insight™

Year 12 *Trial Exam Paper*

2014

MATHEMATICAL METHODS (CAS)

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

Question 1a.

Worked solution

$$\frac{d}{dx}(x\sin(2x)) = \sin(2x) + 2x\cos(2x)$$

Mark allocation: 2 marks

- 1 mark for evidence of using the product rule.
- 1 mark for the correct answer.

Question 1b.

Worked solution

$$f(x) = e^{\sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$$

$$f'(4) = \frac{1}{4}e^{2}$$

Mark allocation: 2 marks

- 1 mark for the correct derivative $f'(x)$.
- 1 mark for the correct answer.

Tip

- *Remember to re-read the question before moving on. Many students differentiate correctly but then forget to evaluate.*

Question 1c.

Worked solution

Average value of a function is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$=\frac{1}{k-1}\int_1^k \frac{1}{5-x}\,dx$$

$$=-\frac{1}{k-1}\left[\log_e|5-x|\right]_1^k$$

$$=-\frac{1}{k-1}\left[\log_e(5-k)-\log_e(4)\right]$$

$$=-\frac{1}{k-1}\log_e\left(\frac{5-k}{4}\right)$$

$$=\frac{1}{k-1}\log_e\left(\frac{4}{5-k}\right)$$

Setting $\frac{1}{k-1}\log_e\left(\frac{4}{5-k}\right)=\frac{1}{2}\log_e(2)$ gives $k=3$.

Mark allocation: 3 marks

- 1 mark for setting up $\frac{1}{k-1}\int_1^k \frac{1}{5-x}\,dx$.
- 1 mark for recognising $\frac{1}{k-1}\left[\log_e|5-x|\right]_1^k$ (or equivalent) as the integral.
- 1 mark for answer $k=3$.

Question 2

Worked solution

$$\begin{bmatrix} 0 & 4 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\left.\begin{aligned} 4y &= x' \\ -3x &= y' \end{aligned}\right\} \Rightarrow \quad \begin{aligned} y &= \frac{x'}{4} \\ x &= -\frac{y'}{3} \end{aligned}$$

Substituting into the equation $2y - 3x = 5$ gives $2\frac{x}{4} - 3\frac{-y}{3} = 5,$ which simplifies to $\frac{x}{2} + y = 5 \Rightarrow y = -\frac{x}{2} + 5.$

Mark allocation: 2 marks

- 1 mark for expanding the matrix to get equations for x and y in terms of y' and x'.
- 1 mark for the answer $y = -\frac{x}{2} + 5$ or equivalent versions.

Question 3a.

Worked solution

$$\begin{aligned} f\left(g(x)\right) &= |x|^2 - 4|x| + 3 \\ &= x^2 - 4|x| + 3 \end{aligned}$$

Or $f\left(g(x)\right) = \begin{cases} x^2 - 4x + 3, & x \geq 0 \\ x^2 + 4x + 3, & x < 0 \end{cases}$

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 3b.

Worked solution

Considering $f\left(g(x)\right)=\begin{cases}x^2-4x+3, & x\geq 0\\ x^2+4x+3, & x<0\end{cases}$

