

Answer is E.

Worked solution

A suitable way to do this question is to choose an arbitrary value for a; e.g. $a = 4$.
A sample graph would be



Owing to the condition that $u < t$, the gradient is positive for $x \in (-\infty, u) \cup (0, t)$.

Question 11

Answer is A.

Worked solution

For this approximation $h = 0.2,\ x = 9$ and $f'(x) = \frac{3}{2}\sqrt{x}$.

So $f(9+h) \approx f(9) + hf'(9)$

$$= \left((9)^{\frac{1}{2}}\right)^3 + 0.2 \times \frac{3}{2}\sqrt{9}$$

$$= 3^3 + 0.2 \times \frac{9}{2} = 27.9$$

Question 12

Answer is A.

Worked solution

Rearranging the equation $y = 1 - 2\sin(3x + \pi)$ gives $\frac{y-1}{-2} = \sin\left(3\left(x + \frac{\pi}{3}\right)\right)$.

So $y = \frac{y'-1}{-2}$ and $x = 3\left(x' + \frac{\pi}{3}\right)$.

Therefore, $y' = -2y + 1$ and $x' = \frac{x}{3} - \frac{\pi}{3}$.

The expansion of the matrix in option A; i.e. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{3} \\ 1 \end{bmatrix}$, gives

$x' = \frac{x}{3} - \frac{\pi}{3}$ and $y' = -2y + 1$, as required.

Question 13

Answer is E.

Worked solution

$X \sim \text{Bi}$

$\text{E}(X) = np = 20$

$\text{var}(X) = npq = 12 = 20q$

So $\quad q = \dfrac{12}{20} = 0.6 \Rightarrow p = 0.4$

$\therefore \; n = \dfrac{20}{0.4} = 50$

Question 14

Answer is A.

Worked solution

$\sum \Pr(X = x) = 1 \;\; \Rightarrow 10a = 1, \;\; \therefore a = 0.1$

$$\begin{aligned}\text{E}(X) &= \sum x \Pr(X = x) \\ &= 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.1 \\ &= 1.1\end{aligned}$$

Question 15

Answer is D.

Worked solution

The transformation represents a reflection in the *y*-axis and a translation of 2 units up.
Operating this transformation on the tangent line gives

$y = 2(-x) - 5 + 2$

$y = -2x - 3$

Question 16

Answer is B.

Worked solution

Using the given information

$\int_{-4}^{5} f(x)\, dx = 4$

$\Rightarrow F(5) - F(-4) = 4 \qquad (1)$

So $\int_{-1}^{2} (f(3x-1))\, dx$

$= \frac{1}{3}(F(3(2)-1) - F(3(-1)-1))$

$= \frac{1}{3}(F(5) - F(-4))$

$= \frac{1}{3}(4) \quad \text{(from equation 1)}$

$= \frac{4}{3}$

Question 17

Answer is C.

Worked solution

$\Pr(G' \mid G) = 0.65$ and $\Pr(G' \mid B) = 0.3$

So the matrix is set up as $\begin{bmatrix} \Pr(G' \mid G) & \Pr(G' \mid B) \\ \Pr(B' \mid G) & \Pr(B' \mid B) \end{bmatrix} = \begin{bmatrix} 0.65 & 0.3 \\ 0.35 & 0.7 \end{bmatrix}$.

Question 18

Answer is C.

Worked solution

$$\begin{aligned} [f(u)]^2 - 2 &= (e^x + e^{-x})^2 - 2 \\ &= e^{2x} + 2 + e^{-2x} - 2 \\ &= e^{2x} + e^{-2x} \\ &= f(2u) \end{aligned}$$

Question 19

Answer is E.

Worked solution

$$\Pr(X > 71) = \Pr\left(Z > \frac{71-50}{7}\right) = \Pr(Z > 3)$$

Using symmetry, $\Pr(Z > 3) = \Pr(Z < -3)$.

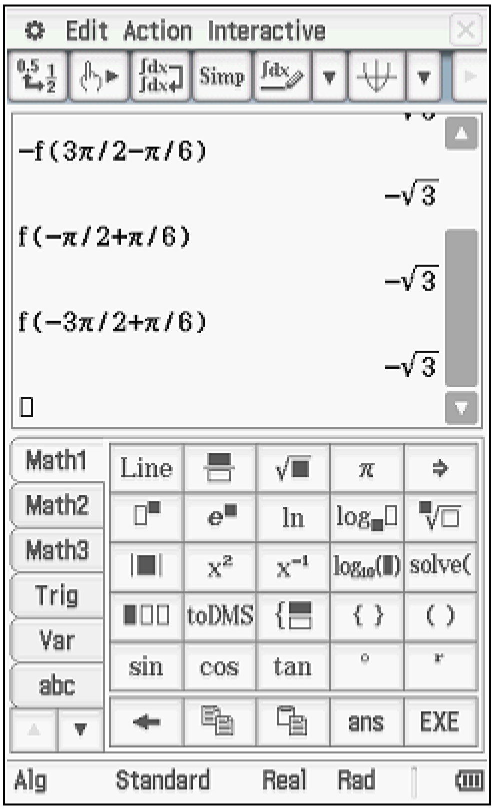
Question 20

Answer is E.

Worked solution

First, choose a value for θ, say $\theta = \frac{\pi}{6}$.

Using CAS gives

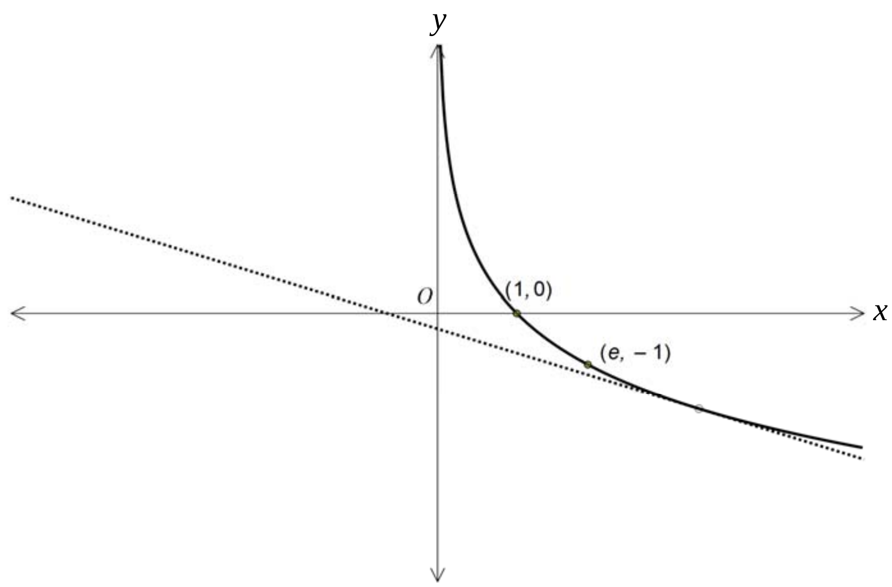


Question 21

Answer is E.

Worked solution

The graph of $y = -\log_e(x)$ is shown below.



