2013 Mathematical Methods (CAS) Trial Exam 1 Solutions

Q1a $f(x)=ax^2+bx+c$, $f'(x)=2ax+b$

Q1b $f(g(x))=a(g(x))^2+b(g(x))+c=a(2x+3)^2+b(2x+3)+c$
$=a(4x^2+12x+9)+b(2x+3)+c$
$=4ax^2+2(6a+b)x+(9a+3b+c)$

Q1c $f(f'(x))=f'(f(x))$
$a(2ax+b)^2+b(2ax+b)+c=2a(ax^2+bx+c)+b$
$4a^3x^2+4a^2bx+ab^2+b^2+c=2a^2x^2+2ac+b$
Compare the coefficients:
$4a^3=2a^2$, $4a^2b=0$ and $ab^2+b^2+c=2ac+b$
Since $f(x)=ax^2+bx+c$ is a quadratic function,
.: $a\neq 0$, .: $b=0$
$4a^3=2a^2$, .: $4a^3-2a^2=0$, $2a^2(2a-1)=0$, .: $a=\frac{1}{2}$
$ab^2+b^2+c=2ac+b$, .: $2ac=c$, $2ac-c=0$, $c(2a-1)=0$
Since $2a-1=0$, .: c can be any real number.

Q2a *Transition matrix* $T=\begin{bmatrix}0.30 & 0.60\\0.70 & 0.40\end{bmatrix}$, *state matrix* $S_3=\begin{bmatrix}0\\1\end{bmatrix}$

$S_5=T^2S_3=\begin{bmatrix}0.30 & 0.60\\0.70 & 0.40\end{bmatrix}^2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0.42\\0.58\end{bmatrix}$ by hand
.: $\Pr(sugar.in.the.fifth)=0.42$

Q2b $S_3=T^2S_1=\begin{bmatrix}0.30 & 0.60\\0.70 & 0.40\end{bmatrix}^2\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0.42\\0.58\end{bmatrix}$
.: $S_1=\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$ by recognition
.: he does not have sugar in his first drink.

Q3a $f(x)=x^3-8x^2-4x+32=(x^3-8x^2)-(4x-32)$
$=x^2(x-8)-4(x-8)=(x^2-4)(x-8)=(x-2)(x+2)(x-8)$

Q3b $2^{3y}-2^{2y+3}-2^{y+2}+2^5=0$, $2^{3y}-8\times 2^{2y}-4\times 2^y+32=0$,
$(2^y)^3-8(2^y)^2-4(2^y)+32=0$
.: $(2^y-2)(2^y+2)(2^y-8)=0$
Since $2^y+2>0$, .: $2^y-2=0$ or $2^y-8=0$
.: $y=1$ or 3

Q4a $y=\log_2 x=\frac{\log_e x}{\log_e 2}$, $\frac{dy}{dx}=\frac{1}{x\log_e 2}$

At $x=2$, *gradient of the tangent* $m_t=\frac{1}{2\log_e 2}$

Q4b At $x=2$, $y=\log_2 2=1$
Equation of the tangent: $y-y_1=m_t(x-x_1)$,
$y-1=\frac{1}{2\log_e 2}(x-2)$

y-intercept: Let $x=0$, $y=1-\frac{1}{\log_e 2}$, .: $\left(0,1-\frac{1}{\log_e 2}\right)$

Q5 The line $y=x$ is a tangent to the graphs of $y=f^{-1}(x)$ and $y=f(x)$ at (a,a).
.: at (a,a), the gradient of $y=f(x)=e^{x-a}+1$ is 1.
The graph of $y=e^{x-a}+1$ is the horizontal translation of the graph $y=e^x+1$. At $(0,2)$, the gradient of the graph $y=e^x+1$ is $\frac{dy}{dx}=e^x=e^0=1$, .: $a=2$

Q6a $\cos\left(\frac{\pi}{12}-\frac{3\pi}{4}\right)=\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}$