For $x\geq 0,\ f(x)=x^2-4x+3,$ so x-intercepts occur at

$x^2-4x+3=0$

$(x-1)(x-3)=0$

So $x=1,\ x=3.$

Turning point at $x=\frac{-b}{2a},$ so

$x=\frac{4}{2}=2$

$y=-1$

Turning point is $(2,\ -1).$

For $x<0,\ f(x)=x^2+4x+3$, so x-intercepts occur at

$x^2+4x+3=0$

$(x+1)(x+3)=0$

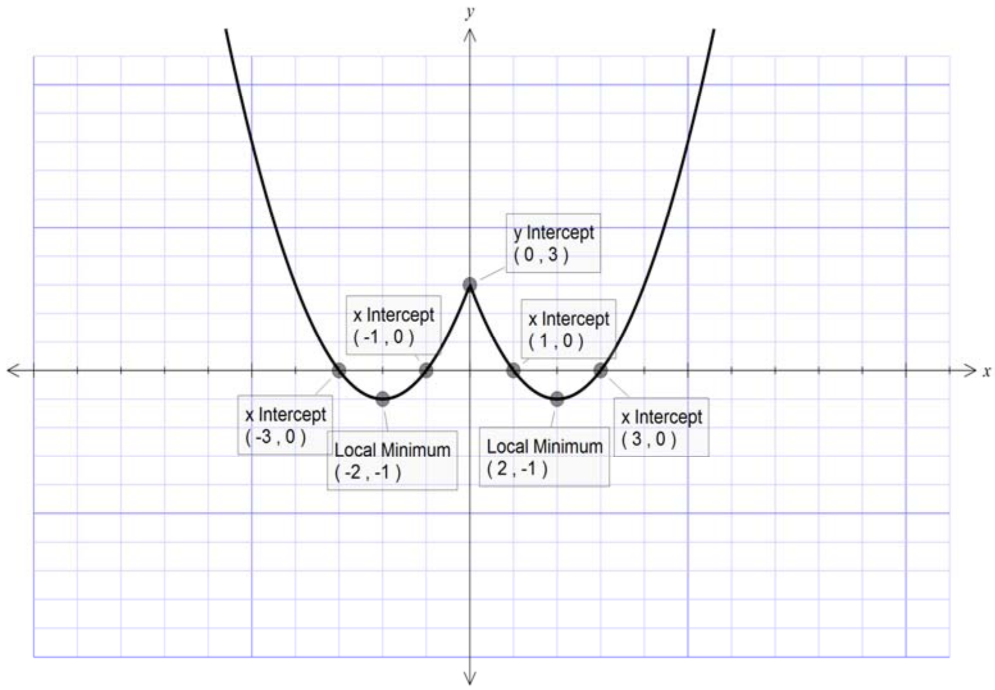
So $x=-1,\ x=-3.$

Turning point at $x=\frac{-b}{2a},$ so

$x=\frac{-4}{2}=-2$

$y=-1$

Turning point is $(-2,\ -1).$



Mark allocation: 3 marks

- 1 mark for shape of graph showing two parabolic sections and a cusp at the *y*-axis.
- 1 mark for all intercepts labelled correctly.
- 1 mark for labelling the turning points correctly.

Tips

- *To sketch graphs of the form* $y = f(|x|)$, *first sketch the graph for* $x \geq 0$, *then reflect the graph in the y-axis and this 'mirror image' becomes the graph for* $x < 0$.
- *Be careful to draw cusps as pointy sections, not as curves.*

Question 3c.

Worked solution

Domain of the derivative is $R \setminus \{0\}$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Tips

- *Graphs are never differentiable at cusps.*
- *Remember to use a 'back slash' or reverse solidus; symbol:* \.

Question 4a.

Worked solution

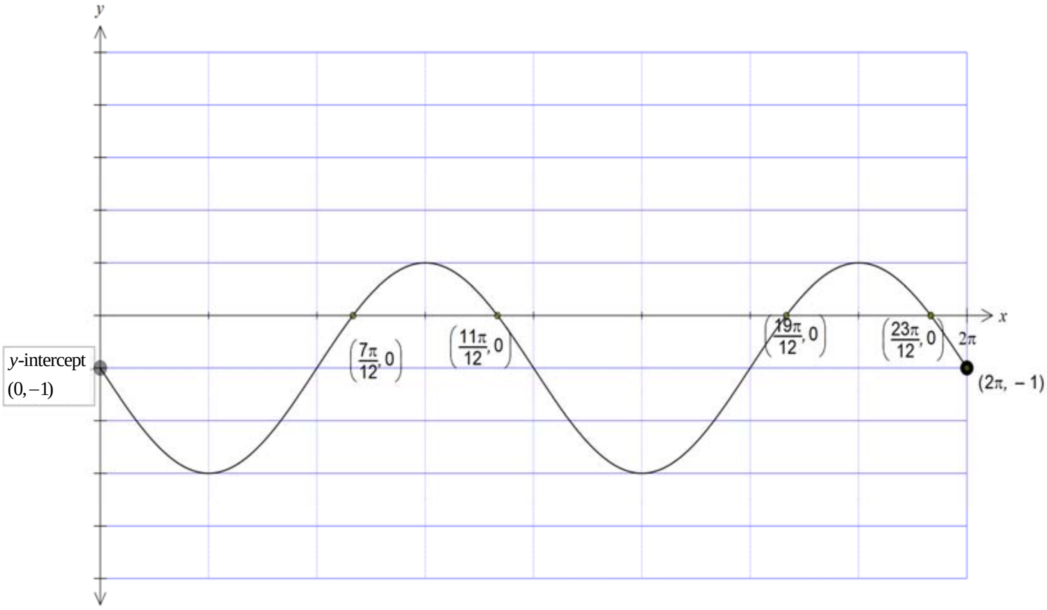
Range is [–3, 1] and period is π.

