QATs
Quality Assessment Tasks

UNIT 4 ■ OUTCOMES 1, 2 & 3

VCE Mathematics Methods (CAS)

SCHOOL-ASSESSED COURSEWORK

Introduction

Outcome 1
Define and explain key concepts as specified in the content from the areas of study, and apply a range of related mathematical routines and procedures.

Outcome 2
Apply mathematical processes in non-routine contexts, and analyse and discuss these applications of mathematics.

Outcome 3
Select and appropriately use technology to develop mathematical ideas, produce results and carry out analysis in situations requiring problem-solving, modeling or investigative techniques or approaches.

Task

Analysis Task – Extended Response Questions on Probability

The task has been designed to allow achievement up to and including the highest level in the performance Descriptors. It covers a broad range of **key knowledge** and **key skills** over the three outcomes for Unit 4.

It will contribute 20 out of the total (40) marks allocated for SAC in Unit 4.

This task will be marked out of 80 and then will be converted to a proportion of the contribution of this task to SAC in this unit.

The marks for each part are indicated in brackets. Answer in space provided or as directed.

You have 120 minutes over no more than two days. Work in progress will be collected.

You can access your logbook and an approved CAS calculator.

T Indicates where use of the technology is specifically required in order to answer the question.

Your teacher will advise you of any variation to these conditions.

NAME:

Task

PART 1

Question 1 Discrete Random Variables **(13 marks)**

Snooker is a game played on a table with 22 balls of various colours. The aim of the game is to knock all of the balls, *except the white*, into pockets around the edges of the table.

The number of balls of each colour and their relative value are given on the right.

Colour	Number of Balls	Value
White (Cue)	1	0
Red	15	1
Yellow	1	2
Green	1	3
Brown	1	4
Blue	1	5
Pink	1	6
Black	1	7

Minh and Jo-Anne have put the balls into a bag and are playing a guessing game where they score points using the same values as for the real game of snooker.

One player selects a ball and the other player has to guess the ball's value.

a) Complete the table below. (2 marks)

Colour	W	R	Y	G	Br	Blue	P	Black
Pr(x)	$\frac{1}{22}$		$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{22}$
Value	0	1	2	3	4	5	6	7
x.Pr(X=x)			$\frac{1}{11}$	$\frac{3}{22}$	$\frac{2}{11}$	$\frac{5}{22}$	$\frac{3}{11}$	$\frac{7}{22}$

b) Calculate the expected value of the distribution. (1 mark)

c) Now calculate the variance of the distribution. Give your answer to two decimal places. (2 marks). **T**

Task

d) After each ball is chosen it is replaced in the bag. What is the probability of either player selecting a ball with a value greater than 3? (1 mark)

e) Calculate the median of the distribution. (2 marks)

f) By comparing your answers for expected value and median, what conclusions can you make about the distribution? (2 marks)

g) Minh takes his turn. What is the probability that he has chosen a coloured (non-red) ball? (1 mark)

h) When Minh takes his turn he tells Jo-Anne that he has chosen a coloured (non-red) ball. What is the probability that the ball he has chosen has a value less than 4? (2 marks)

Task

Question 2 Bernoulli Distribution **(18 marks)**

Axelle has found that she has a 3 in 5 chance of receiving mail on any given day of the working week (Monday to Friday).

a) What is the probability that Axelle receives mail only on *Monday*, *Tuesday* and *Wednesday* of a particular week? (2 marks)

b) What is the probability that Axelle receives mail on exactly *three* of the five days in a given week? Give your answer correct to 4 decimal places. (2 marks)

c) In any given four week period with 20 working days, what is the expected value and standard deviation for Axelle receiving mail? Give your answers correct to 3 decimal places where appropriate. (2 marks)

d) What is the expected number of days she should receive mail in a calendar month with 31 days: Give your answers correct to 1 decimal place.

i. If the 1st of the month is a Monday (2 marks) **T**

ii. If the 1st of the month is a Friday. (2 marks). **T**

Task

e) There is a third possible answer for expected value, 13.2. How many working days would such a month have to have in order to achieve this result? (2 marks)

f) Calculate the standard deviation for all three possible situations, giving your answers correct to 3 decimal places. (3 marks) **T**

g) Without calculating the result, describe how you could calculate the expected value of Axelle receiving mail during any month with 31 days. (3 marks)

Task

Question 3 Markov Chains **(11 marks)**

Meaghan is a keen reader and borrows a new novel from her school library every Wednesday afternoon. Unfortunately, she does not always remember to return her previous book at the same time. The probability of her returning the book she currently has on time depends on whether she returned her last book on time or not.

