INSIGHT

YEAR 12 *Trial Exam Paper*

2013

MATHEMATICAL METHODS (CAS)

Written examination 2

STUDENT NAME:

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
		Total	80

- Students are permitted to bring the following items into the examination: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer based CAS, their full functionality may be used.
- Students are NOT permitted to bring the following items into the examination: blank sheets of paper and/or white out liquid/tape.

Materials provided

- The question and answer book of 31 pages, with a separate sheet of miscellaneous formulas.
- An answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the box provided and on the answer sheet for multiple-choice questions.
- Remove the formula sheet during reading time.
- You must answer the questions in English.

At the end of the exam

- Place the answer sheet for multiple-choice questions inside the front cover of this question book.

Students are NOT permitted to bring mobile phones or any other electronic devices into the examination.

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SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the multiple-choice answer sheet.
Select the response that is **correct** for the question.
A correct answer scores 1 mark, whereas an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
If more than one answer is selected, no marks will be awarded.

Question 1

The function $f:[0,2\pi]\to R,\ f(x)=2-\sin(\frac{\pi}{3}-2x)$ has an amplitude and period, respectively, of

A. 3 and 2

B. 3 and 2π

C. 1 and π

D. 2 and π

E. 1 and $\frac{\pi}{2}$

Question 2

The range of the function $f(x)=\left|\log_e(a+x)\right|+b$ for $a,b\in R$ is

A. $[0,\infty)$

B. R^+

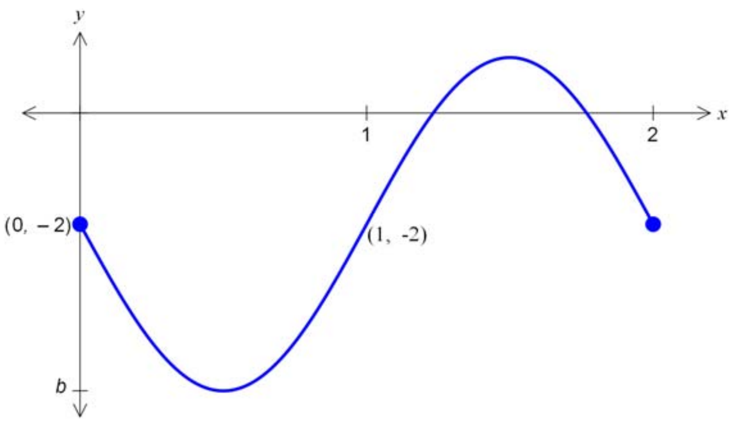
C. $(-\infty,0]$

D. $[b,\ \infty)$

E. (b,∞)

Question 3

The diagram below shows one cycle of the graph of a circular function, where $b < -2$.



A possible equation for the function whose graph is shown is

A. $y = -(b-2)\sin(2x) - 2$

B. $y = 2 - b\sin(2x)$

C. $y = 1 - (b+2)\sin(2\pi x)$

D. $y = (b+2)\sin(\pi x) - 2$

E. $y = (-b-2)\sin(2x) + 2$

Question 4

One cycle of the graph of the function with the equation $y = \tan(ax)$ has successive vertical asymptotes at $x = \frac{1}{12}$ and $x = \frac{1}{4}$.

A possible value for a is

A. 12

B. 12π

C. 4

D. 4π

E. 6π

Question 5

For the system of simultaneous linear equations

$$\begin{aligned} z &= 3 \\ x - z &= -5 \\ x + y &= 0 \end{aligned}$$

an equivalent matrix equation is

A. $\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} z \\ y \\ x \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$

D. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$

E. $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$

Question 6

The transformation $T : R^2 \to R^2$ that maps the curve with the equation $y = e^x$ to the curve with the equation $y = e^{2x+4} - 3$ could have the rule

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

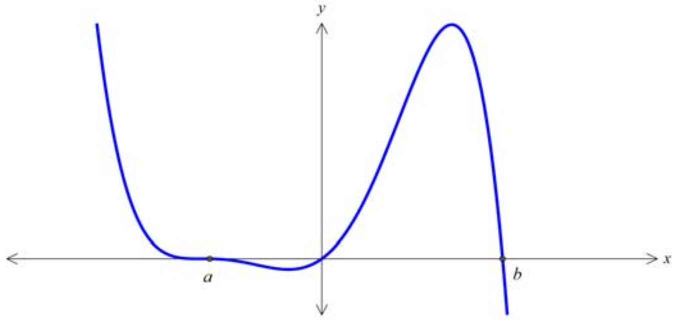
B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Question 7