Mark allocation: 2 marks

- 1 mark for the range.
- 1 mark for the period.

Question 4b.

Worked solution



To find the x-intercepts, first solve $-2\sin(2x)-1=0$:

$$-2\sin(2x)-1=0$$
$$\sin(2x)=-\frac{1}{2}$$
$$2x=\pi+\frac{\pi}{6},\ 2\pi-\frac{\pi}{6}$$
$$2x=\frac{7\pi}{6},\ \frac{11\pi}{6}$$
$$x=\frac{7\pi}{12},\ \frac{11\pi}{12}$$

Additional intercepts are found by adding the period of π to both answers.
This gives x-intercepts of $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

Mark allocation: 3 marks

- 1 mark for showing two cycles.
- 1 mark for all x-intercepts labelled correctly.
- 1 mark for both end points labelled correctly.

Question 4c.

Worked solution

This is best done graphically.
Look to place a horizontal line through the graph and have this line intersect the graph in *four* places.
It can be observed that this happens when $-3 < p < -1 \cup -1 < p < 1$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 5a.

Worked solution

Interchange x and y:

$$x = 3\log_e(4 - y)$$

$$\frac{x}{3} = \log_e(4 - y)$$

$$e^{\frac{x}{3}} = 4 - y$$

$$y = 4 - e^{\frac{x}{3}}$$

So $f^{-1}(x) = 4 - e^{\frac{x}{3}}$.

Mark allocation: 2 marks

- 1 mark for swapping x and y.
- 1 mark for the correct rule.

Tip

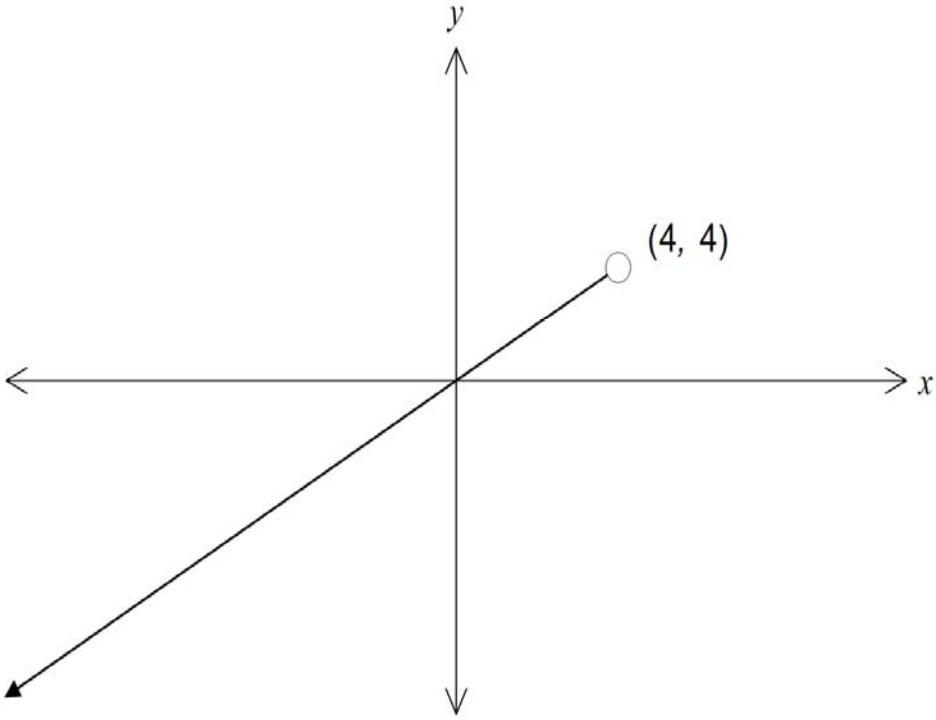
- *You must use the correct notation. In this case '$y =$' is not acceptable; the answer must be written with* f^{-1}.