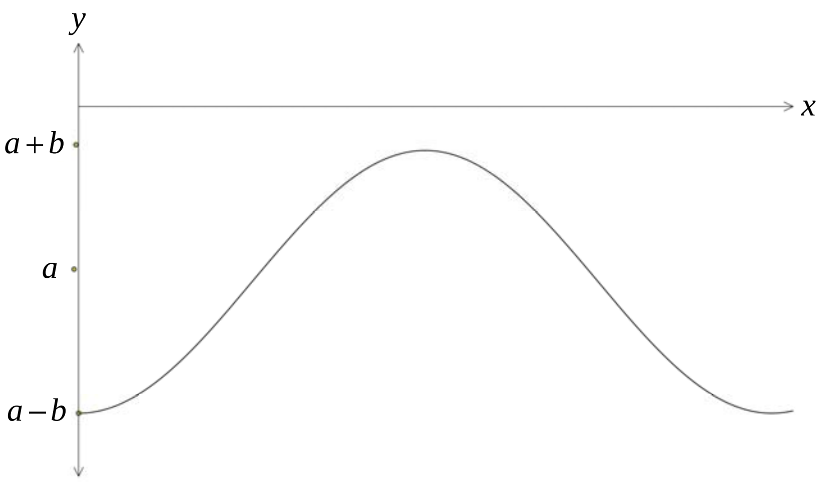
By choosing a value greater than *e* the *x*-intercept of the tangent line to the graph is negative, which means that option E is false.

Question 22

Answer is E.

Worked solution

A graph of the general curve looks like



So for $f(x) < 0$, then $a + b < 0$; i.e. $a < -b$.

Tip

- *If it helps, choose arbitrary values for* a *and* b *to produce a sketch.*

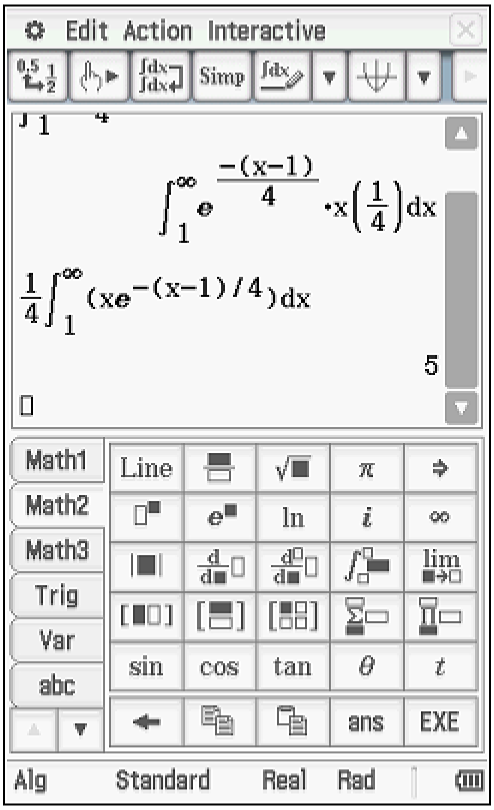
SECTION 2

Question 1a.

Worked solution

$$\text{Mean} = \int_1^{\infty}\left(x \times \frac{1}{4}e^{\frac{-(x-1)}{4}}\right)dx$$

Using CAS, we get



This gives the mean of 5 years.

Mark allocation: 2 marks

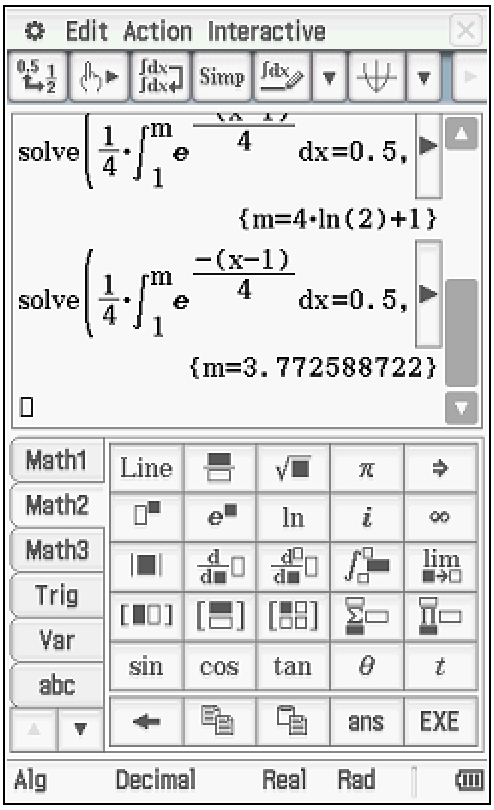
- 1 mark for writing: $\text{Mean} = \int_1^{\infty}\left(x \times \frac{1}{4}e^{\frac{-(x-1)}{4}}\right)dx$.
- 1 mark for answer of 5 years.

Question 1b.

Worked solution

Median = m, such that $\int_1^m \frac{1}{4} e^{\frac{-(x-1)}{4}}\, dx = 0.5.$

Using CAS to solve for m gives m = 3.77 years.



Mark allocation: 2 marks

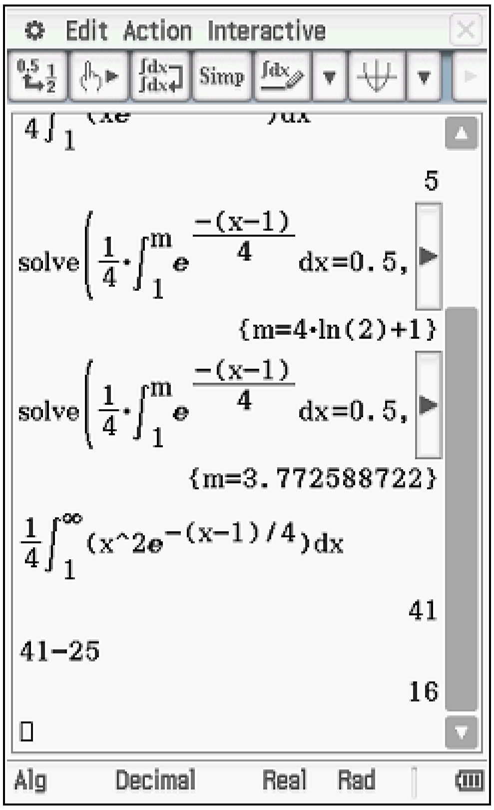
- 1 mark for writing $\int_1^m \frac{1}{4} e^{\frac{-(x-1)}{4}}\, dx = 0.5.$
- 1 mark for answer of m = 3.77 years.

Question 1c.

Worked solution

$$\text{var}(X) = \text{E}(X^2) - (\text{E}(X))^2$$
$$= \int_1^\infty x^2 f(x)dx - \left(\int_1^\infty x f(x)\, dx\right)^2$$
$$= \int_1^\infty x^2 \frac{1}{4} e^{\frac{-(x-1)}{4}}\, dx - \left(\int_1^\infty x \frac{1}{4} e^{\frac{-(x-1)}{4}}\, dx\right)^2$$

Using CAS, this gives var(X) = 16 years.



Mark allocation: 2 marks

- 1 mark for stating $\text{var}(X) = \int_1^\infty x^2 \frac{1}{4} e^{\frac{-(x-1)}{4}}\, dx - \left(\int_1^\infty x \frac{1}{4} e^{\frac{-(x-1)}{4}}\, dx\right)^2$.
- 1 mark for answer var(X) = 16 years.