If Meaghan returned her last book on time, then the probability that she will return her current book on time is $\frac{4}{9}$. On the other hand, if she failed to return her last book on time the chance of her returning her current book on time increases to $\frac{2}{3}$. The reason for this is that when she is late returning a book, the library sends reminder notices.

Terms are ten weeks long and students are not allowed to borrow books over the holidays.

a) Complete the transition matrix for this situation. (2 marks)

$$T = \begin{bmatrix} \frac{4}{9} & \frac{2}{3} \\ & \end{bmatrix}$$

b) If Meaghan returned the first book she borrowed for the term on time, what is the probability that she returned the last book of the term on time? Give your answer correct to 2 decimal places. (2 marks) <u>**T**</u>

c) If Meaghan returned the first book she borrowed for the term ***late***, what is the probability that she returned the last book of the term on time? Give your answer correct to 2 decimal places. (2 marks) <u>**T**</u>

d) Does the distribution reach a steady state by the end of a ten week term? Give a reason for your answer and justify this reason by quoting relevant values. Quote your answer to 6 decimal places. (2 marks)

Task

e) What is the probability of Meaghan returning all of her books on time? Give your answer correct to 3 decimal places. (1 mark)

f) What is the probability of Meaghan having a run of returning the first seven books on time but then being late with her eighth book? Give your answer correct to 3 decimal places. (2 marks) **T**

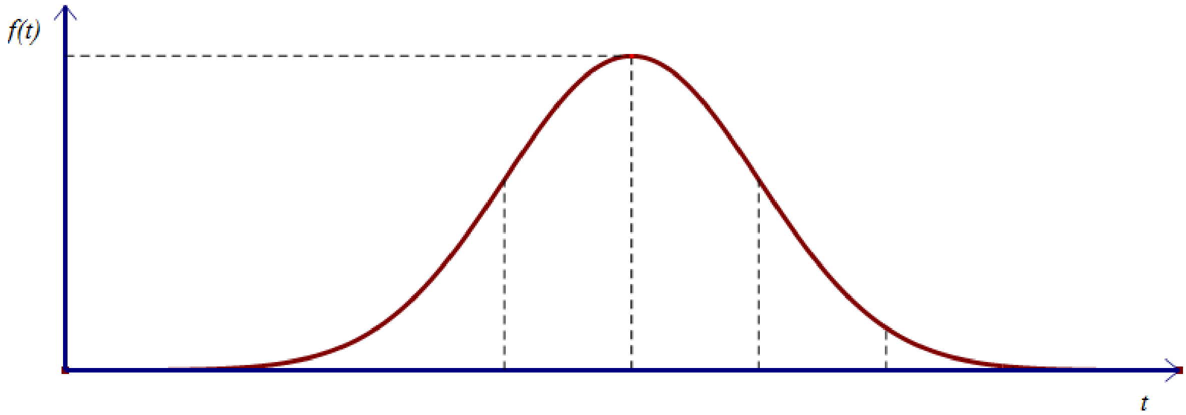
Task

PART 2

Question 4 Normal Distribution **(21 marks)**

Janai and Aiden are investigating how long it takes for popcorn kernels to pop when heated. In experiments they find that the time can be modelled by a normal distribution with a mean time is 15.01 seconds with a standard deviation of 3.37 seconds.

a) Use these figures to fill in the spaces under the graph below. (2 marks)



$\mu - \sigma =$ ______ $\mu + 2\sigma =$ ______

b) The equation of this function is $f(t) = \frac{1}{3.37\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{t\ -\ 15.01}{3.37}\right)^2}$. What is the area under this curve over the domain $x \in R$? (1 mark)

c) In standardising the general normal distribution, the formula for z scores is used to make two transformations. What two effects does the σ term have on the shape of the graph? (2 marks)

d) Calculate the probability of a particular kernel popping in under 20 seconds? Give your answer correct to 4 decimal places. (2 marks) **T**