Q6b $\sin\left(\frac{\pi}{2}-3x\right)+\frac{1}{2}=0$, $\frac{\pi}{6}<x<\frac{\pi}{2}$
$\cos(3x)+\frac{1}{2}=0$, $\frac{\pi}{2}<3x<\frac{3\pi}{2}$
.: $\cos(3x)=-\frac{1}{2}$, $3x=\frac{2\pi}{3}$, $\frac{4\pi}{3}$
.: $x=\frac{2\pi}{9}$, $\frac{4\pi}{9}$

Q7a $\int_0^a \frac{x+1}{4}dx=1$, $\left[\frac{x^2}{8}+\frac{x}{4}\right]_0^a=1$, $\frac{a^2}{8}+\frac{a}{4}=1$
$a^2+2a-8=0$, $(a-2)(a+4)=0$
Since $a>0$, .: $a=2$

Q7b *Average value of* $f(x)$ $=\frac{\int_0^2 \frac{x+1}{4}dx}{2-0}=\frac{1}{2}$ in $0\le x\le 2$.

Q7c *Average value of X* $=\int_0^2 x\left(\frac{x+1}{4}\right)dx=\int_0^2 \frac{x^2+x}{4}dx$

$=\left[\frac{x^3}{12}+\frac{x^2}{8}\right]_0^2=\frac{7}{6}$

Q8a The outcomes of distinct probability values are:
3 reds, 3 blues, 3 greens, 2 reds and 1 blue, 2 reds and 1 green, 2 blues and 1 red, 2 blues and 1 green, 2 greens and 1 red, 2 greens and 1 blue, 1 of each colour.
.: 10 distinct probability values.

Q8b There are six possible outcomes consisting of different colours.

$$\Pr(different.colours) = 6 \times \frac{1}{6} \times \frac{2}{6} \times \frac{3}{6} = \frac{1}{6}$$

Q8c Let X be the random variable of number of blue faces.

X has binomial distribution: $n = 3$, $p = \frac{2}{6} = \frac{1}{3}$

$$\Pr(X = 1) = {}^3C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Q9a $f(x) = \frac{1}{5}\sin(5x-1)$, $f'(x) = \cos(5x-1)$, $f'(0.2) = \cos 0 = 1$

Q9b $f(a+h) \approx f(a) + h \times f'(a)$

$f(0.21) = f(0.20 + 0.01)$

$\approx f(0.20) + 0.01 \times f'(0.20) = 0 + 0.01 \times 1 = 0.01$

Q10a $y = 2 - f(1-x) \xrightarrow{(1)} x = 2 - f(1-y)$

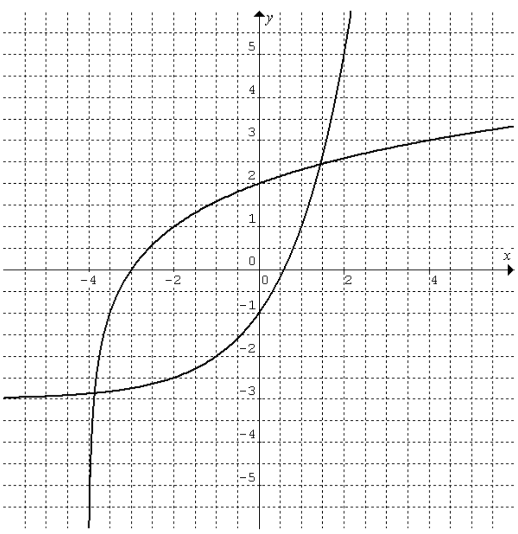
$\xrightarrow{(2)} x = 2 - f(2-y)$

$\xrightarrow{(3)} x = 1 - f(2-y)$

A sequence: (1) Reflection in the line $y = x$ (2) Upward translation by 1 unit (3) Translation to the left by 1 unit.

There are other possible sequences.

Q10b



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