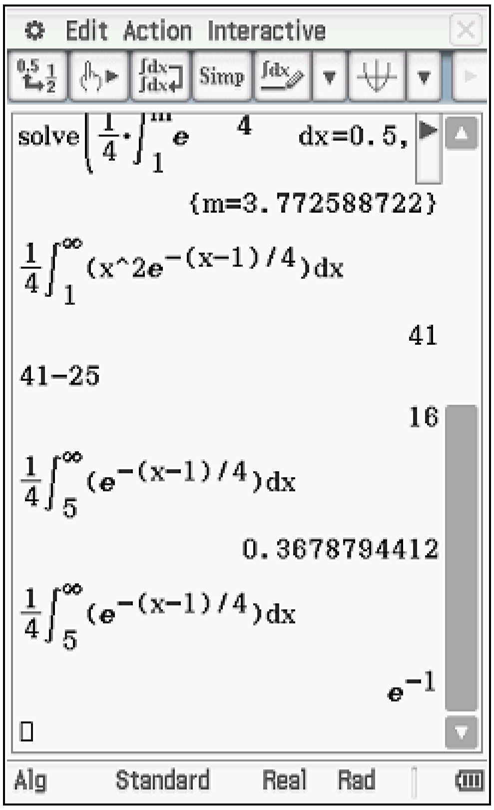
Question 1d.

Worked solution

$$\Pr(X > 5) = \int_5^\infty \frac{1}{4} e^{\frac{-(x-1)}{4}}\,dx$$

$$= \frac{1}{e} \quad \text{(Found using CAS.)}$$

Using CAS gives



Mark allocation: 2 marks

- 1 mark for stating $\Pr(X > 5) = \int_5^\infty \frac{1}{4} e^{\frac{-(x-1)}{4}}\,dx$.
- 1 mark for answer $\frac{1}{e}$.

Tip

- *The answer requires an exact answer, so be sure to have your CAS calculator in exact/standard mode. A decimal answer, no matter how accurate, will not be accepted in this instance.*

Question 1e.

Worked solution

$$Y \sim \text{Bi}\left(n = 4,\ p = \Pr(X > 5) = \frac{1}{e}\right)$$

$$\text{E}(Y) = n \times p = 4 \times \frac{1}{e} = \frac{4}{e}$$

Mark allocation: 2 marks

- 1 mark for identifying the binomial and the parameters.
- 1 mark for answer $\text{E}(Y) = \frac{4}{e}$.

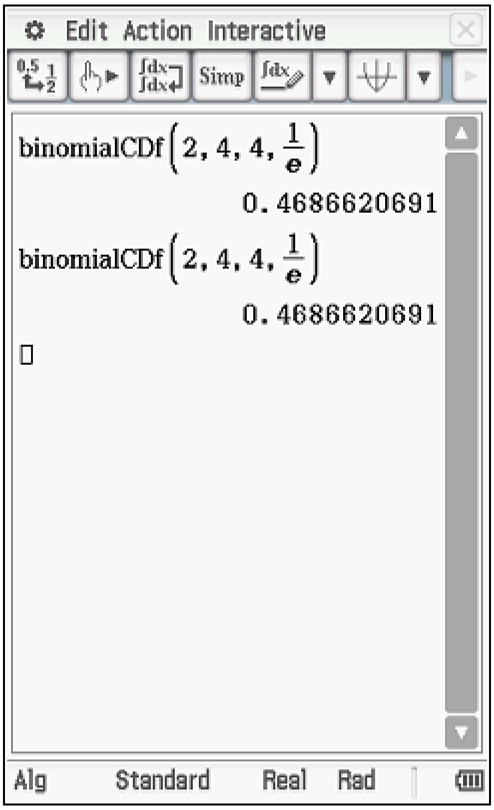
Tip

- *Although not explicitly asked for, this question also requires an exact answer.*

Question 1f.

Worked solution

To be operational, the television requires at least two switches to be working; i.e. $\Pr(Y \geq 2)$.
Using CAS, this gives $\Pr(Y \geq 2) = 0.4687$.



Mark allocation: 2 marks

- 1 mark for stating $\Pr(Y \geq 2)$.
- 1 mark for answer 0.4687.

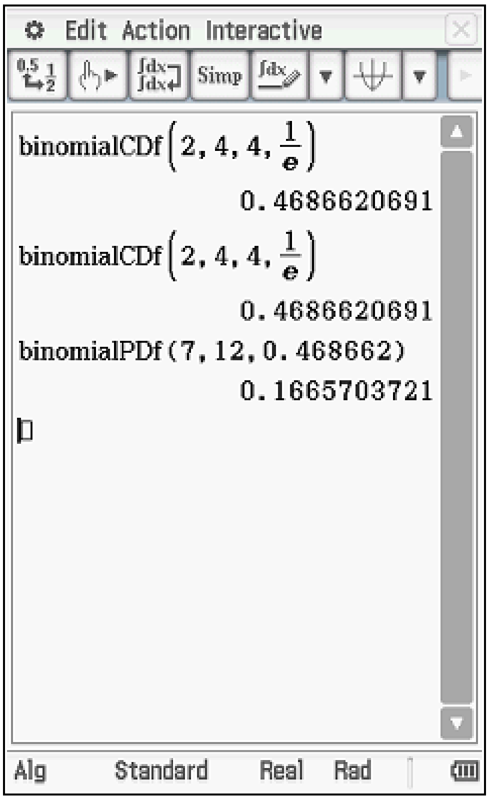
Question 1g.

Worked solution

$N \sim (n = 12,\ p = 0.468662)$

$\Pr(N = 7) = 0.1666$ (Found using CAS.)

Using CAS gives



Mark allocation: 2 marks

- 1 mark for recognising the binomial with n = 12 and p = 0.4688.
- 1 mark for answer 0.1666.

Tip

- *To answer correct to 4 decimal places requires that your calculations be carried out to at least 5 decimal places.*

Question 2a.

Worked solution

Let $4a\sqrt{x} - x = 0.$

So $4a\sqrt{x} = x$

$16a^2x = x^2$

$16a^2x - x^2 = 0$

$x(16a^2 - x) = 0$

$x = 0$ or $x = 16a^2$

Since $c \neq 0$, $c = 16a^2$.

Mark allocation: 2 marks

- 1 mark for setting $4a\sqrt{x} - x = 0.$
- 1 mark for $c = 16a^2$.

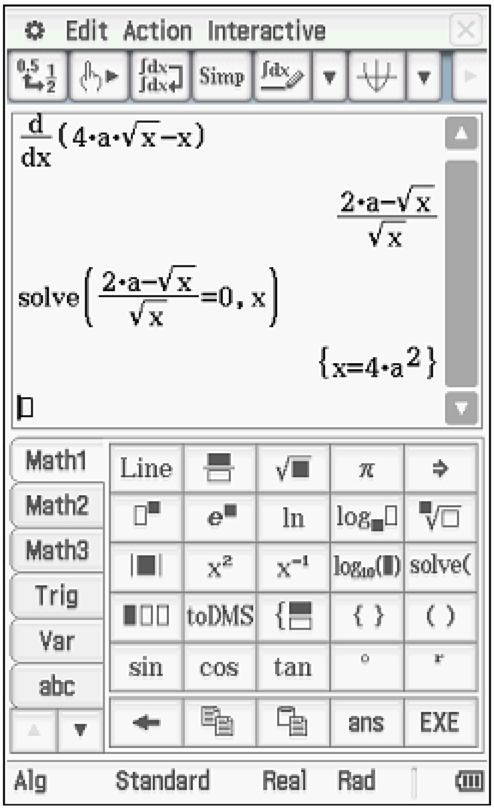
Question 2b.

Worked solution

f_a is strictly decreasing for $f_a{}' \le 0$.

Using CAS, we get $f'(x) = 0$ for $x = 4a^2$.

So, it is strictly decreasing for $x \in [4a^2, \infty)$.



Mark allocation: 2 marks

- 1 mark for finding $x = 4a^2$.
- 1 mark for $x \in [4a^2, \infty)$.

Tip

- *For strictly increasing/decreasing conditions, turning points must be included.*

Question 2c.