Question 5b.

Worked solution

$f^{-1}(f(x)) = x$ for $x \in \text{dom}(f(x))$.

So in this case $f^{-1}(f(x)) = x$ for $x \in (-\infty, 4)$.



Mark allocation: 1 mark

- 1 mark for correctly drawn graph with correct domain.

Tip

- *Always consider the domain of a function.*

THIS PAGE IS BLANK

Question 6

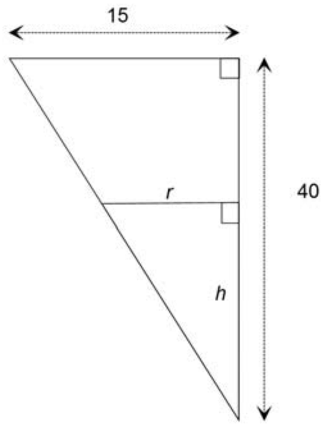
Worked solution

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$, where $\frac{dV}{dt} =$ 6 cm^3/min.

$\frac{dh}{dV}$ will need to be found by developing a relationship between *h* and *V*.

For a cone, $V = \frac{1}{3}\pi r^2 h$.

For this cone, the following pair of similar triangles apply:



This gives $\frac{r}{15} = \frac{h}{40}$, so $r = \frac{3h}{8}$.

The formula for a cone is $V = \frac{1}{3}\pi r^2 h$, so for this cone $V = \frac{1}{3}\pi\left(\frac{3h}{8}\right)^2 h = \frac{3\pi h^3}{64}$.

Therefore, a volume of 24π has a height of

$$24\pi = \frac{3\pi h^3}{64}$$

$$8 \times 64 = h^3$$

$$h = 8\,\text{cm}$$

So differentiating $V = \frac{3\pi h^3}{64}$ gives $\frac{dV}{dh} = \frac{9\pi h^2}{64}$ and $\frac{dh}{dV} = \frac{64}{9\pi h^2}$.

Therefore, $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$.

Substituting, this gives $\frac{dh}{dt} = 6 \times \frac{64}{9\pi h^2}$

So, at a height of 8 cm, $\frac{dh}{dt} = 6 \times \frac{64}{9 \times 64\pi}$

$$= \frac{6}{9\pi} = \frac{2}{3\pi} \text{ cm/min}$$

Mark allocation: 3 marks

- 1 mark for setting up the rate equation $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$.
- 1 mark for obtaining $\frac{dh}{dV} = \frac{64}{9\pi h^2}$.
- 1 mark for the answer $\frac{2}{3\pi}$ cm/min.

Question 7

Worked solution

$$\begin{array}{cc} & \begin{array}{cc} A_i & B_i \end{array} \\ \begin{array}{c} A_{i+1} \\ B_{i+1} \end{array} & \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix} \end{array}$$

$$\Pr(B, A, A, A, B) = 1 \times 0.6 \times 0.3 \times 0.3 \times 0.7 = \frac{3}{5} \times \frac{3}{10} \times \frac{3}{10} \times \frac{7}{10} = \frac{189}{5000}$$

Mark allocation: 2 marks

- 1 mark for either writing the transition matrix or for identifying the chain of probabilities.
- 1 mark for the correct answer.

Question 8a.

Worked solution

Note that c will be a value that is one standard deviation above the mean of X.
So $c = 36 + 8 = 44$.

Alternatively, using symmetry, $\frac{c - 36}{8} = 1$, so $c = 44$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 8b.