Task

e) How long is it likely to take for 75% of the kernels to pop? Give your answer correct to 2 decimal places. (2 marks) **T**

f) In a particular sample of twenty kernels, the first one popped after 8.8 seconds and the last one popped at 25.38 seconds. In comparison to the normal distribution where do these values fit in terms of percentiles? (3 marks) **T**

g) In a bag of 1000 kernels, how many would be expected to pop:

i. before 10 seconds. (1 mark) **T**

ii. in the first 24 seconds. (1 mark) **T**

Janai and Aiden have also found that popped kernels will begin to burn according to another normal distribution with a mean 10 seconds higher than the first but the same standard deviation.

h) Write down the equation for the new function. (1 mark)

Task

i) For a bag of 1000 kernels, calculate how long it will be before 100 kernels have been burned. Give you answer to 2 decimal places. (2 marks) **T**

In general, finding the time when the number of unpopped kernels is the same as the number of burnt kernels is difficult to do. In this particular case, however, there is one property that makes finding the solution much simpler.

j) Name the property that the two distributions share and explain why it makes the calculation easier. (2 marks)

k) Now write down the time when the number of popped kernels is the same as the number of burnt kernels. Quote your answer to two decimal places. (1 mark)

l) Finally, calculate the percentage of burnt kernels at this time. Quote your answer to 2 decimal places. (1 mark) **T**

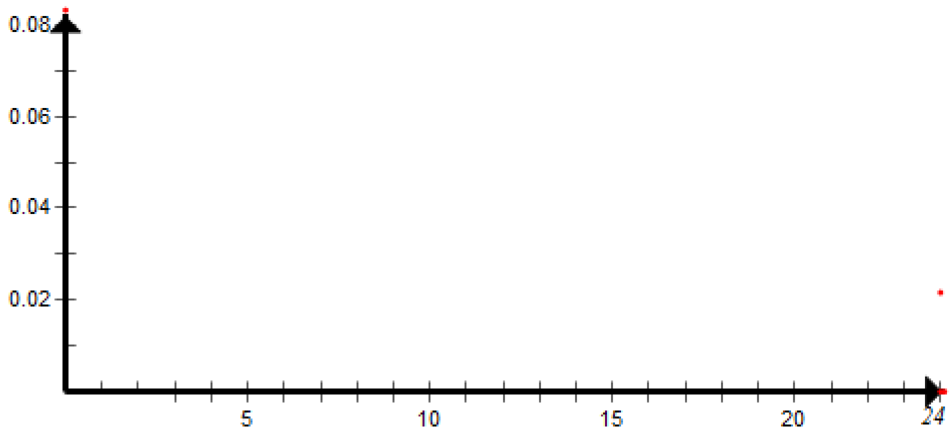
Task

Question 5 Continuous Random Variables **(17 marks)**

The probability of many natural events is based around the length of a day. Consider the relatively simply example of:

$$f(x) = \frac{1}{50}sin\left(\frac{\pi(x-6)}{12}\right) + \frac{1}{24},\quad 0 \le x \le 24$$

a) Draw this graph on the axes provided. Include maximum and minimum values as fractions. <u>**T**</u>
(3 marks)



b) Show using calculus that the area under the curve is indeed equal to one unit. (2 marks) <u>**T**</u>

c) Now consider the effect of changing the value of the fraction $\frac{1}{50}$ to say, $\frac{1}{25}$. Describe what happens to the curve. (1 mark)

Task

d) Can this new curve still be used as a continuous probability distribution? Provide evidence for your conclusion in the form of a calculation. (2 marks) **T**

e) What if the fraction at the end of the equation was changed instead, from $\frac{1}{24}$ to $\frac{1}{12}$ for example? (2 marks)

Next, consider the hybrid function:

$$g(x) = \begin{cases} \frac{1}{50} sin\left(\frac{\pi(x-6)}{12}\right) + \frac{1}{24}, & 0 \le x < 12 \\ \frac{-1}{600}x + \frac{43}{600}, & 12 \le x \le 24 \end{cases}$$

f) Calculate the mean for $g(x)$. (1 mark) **T**

Task

g) Write down values for the median and the mode of g(x). (2 marks)

h) Which of the following values remain unchanged between *f(x)* and *g(x)*? Circle your choice/s. (1 mark)

Mean Median Mode

i) What is the difference in the probabilities for *f(x)* and *g(x)* over the domain $12 \leq x \leq 18$? Quote your answer to four decimal places. (3 marks)

Teacher Advice

Assessment planning

This is the Analysis Task, as suggested to be undertaken in weeks 12 and 13 in the sample teaching sequence on page 193 of the VCAA Study Design.