Worked solution

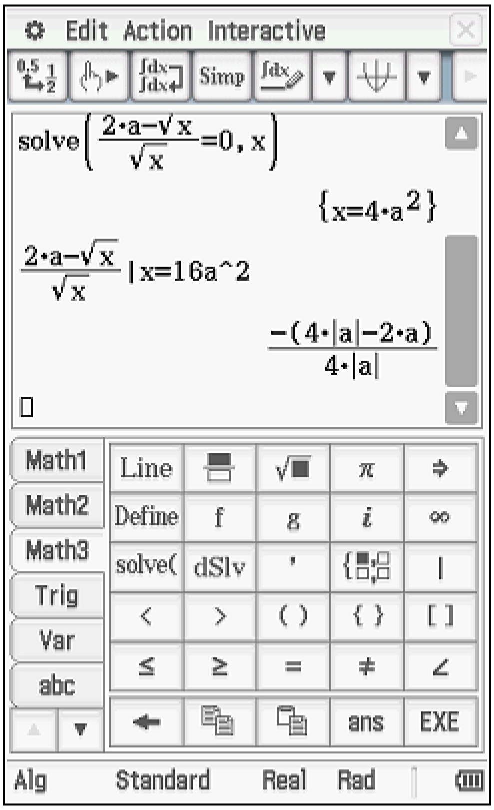
At $x = c$, $x = 16a^2$ and $y = 0$ (as determined from part **a**).

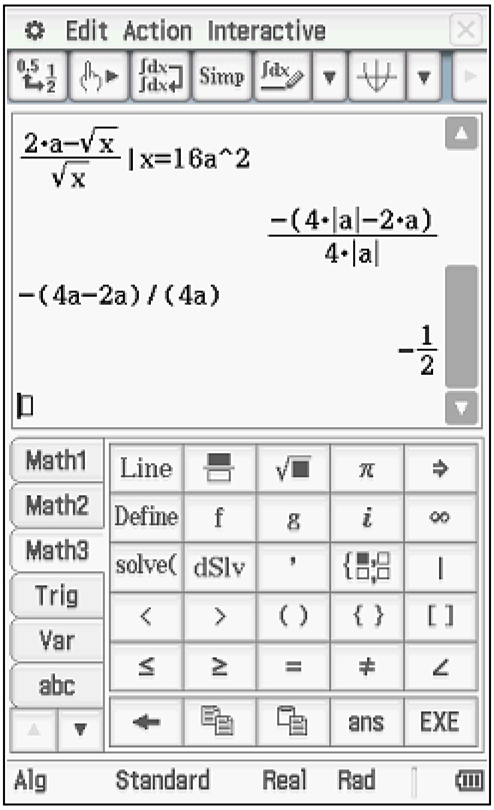
Using CAS, we get $f'(x) = \dfrac{2a - \sqrt{x}}{\sqrt{x}}$.

At $x = 16a^2$, $f'(16a^2) = \dfrac{-(4|a| - 2a)}{4|a|}$

$$= -\frac{1}{2}, \text{ since } a > 0.$$

So, $m = -\dfrac{1}{2}$.





So equation of the tangent line is

$y - y_1 = m(x - x_1)$

$y - 0 = -\dfrac{1}{2}(x - 16a^2)$

$y = -\dfrac{1}{2}(x - 16a^2)$

Mark allocation: 3 marks

- 1 mark for finding $m = -\frac{1}{2}$.
- 1 mark for finding *y*-intercept.
- 1 mark for finding tangent line equation.

Tip

- *The question specifically stated 'show that ...', therefore your working must show step-by-step how you obtained your answer.*

Question 2d.

Worked solution

$$y = -\frac{1}{2}(x - 16a^2)$$
$$= -\frac{1}{2}x + 8a^2$$

The tangent line drawn to $f_a(x)$ at $x = c$ passes through the point $(0, 8a^2)$.

So in order for the tangent drawn to $g_a(x)$ to pass through the origin, the graph tangent line must be translated down by $8a^2$, so $b = 8a^2$.

Mark allocation: 2 marks

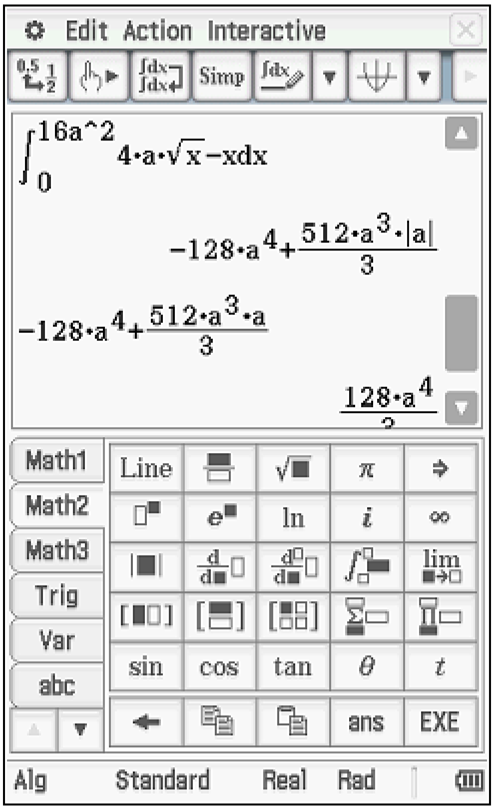
- 1 mark for finding *y*-intercept $(0, 8a^2)$.
- 1 mark for finding $b = 8a^2$.

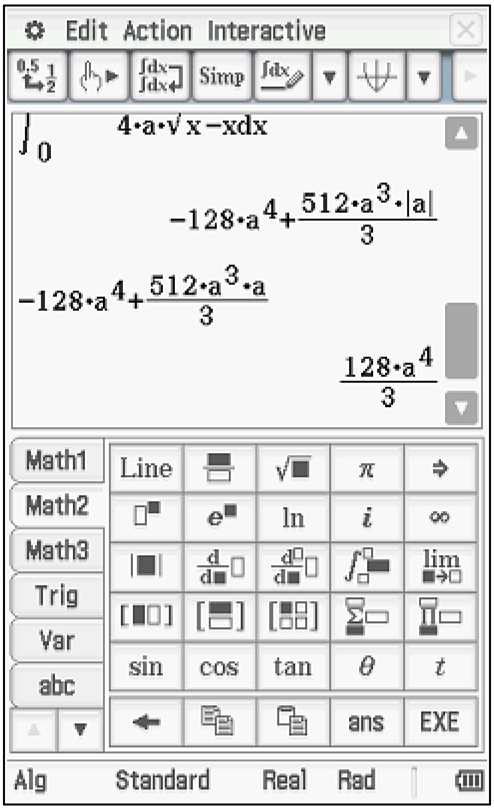
Question 2e.i.

Worked solution

The area under the curve is given by $\int_0^{16a^2} 4a\sqrt{x} - x \, dx$.

Evaluating this using CAS gives the area is equal to $\dfrac{128a^4}{3}$ square units, since $a > 0$.





Mark allocation: 2 marks

- 1 mark for setting up the integral.
- 1 mark for the answer $\dfrac{128a^4}{3}$ square units.

Tip

- *Make use of the dilation factor and understand what effect this has on the graph and the resulting area.*

Question 2e.ii.

Worked solution

$f\left(\frac{x}{2}\right)$ represents a dilation of factor of 2 parallel to the x-axis. This means the area under the curve is doubled, so the area equals $\frac{256a^4}{3}$ square units.