Worked solution

Since 20 is 2 standard deviations below the mean of X, d will be an equivalent value that is 2 standard deviations above the mean of Z; so $d = 2$.

Alternatively, $z = \frac{\mu - X}{\sigma} = \frac{36 - 20}{8} = 2$

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 9a.

Worked solution

Let $\int_0^4 k(2x+3)\,dx = 1.$

$$\begin{aligned}\text{LHS} &= k\int_0^4 (2x+3)\,dx \\ &= k\left[x^2+3x\right]_0^4 \\ &= k\left[28-0\right] \\ &= 28k\end{aligned}$$

So $28k = 1 \Rightarrow k = \dfrac{1}{28}$

Mark allocation: 2 marks

- 1 mark for setting up the integral equal to 1 or for using the area of a triangle.
- 1 mark for the correct antiderivative, leading to the correct result of *k*.

Question 9b.

Worked solution

$$\begin{aligned}\Pr(X \le 2 \mid X < 3) &= \frac{\Pr(X \le 2 \cap X < 3)}{\Pr(X < 3)} \\ &= \frac{\Pr(X \le 2)}{\Pr(X < 3)} \\ &= \frac{\frac{1}{28}\int_0^2 (2x+3)\,dx}{\frac{1}{28}\int_0^3 (2x+3)\,dx} \\ &= \frac{\left[x^2+3x\right]_0^2}{\left[x^2+3x\right]_0^3} = \frac{10}{18} = \frac{5}{9}\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for setting up a conditional probability.
- 1 mark for the correct answer.

Question 10a.

Worked solution

$$m_{AB} = \frac{f(0) - f(-p)}{0 - (-p)} = \frac{f(0) - f(-p)}{p}$$

$$m_{BC} = \frac{f(p) - f(0)}{p - 0} = \frac{f(p) - f(0)}{p}$$

Let $m_{AB} = m_{BC}$, giving:

$$\frac{f(p) - f(0)}{p} = \frac{f(0) - f(-p)}{p}$$

$$f(p) - f(0) = f(0) - f(-p)$$

$$f(p) + f(-p) = 2f(0)$$

$$\frac{f(p) + f(-p)}{2} = f(0)$$

Mark allocation: 1 mark

- 1 mark for the correct working, leading to the required answer.

Question 10bi.

Worked solution

$$f(3) = -27 + 9b + 3c + d$$

$$f(-3) = 27 + 9b - 3c + d$$

$$f(0) = d$$

Using the result from part **a** gives:

$$\frac{f(p) + f(-p)}{2} = f(0)$$

$$\frac{-27 + 9b + 3c + d + 27 + 9b - 3c + d}{2} = d$$

$$\frac{18b + 2d}{2} = d$$

$$18b + 2d = 2d$$

$$18b = 0$$

$$b = 0$$

Mark allocation: 2 marks

- 1 mark for $f(3)$, $f(-3)$ and $f(0)$.
- 1 mark for correct working, leading to the required result.

Question 10bii.

Worked solution

When $b = 0$, $f(x) = -x^3 + cx + d$, so $f'(x) = -3x^2 + c$.

$$m_{AC} = \frac{f(3) - f(0)}{3} = \frac{-27 + 3c + d - d}{3} = -9 + c$$

Let $f'(x) = m_{AC}$.

So, $-9 + c = -3x^2 + c$

$3 = x^2$

$x = \pm\sqrt{3}$

$x = \sqrt{3}$ gives $f(\sqrt{3}) = -3\sqrt{3} + \sqrt{3}c + d$.

$x = -\sqrt{3}$ gives $f(-\sqrt{3}) = 3\sqrt{3} - \sqrt{3}c + d$.

So, the coordinates are $(\sqrt{3}, -3\sqrt{3} + \sqrt{3}c + d)$ and $(-\sqrt{3}, 3\sqrt{3} - \sqrt{3}c + d)$.

Mark allocation: 3 marks

- 1 mark for finding $f'(x)$.
- 1 mark for setting $f'(x) = m_{AC}$.
- 1 mark for the correct coordinates.

END OF WORKED SOLUTIONS BOOK