The graph shown could be that of a function f with the equation

A. $y = -x(x+a)^3(x-b)$

B. $y = -x(x+a)^2(x-b)$

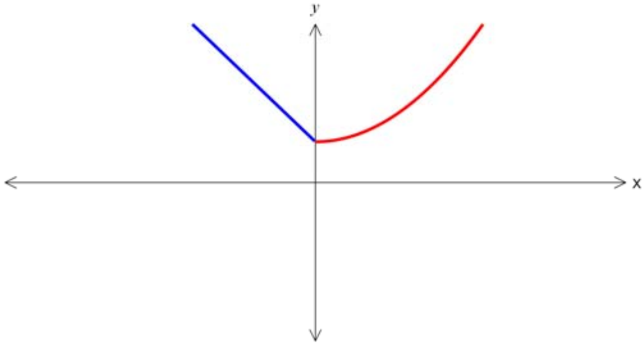
C. $y = x(x-a)^3(x-b)$

D. $y = -x(x+a)^3(b-x)$

E. $y = x(x-a)^3(b-x)$

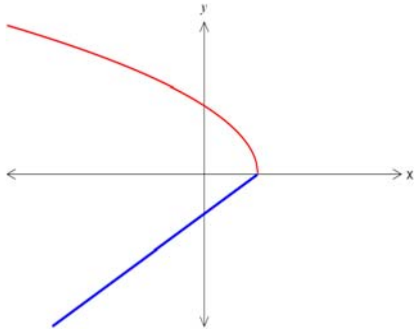
Question 8

The graph of the function with the equation $y = f(x)$ is shown below.

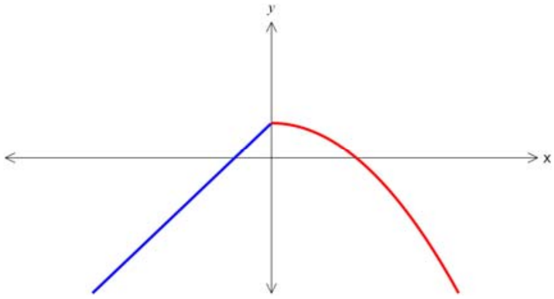


Which of the following is most likely to be the graph of $y = f^{-1}(-x)$?

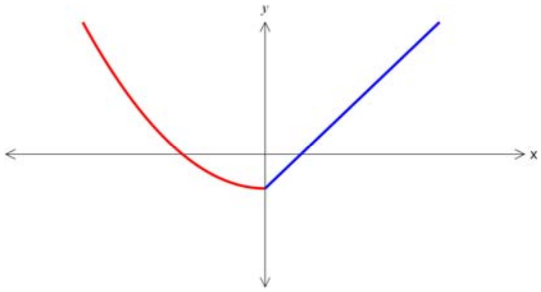
A.



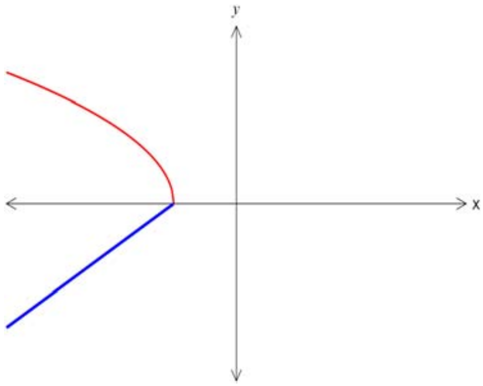
B.



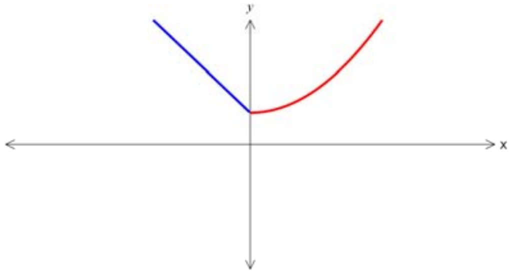
C.



D.



E.



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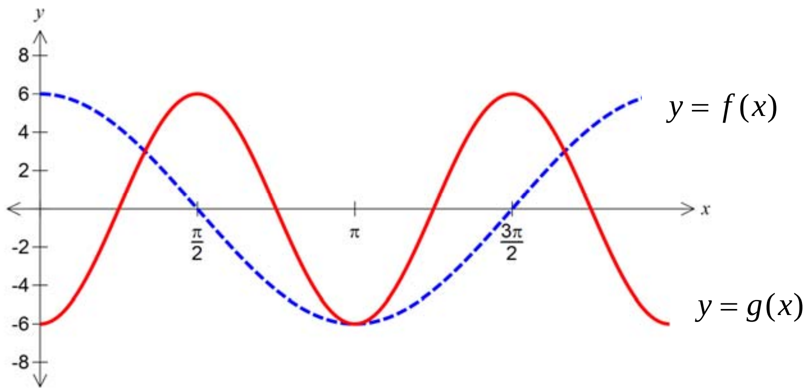
Question 9

The diagram below shows the graphs of two circular functions, f and g.