Mark allocation: 2 marks

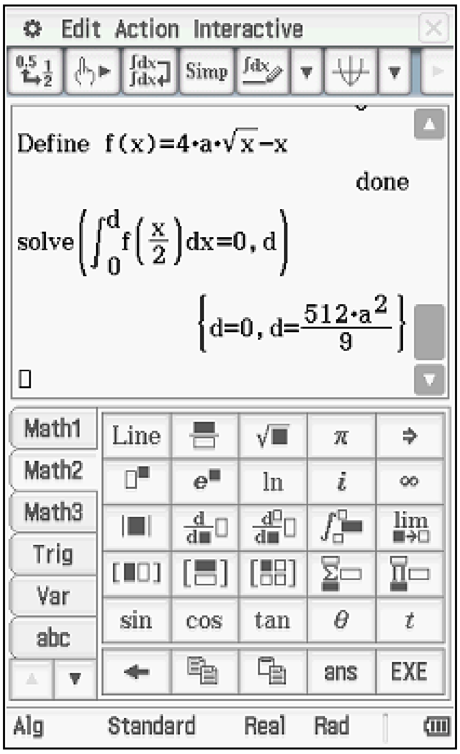
- 1 mark for identifying the transformation correctly.
- 1 mark for the answer $\frac{256a^4}{3}$ square units.

Question 2f.

Worked solution

Find d such that $\frac{1}{d}\int_0^d h_a(x)\ dx = 0$; i.e. $\int_0^d h_a(x)\ dx = 0$, since $d \neq 0$.

Using CAS to solve, gives



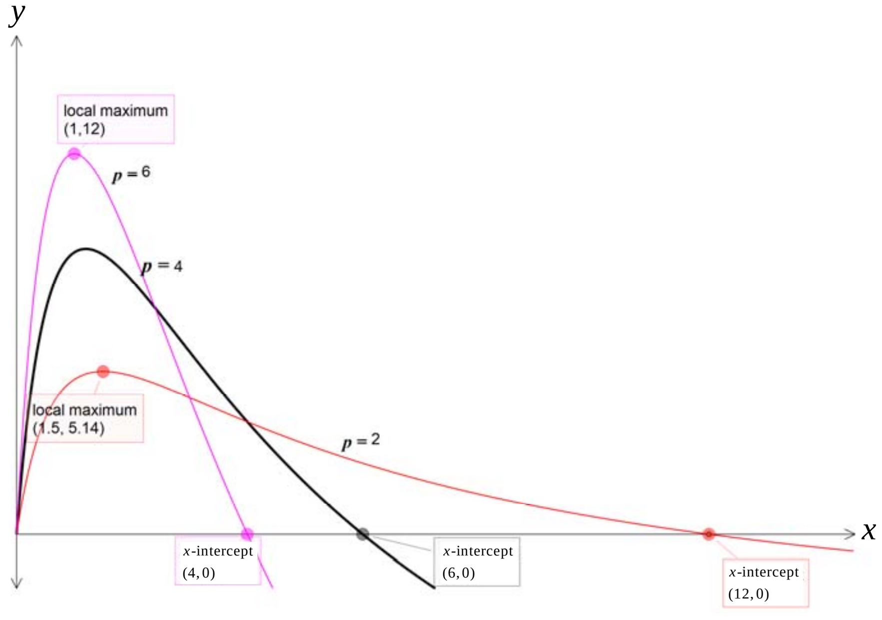
So $d = \frac{512a^2}{9}$.

Mark allocation: 2 marks

- 1 mark for setting up $\frac{1}{d}\int_0^d h_a(x)\ dx = 0$.
- 1 mark for the answer $d = \frac{512a^2}{9}$.

Question 3a.

Worked solution



Mark allocation: 2 marks

- 1 mark for correct shapes and labelling x-intercept correctly.
- 1 mark for labelling turning points correctly.

Question 3b.

Worked solution

Firework is extinguished when $F = 0$.

When $F = 0,$ then
$\Rightarrow -pt(pt - 24) = 0$
$\Rightarrow t = 0$ or $pt = 24$
So $t = \dfrac{24}{p}.$

Mark allocation: 2 marks

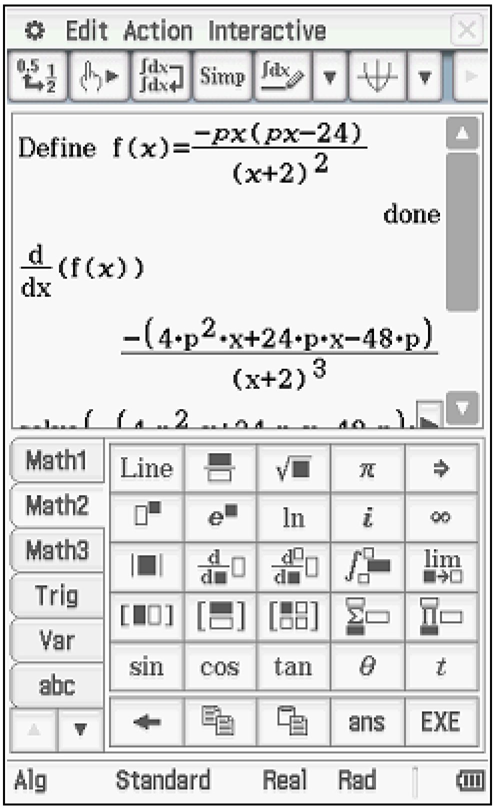
- 1 mark for setting $F = 0$.
- 1 mark for the answer $t = \dfrac{24}{p}.$

Question 3c.i.

Worked solution

Maximum brightness is achieved when $F'(x) = 0$.

Using CAS gives

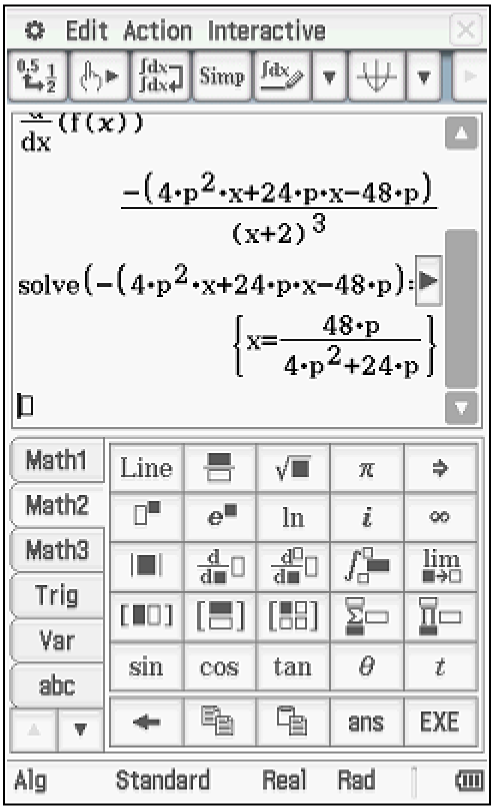


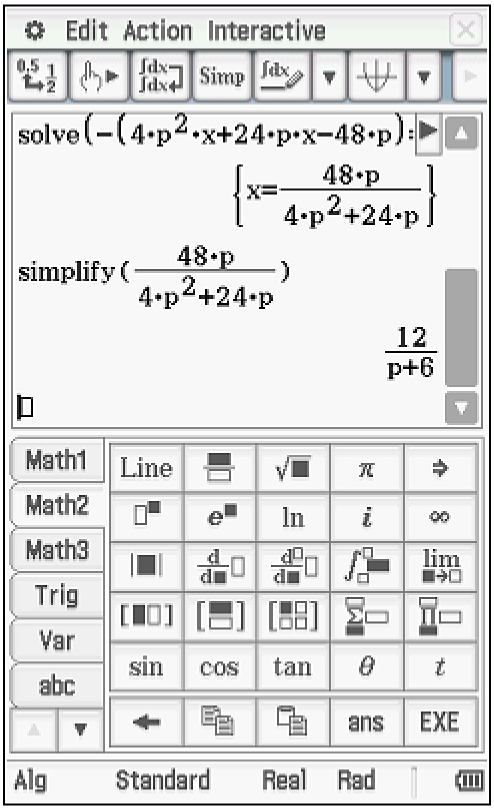
So the maximum brightness will occur when $\dfrac{-(4p^2t + 24pt - 48p)}{(t+2)^3} = 0.$

Mark allocation: 2 marks

- 1 mark for finding the derivative.
- 1 mark for setting it equal to zero.