This task covers assessment in:

- properties of anti-derivatives and definite integrals;
- the concepts of a random variable (discrete and continuous), bernoulli trials and markov chains and probability distributions, the parameters used to define a distribution and properties of probability distributions and their graphs;
- the conditions under which a bernoulli trial or markov chain, or a probability distribution, may be selected to suitably model various situation.

This task contributes 20 of the 40 SAC marks in Unit 4.
The coursework scores for this task are:

Outcome 1	8 marks	40%
Outcome 2	7 marks	35%
Outcome 3	5 marks	25%
TOTAL	20 marks	

This weighting can be used in the conversion of their mark out of 50.
For example, a score of 40 results in:

OUTCOME 1	OUTCOME 2	OUTCOME 3
40/80*20*0.4	40/80*20*0.35	40/80*20*0.25
= 4	= 3.5	= 2.5
	= 4 (rounded up)	= 4 (rounded up)

The above can be established in an Excel file.

Alternatively, the mark breakdown given in the following table can be used.

Teacher Advice

Question	Outcome 1 (8)	Outcome 2 (7)	Outcome 3 (5)
1 a)	2		
b)	1		
c)	1		1
d)		1	
e)	1	1	
f)		2	
g)	1		
h)	1	1	
2 a)	2		
b)	1	1	
c)	2		
d) i)		1	1
ii)		1	1
e)	1	1	
f)	1		2
g)	1	2	
3 a)	2		
b)	1		1
c)	1		1
d)		2	
e)	1		
f)	1		1
4a)	2		
b)		1	
c)		2	
d)	1		1
e)	1		1
f)		1	2
g) i)			1
ii)			1
h)	1		
i)		1	1
j)		2	
k)	1		
l			1

Teacher Advice

5 a)		2	1
b)	1		1
c)		1	
d)	1		1
e)		2	
f)			1
g)	2		
h)		1	
i)	1	2	

This QAT has been designed to meet the highest level in the performance descriptors provided by VCAA for each outcome in Unit 4 in the Assessment Handbook.

Notes on the two parts of the task

Questions 1-3 are based on the Discrete Random Variables, the Binomial Distribution and Markov Chains These are worth 42 marks.

Questions 4-5 are on continuous Probability distributions and the Normal distribution. These are worth 38 marks.

Solution Pathway

Question 1 Discrete Random Variables **(13 marks)**

a) (2 marks)

Colour	W	R	Y	G	Br	Blue	P	Black
Pr(x)	$\frac{1}{22}$	$\frac{15}{22}$	$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{22}$
Value	0	1	2	3	4	5	6	7
x.Pr(X=x)	0	$\frac{15}{22}$	$\frac{1}{11}$	$\frac{3}{22}$	$\frac{2}{11}$	$\frac{5}{22}$	$\frac{3}{11}$	$\frac{7}{22}$

b) $E(X) = \sum x \Pr(x)$

$= \frac{42}{22} = \frac{21}{11}$ (1 mark)

c)

x^2.Pr(X=x)	0	$\frac{15}{22}$	$\frac{2}{11}$	$\frac{9}{22}$	$\frac{8}{11}$	$\frac{25}{22}$	$\frac{18}{11}$	$\frac{49}{22}$

(1 mark)

$\sum x^2 \Pr(x) = \frac{154}{22}$ (1 mark)

$Var(x) = \frac{154}{22} - \frac{441}{121}$

$= \frac{406}{121} \approx 3.35$ (1 mark)

d) Pr(X= 4,5,6,7) = $\frac{4}{22} = \frac{2}{11}$ (1 mark)

e) Add the probabilities until the sum is greater than 0.5

	0	1
$\sum \Pr(X = x)\_$	$\frac{1}{22}$	$\frac{16}{22}$

(1 mark)

Median = 1 (1 mark)

Solution Pathway

f) It is negatively skewed (1 mark)
Both values are on the left side of the distribution (1 mark)

g) $\frac{7}{22}$. (1 mark)

h) This is conditional probability. There are 7 coloured balls in the bag.