The graph of the function with the equation $y = f(x)$ is transformed into the graph of the function with the equation $y = g(x)$ by

A. a dilation by a scale factor of $\frac{1}{2}$ from the x-axis followed by a reflection in the x-axis.

B. a dilation by a scale factor of 2 from the y-axis followed by a reflection in the x-axis.

C. a dilation by a scale factor of 2 from the y-axis followed by a reflection in the y-axis.

D. a dilation by a scale factor of $\frac{1}{2}$ from the y-axis followed by a reflection in the x-axis.

E. a dilation by a scale factor of 2 from the x-axis followed by a reflection in the y-axis.

Question 10

The function defined by $f : A \to R,\ f(x) = e^{|x+b|},\ b \in R$, will have an inverse function for all values of b when its domain A is

A. R

B. $R \setminus \{b\}$

C. $[b, \infty)$

D. R^+

E. $[-b, \infty)$

Question 11

If $f(x) = e^x$ and $g(x) = \dfrac{1}{\sqrt{2x}}$, then $g(f(0))$ is

A. $\dfrac{1}{2e}$

B. undefined

C. $\dfrac{1}{2\sqrt{e}}$

D. $\dfrac{1}{\sqrt{2}}$

E. $\dfrac{1}{\sqrt{2e}}$

Question 12

If $y = \log_e(\sqrt{f(x)})$ then $\dfrac{dy}{dx}$ is equal to

A. $\dfrac{1}{\sqrt{f(x)}}$

B. $\dfrac{1}{2\sqrt{f(x)}}$

C. $\dfrac{f'(x)}{2\sqrt{f(x)}}$

D. $\dfrac{f'(x)}{2f(x)}$

E. $\dfrac{1}{2f(x)}$

Question 13

The equation of the normal to the curve with equation $y = x\sin(x)$, at the point on the curve with x-coordinate π is

A. $y = \dfrac{1}{\cos(x)}$

B. $y = -\pi$

C. $\pi(y - x) = 1$

D. $\pi(y + 1) = x$

E. $\pi y + 1 - x = 0$

Question 14

The average value of $y = 2e(e^x - 1)$ over the interval [0, 2] is

A. $2e^3 - 6e$

B. 0

C. $\dfrac{(e^2-1)^2}{2}$

D. $e^3 - 3e$

E. $(e^2-1)^2$

Question 15

The interval $[0, 8]$ is divided into n equal subintervals by the points $x_0,\ x_1,\ .....\, x_{n-1},\ x_n,$ where $0 = x_0 < x_1 < .... < x_{n-1} < x_n = 8.$ Let $\delta x = x_i - x_{i-1}$ for $i = 1,\ 2,\ ......,\ n.$

Then $\lim\limits_{\delta x \to 0} \sum\limits_{i=1}^{n} (4x_i \delta x)$ is equal to

A. $\displaystyle\int_8^0 4x\ dx$

B. $\displaystyle\int_0^8 2x^2\ dx$

C. 128

D. $\dfrac{1024}{3}$

E. 32

Question 16

If $f'(x) = \dfrac{5}{\cos^2(k\,x)}$ and k and c are real constants, then $f(x)$ is equal to

A. $5\tan(k\,x) + c$

B. $5k\tan(x) + c$

C. $5\sin^2(k\,x) + c$

D. $\dfrac{5}{k}\tan(k\,x) + c$

E. $5\log_e(\cos(k\,x)) + c$

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Question 17

Let p be a function defined for the interval $[a,\ b]$ and q a function such that $q'(x) = p(x)$, for all $x \in [a,\ b]$.