Question 3c.ii.





So the maximum occurs when $t = \dfrac{12}{p+6}$.

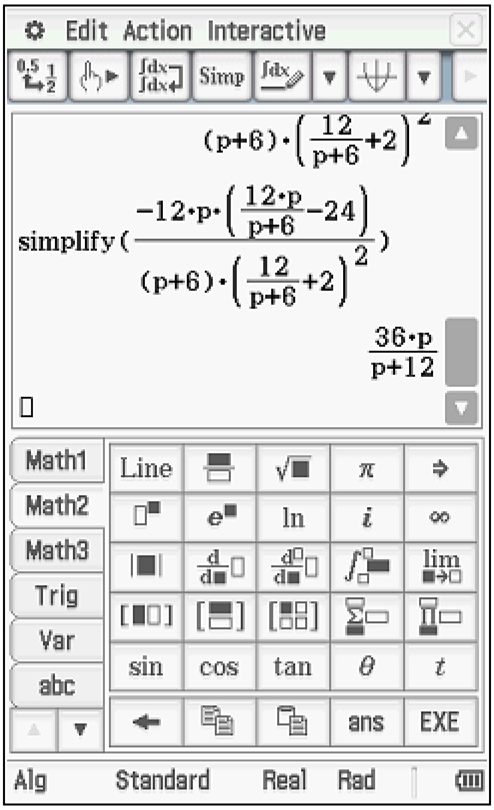
Mark allocation: 2 marks

- 1 mark for finding t in terms of *p*.
- 1 mark for simplifying to get $t = \dfrac{12}{p+6}$.

Question 3d.

Worked solution

Maximum brightness is at $x=\dfrac{12}{p+6}$ and $F\left(\dfrac{12}{p+6}\right)=\dfrac{36p}{p+12}$, which was found using CAS.



Mark allocation: 2 marks

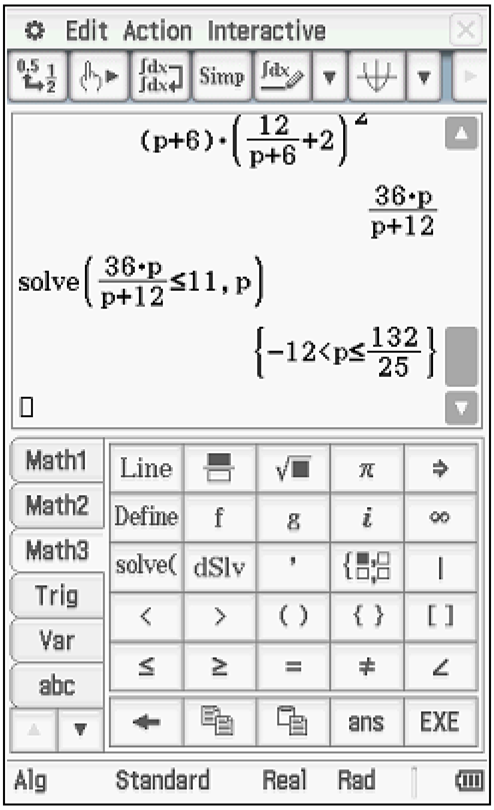
- 1 mark for finding $F\left(\dfrac{12}{p+6}\right)$.
- 1 mark for the answer $\dfrac{36p}{p+12}$.

Question 3e.

Worked solution

For $F \leq 11$, equation becomes $\dfrac{36p}{p+12} \leq 11$.

Using CAS to find this value of p gives



So the value of K is $\dfrac{132}{25} = 5.28$.

Mark allocation: 2 marks

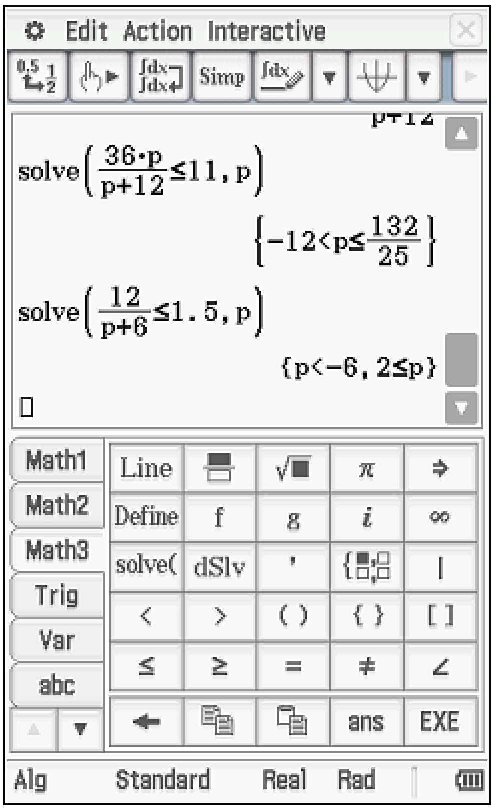
- 1 mark for setting $\dfrac{36p}{p+12} \leq 11$.
- 1 mark for the answer $\dfrac{132}{25}$ or 5.28.

Question 3f.

Worked solution

Maximum brightness occurs at $t = \dfrac{12}{p+6}$.

We want $t \le 1.5$, so equation becomes $\dfrac{12}{p+6} \le 1.5$.



So the values for *p* that satisfy the conditions are $2 \le p \le 5.28$.

Mark allocation: 2 marks

- 1 mark for finding $p \ge 2$.
- 1 mark for $2 \le p \le 5.28$.

Question 4a.

Worked solution

The x-intercepts of the graph are at (0, 0) and (2, 0). These are found when $y = 0$.
Setting $y = 0$ gives

$$\frac{\pi x}{4}\cos(nx) = 0$$

$$\Rightarrow \frac{\pi x}{4} = 0 \text{ or } \cos(nx) = 0$$

$$\Rightarrow x = 0 \text{ or } nx = \frac{\pi}{2}$$

So when $x = 2 \Rightarrow 2n = \frac{\pi}{2}$.

$$\therefore n = \frac{\pi}{4}$$

Mark allocation: 2 marks

- 1 mark for setting $\frac{\pi x}{4}\cos(nx) = 0$.
- 1 mark for correctly simplifying to give $n = \frac{\pi}{4}$.

Question 4b.