The probability of Minh having any one of the coloured balls is now $\frac{1}{7}$. (1 mark)

The probability of choosing a ball with a value less than 4 is $\frac{3}{7}$. (1 mark)

Solution Pathway

Question 2 Bernoulli Distribution **(18 marks)**

a) $\Pr(X = M, T, W) = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{108}{3125}$ (Method 1, Answer 1)

b) n = 5 and P = 0.6 (1 mark)
Pr (X = 3) = ${}^5C_3\,(0.6)^3\,(0.4)^2 = 0.3456$ (1 mark)

(Students have to interpret the wording and show understanding of what n and p are in order to arrive at this. It is hence not a simple calculation question, hence the extra mark)

c) N = 20, p = 0.6

E(X) = 20 x 0.6 = 12 (1 mark)
Var(X) = 0.6 x 0.4 x 20 = 4.8 (1 mark)

$SD(X) = \sqrt{4.8}$

d)

i. If the 1st of the month is a Monday.

If the first of the month is a Monday then there are 23 working days. (1 mark)

$E(X) = 23 \times 0.6 = 13.8$ (1 mark)

ii. If the first of the month is a Friday then there are 21 working days. (1 mark)

$E(X) = 21 \times 0.6 = 12.6$ (1 mark)

e) Let $13.2 = n \times 0.6$

$E(X) = \frac{13.2}{0.6}$ (1 mark)

22 days. (1 mark)

f) For $n_1 = 23$, $SD(X) = \sqrt{23 \times 0.6 \times 0.4} = 2.349$ (1 mark)

For $n_2 = 21$, $SD(X) = \sqrt{21 \times 0.6 \times 0.4} = 2.245$ (1 mark)

For $n_3 = 22$, $SD(X) = \sqrt{22 \times 0.6 \times 0.4} = 2.298$ (1 mark)

Solution Pathway

g) Work out the number of days per week that give each of the three working day amounts. (1 mark)
Multiply each of these by their respective probabilities. (1 mark)
Find the average of the three values. (1 mark)

Solution Pathway

Question 3 Markov Chains **(11 marks)**

a) $T = \begin{bmatrix} \frac{4}{9} & \frac{2}{3} \\ \frac{5}{9} & \frac{1}{3} \end{bmatrix}$ (1 mark for each value)

b) $\begin{bmatrix} \frac{4}{9} & \frac{2}{3} \\ \frac{5}{9} & \frac{1}{3} \end{bmatrix}^8 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.54547 \\ 0.45453 \end{bmatrix}$ (1 mark for method, note that there are 8 transitions)

Pr(last book on time) = 0.54547 (1 mark)

c) $\begin{bmatrix} \frac{4}{9} & \frac{2}{3} \\ \frac{5}{9} & \frac{1}{3} \end{bmatrix}^8 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.545451 \\ 0.454549 \end{bmatrix}$ (1 mark for either representation of the method)

Pr(last book on time) = 0.545451 (1 mark)

d) No. The values from parts c) and d) are not yet equal to one another. (2 marks)

e) $\left(\frac{4}{9}\right)^8$ (1 mark). No need to go any further.

f) $\left(\frac{4}{9}\right)^7 \frac{5}{9} = 0.0019$ (1 mark for method, 1 mark for value)

Solution Pathway

Question 4 Normal Distribution **(21 marks)**

a) $\mu - \sigma = 15.01 - 3.37 = 11.64$ (1 mark)
$\mu + 2\sigma = 15.01 + 6.74 = 21.75$ (1 mark)

b) 1 unit (1 mark)
The distribution is divided by the standard deviation. This dilates the curve into a probability distribution.

c) σ is a dilation factor. It has the effect of reducing the maximum value of the curve and changing the area under the curve to one unit.
(1 mark for each reason)

d) $normcdf(-\infty, 20, 15.01, 3, 37) = 0.9307$ 1 method, 1 answer

e) $invnorm(0.75, 15.01, 3.37) = 17.283$ 1 method, 1 answer

f) $normcdf(-\infty, 8.8, 15.01, 3, 37) = 0.03268$ This is between the 3rd and 4th percentiles

normCdf($-\infty$,25.38,15.01,3.37) = 0.9990 This is between the 99th and 100th percentiles

1 mark for each answer and a mark for writing in terms of percentiles.

g)
i. before 10 seconds (1 mark)

Percentage = normCdf($-\infty$,10,15.01,3.37) = 0.068544

N = 1000,

So 1000 x 0.068544 = 68.5
68 kernels would be expected to pop in the first 10 seconds. (1 mark)

ii. after 24 seconds (1 mark)