Worked solution

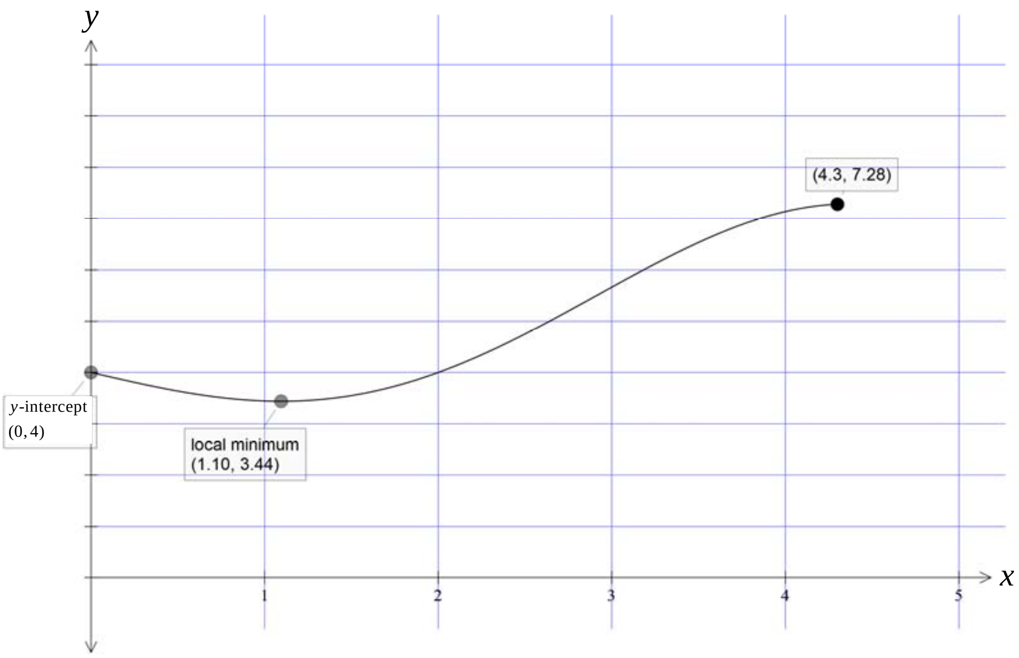
Using CAS, the second x-intercept occurs at (6, 0); so $b = 6$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 4c.i.

Worked solution



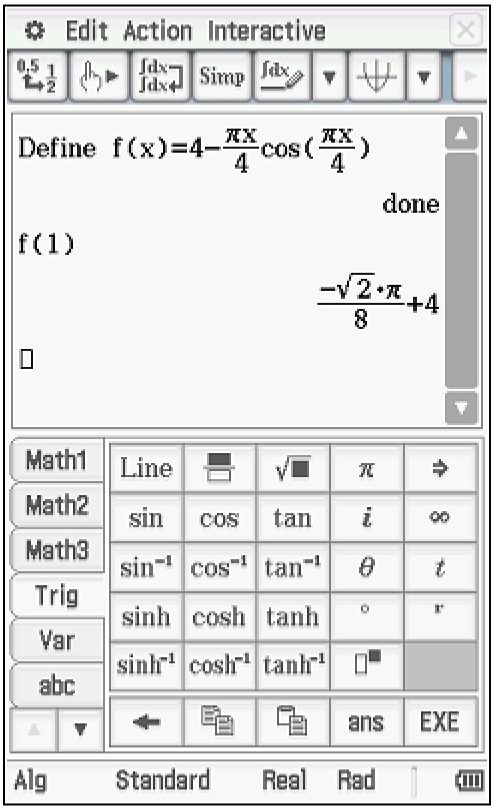
Mark allocation: 3 marks

- 1 mark for shape passing through (0, 4) and (2, 4).
- 1 mark for labelling endpoints correctly.
- 1 mark for labelling turning point correctly.

Question 4c.ii.

Worked solution

Using CAS, at $x = 1$, $y = \frac{-\sqrt{2}\,\pi}{8} + 4$.



Mark allocation: 1 mark

- 1 mark for the correct answer.

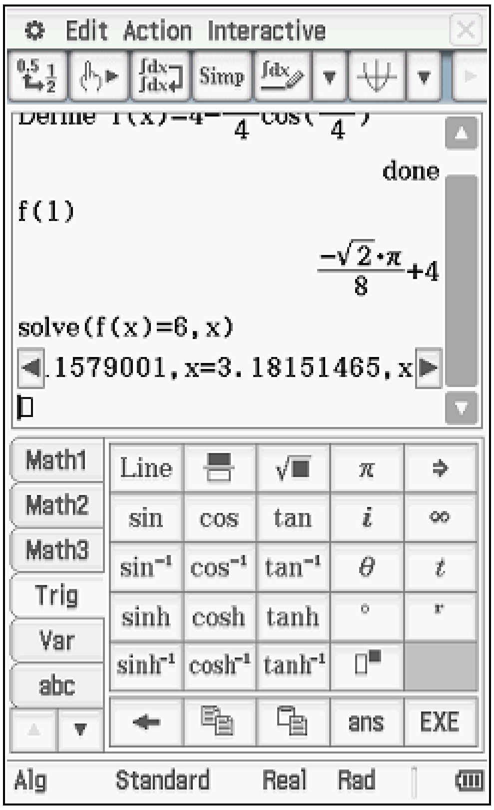
Tip

- *Note that an exact value is required here.*

Question 4c.iii.

Worked solution

Use CAS to find when $f(x) = 6$.



So $f(x) \geq 6$ for $3.1815 \leq x \leq 4.3,$ so for 1.119 km.

Mark allocation: 2 marks

- 1 mark for finding the points and intersection.
- 1 mark for the correct answer 1.119.

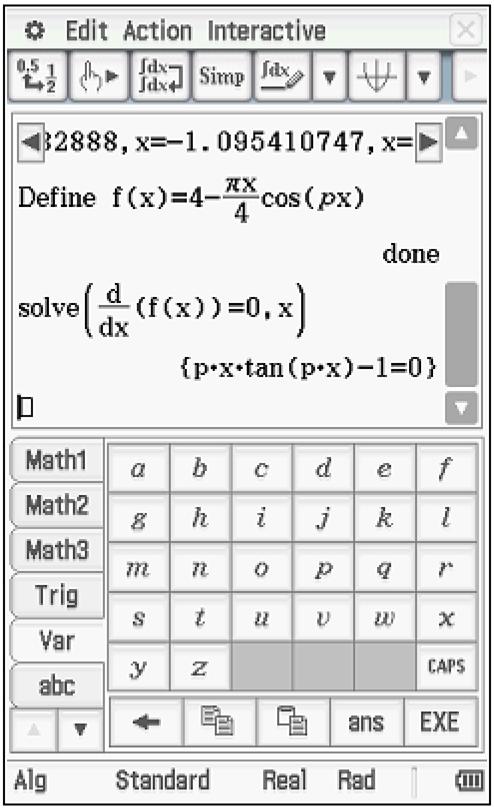
Tips

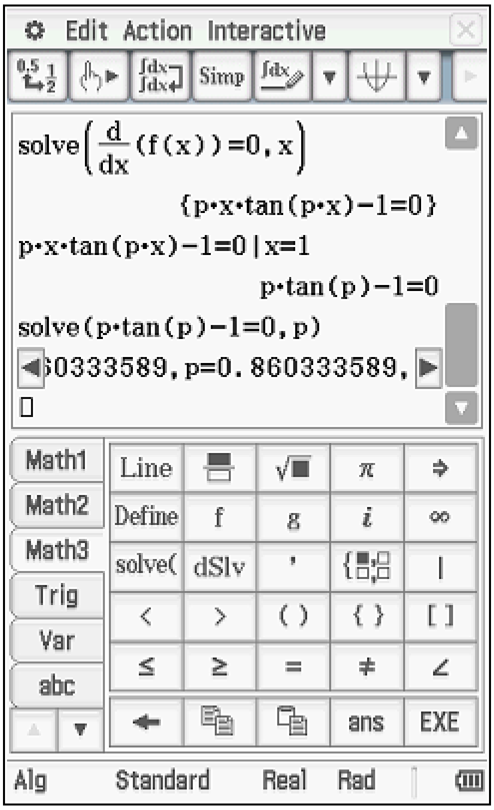
- *Be careful of the endpoint! The graph stops at x = 4.3, so the upper endpoint of the interval is 4.3.*
- *To give an answer that is correct to 3 decimal places requires that your working be done to at least 4 decimal places.*

Question 4d.i.

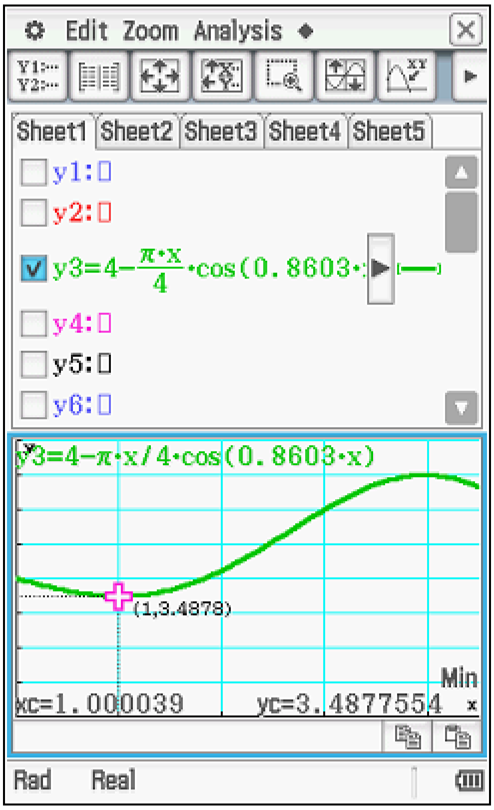
Worked solution

Use CAS to find p for when $\frac{d}{dx}\left(4-\frac{\pi x}{4}\cos(px)\right)=0$ with $x=1$.





Although this gives a number of values for p, $p = 0.8603$ gives the correct outcome. This can be verified by checking the graph $y = 4 - \frac{\pi x}{4}\cos(px)$ with $p = 0.8603$.



Mark allocation: 2 marks

- 1 mark for setting $\frac{d}{dx}\left(4 - \frac{\pi x}{4}\cos(px)\right) = 0$ with $x = 1$.
- 1 mark for the answer $p = 0.8603$.

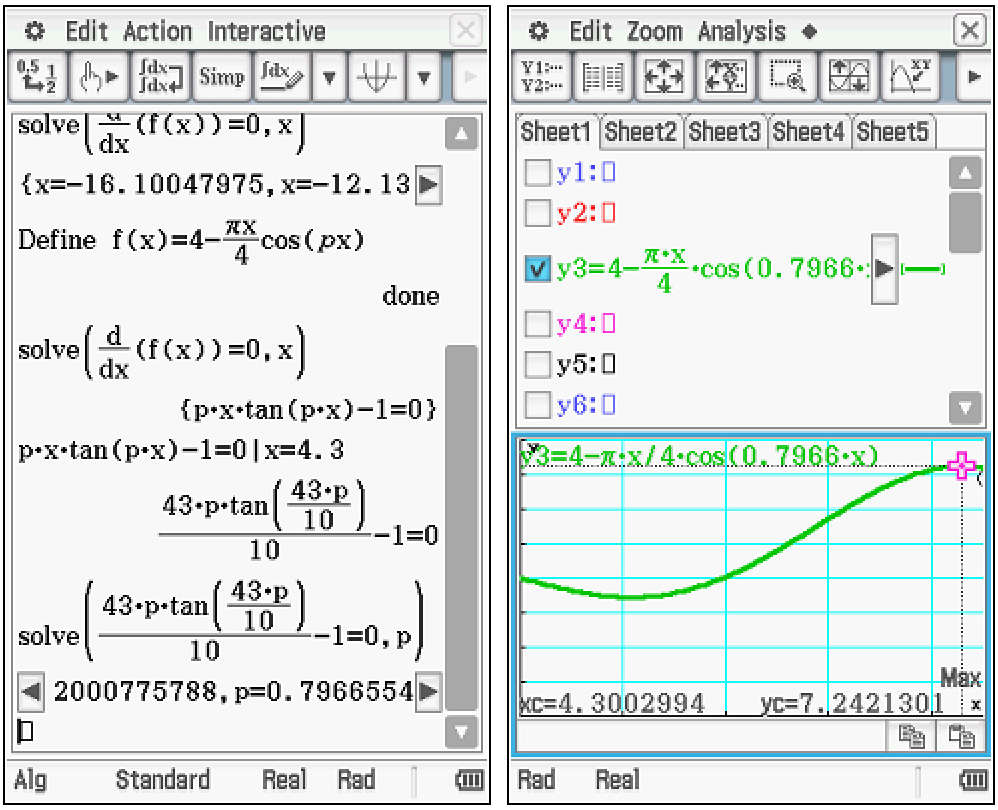
Question 4d.ii.

Worked solution

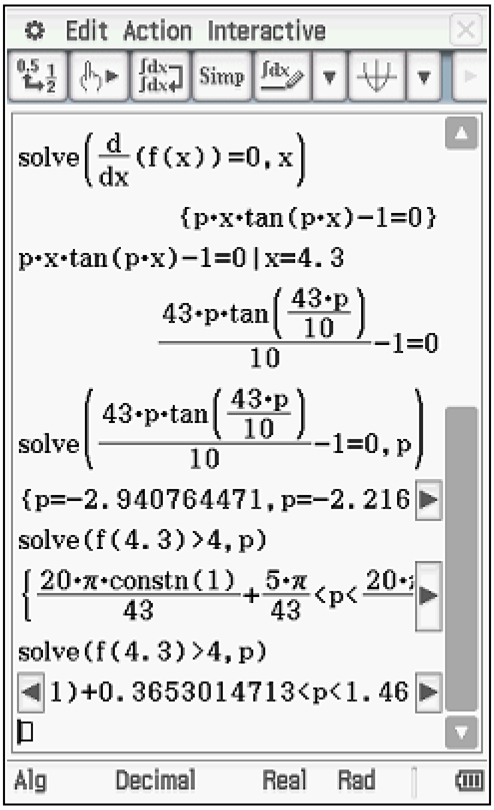
For the maximum to occur at the endpoint, this requires that the maximum turning point occurs at $x \geq 4.3$ and that $f(4.3) > f(0) = 4$.

Use CAS to find p such that $\frac{d}{dx}\left(4 - \frac{\pi x}{4}\cos(px)\right) = 0$ with $x = 4.3$.

This gives a number of values for p. Verifying the correct value for p by checking the graph gives $p = 0.7966$. (Note: The value 0.7967 gives a turning point just before the endpoint.)



Using CAS to solve $f(4.3) > 4$ for p gives $p > 0.3653$.



So for the maximum to occur at $x = 4.3,$ then $0.3653 < p < 0.7966.$

Mark allocation: 4 marks

- 1 mark for setting $\dfrac{d}{dx}\left(4-\dfrac{\pi x}{4}\cos(px)\right)=0$ with $x = 4.3$.
- 1 mark for $p < 0.7966$.
- 1 mark for $f(4.3) > 4$.
- 1 mark for $0.3653 < p < 0.7966$.

Tip

- *Don't assume that the normal rules for rounding apply. Always check your answer by sketching a graph and seeing whether it complies with the requirements. In this case, a 'rounded down' answer of 0.7966 was required.*

END OF WORKED SOLUTIONS BOOK