Percentage = normCdf($-\infty$,24,15.01,3.37) = 0.996181

N = 1000,

So 1000 x 0.996181 = 68.5
996 kernels would be expected to pop in the first 24 seconds. (1 mark)

Solution Pathway

h)

$$f(t) = \frac{1}{3.37\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{t\ -\ 25.01}{3.37}\right)^2}$$ (1 mark)

i) $invNorm\left(\frac{100}{1000}, 25.01, 3.37\right) = 20.69$ seconds (area - 1 mark, answer – 1 mark)

j) The standard deviation is the same in both distributions. (1 mark)
This is the term that makes the area under the curve and the height of the curve the same for both distributions. (1 mark)

k) $\frac{25.01+15.01}{2} = 20.01$ (1 mark)

l)

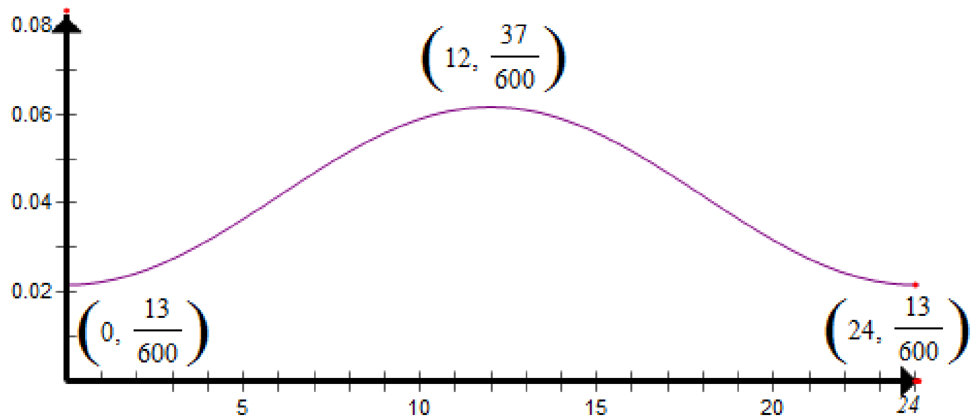
i. $normcdf(-\infty, 20.01, 25.01, 3.37) = 0.0689$

6.89% of the kernels are burnt. (1 mark)

Solution Pathway

Question 5 Continuous Random Variables **(17 marks)**

a)



Shape – 1, position – 1, values – 1

b) $\int_0^{24} \frac{1}{50} \, sin\left(\frac{\pi(x-6)}{12}\right) + \frac{1}{24} dx = \left[\frac{-6}{25\pi} \cos\left(\frac{\pi(x-6)}{12}\right) + \frac{x}{24}\right]_0^{24}$ (1 mark)

$= 1$ unit (1 mark)

c) It changes the dilation factor and hence the maximum and minimum values of the curve. (1 mark)

d) For it to be a probability density function it needs:

1) All values to be positive. This is so for this new function. (1 mark)
2) The area under the curve to still be equal to 1.

Check $\int_0^{24} \frac{1}{25} \, sin\left(\frac{\pi(x-6)}{12}\right) + \frac{1}{24} dx = 1$ (1 mark)

Yes it can still be used as a probability density function.

e) Changing the end fraction translates the curve parallel to the y-axis. (1 mark)

This in turn changes the area under the curve so no, this will not result in a new probability density function. (1 mark)

f) $\int_0^{12} x\left(\frac{1}{50}\left(\frac{\pi(x-6}{12}\right) + \frac{1}{24}\right) dx + \int_{12}^{24} x\left(\frac{-x}{600} + \frac{43}{600}\right) dx = 12.34$ (1 mark)

Solution Pathway

g) Median = 12
Mode = 12 (2 marks)

h) The median and the mode stay the same. *Both required*. (1 mark)

i) $f(t) = \int_{12}^{18} \left[\frac{1}{25} \sin\left(\frac{\pi(x-6)}{12}\right) + \frac{1}{24}\right] dx = 0.3264$ (1 mark)

$g(t) = \int_{12}^{18} \left(\frac{-t}{600} + \frac{43}{600}\right) dx = 0.28$ (1 mark)

Difference = 0.3264 – 0.28 = 0.0464 (1 mark)