Hence, $\int_a^b p(2x)\,dx$ is equal to

A. $q(2x) + c$

B. $p(2b) - p(2a)$

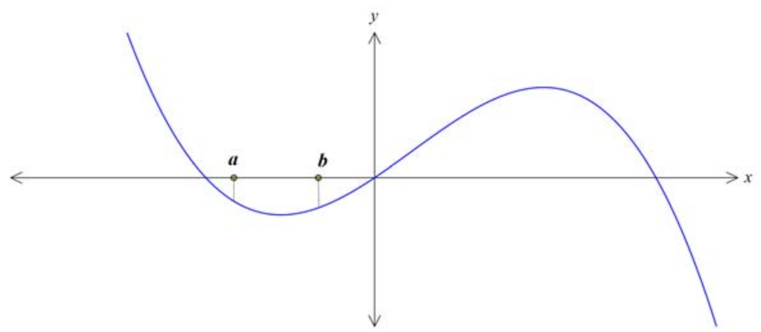
C. $\frac{1}{2}\left(q(2b) - q(2a)\right)$

D. $\frac{1}{2}\left(q'(2b) - q'(2a)\right)$

E. $q(2b) - q(2a)$

Question 18

The graph of the function with equation $y = f(x)$ is shown below.



Let g be a function such that $g'(x) = -2f(x)$.

For the interval $(a,\ b)$, the graph of g will have a

A. maximum turning point.

B. minimum turning point.

C. negative gradient.

D. positive gradient.

E. stationary point of inflection.

Question 19

The number of defective batteries in a box of batteries ready for sale is a random variable having a binomial distribution with a mean of 12 and a variance of 10.

If a battery is drawn at random from the box, the probability that it is not defective is

A. $\frac{1}{6}$

B. $\frac{5}{6}$

C. $\frac{1}{72}$

D. $\frac{1}{12}$

E. $\frac{9}{10}$

Question 20

Let X be a continuous random variable with mean μ and standard deviation σ. Which one of the following is not always true?

A. $\Pr(X > a) = 1 - \Pr(X < a)$

B. $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$

C. $\Pr(\mu - \sigma < X < \mu + \sigma) \approx 0.68$

D. $\Pr(X < \mu) = 0.5$

E. $\Pr(a < X < b) = \Pr(X < b) - \Pr(X < a)$

Question 21

The random variable X has the following probability distribution.

X	2	4	6
$\Pr(X = x)$	a	b	0.1

If the mean of X is 2.6, then the values of a and b, respectively, are

A. 0.5, 0.4

B. 0.8, 0.1

C. 0.1, 0.8

D. 0.3, 0.7

E. 0.4, 0.5

Question 22

Fresh Navels orange juice is packed in small glass bottles labelled as containing 175 mL. The packing process produces bottles that are normally distributed with a mean of 176 mL. In order to guarantee that only 1% of bottles are under-volume, the standard deviation for the volume, in mL, would be required to be closest to

A. 0.01

B. 0.99

C. 0.43

D. 176.01

E. 175.01

END OF SECTION 1
TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal answer will not be accepted if an **exact** answer is required to a question.
For questions where more than one mark is available, appropriate working must be shown.
Unless otherwise stated, diagrams are not drawn to scale.

Question 1 (14 marks)

Rebecca buys cut flowers once a week on a Saturday. Each week she buys one of two types of flowers, lilies or roses. Her choice of flower each week depends only on which flower she has bought the previous week. If she buys lilies one week then the chance of her buying lilies the next week is $\frac{3}{5}$ and if she buys roses one week then the chance of her buying roses the next week is $\frac{2}{3}$.

The transition matrix for the probabilities of Rebecca buying either flower given the flower she bought the previous week is

$$\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}$$

a. If Rebecca buys roses one week, what is the chance that she buys lilies the next week?

1 mark

Consider the next five Saturdays.

Suppose Rebecca buys lilies on the first Saturday.

b. **i.** What is the exact probability that she buys lilies on the fifth Saturday?

2 marks

SECTION 2 – Question 1 – continued

b. **ii.** What is the *exact* probability that Rebecca does not buy lilies again until the fifth Saturday?

1 mark

c. In the long term, what is the exact probability that Rebecca buys roses?

2 marks

The time, in days, that the flowers stay fresh is determined by distinct probability density functions, unique to the type of flower.

For roses, the time T_R, in days, is a random variable with the probability density function

$$f(t) = \begin{cases} \dfrac{1}{288}t(12-t) & \text{if } 0 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$$

d. State the mean and median of this probability density function.

2 marks

e. What is the probability that, on one particular occasion, the roses will last longer than 9 days, if it is known they have lasted longer than 6 days?

2 marks

For lilies, the time T_L, in days, that they stay fresh is normally distributed with a mean of 7.2 days and a standard deviation of 1 day.

Close to the holidays, Rebecca buys two bunches of flowers; i.e. a bunch of roses and a bunch of lilies.

f. For each flower type, what is the probability that they last no longer than 9 days?
Give your answers correct to 4 decimal places.

2 marks

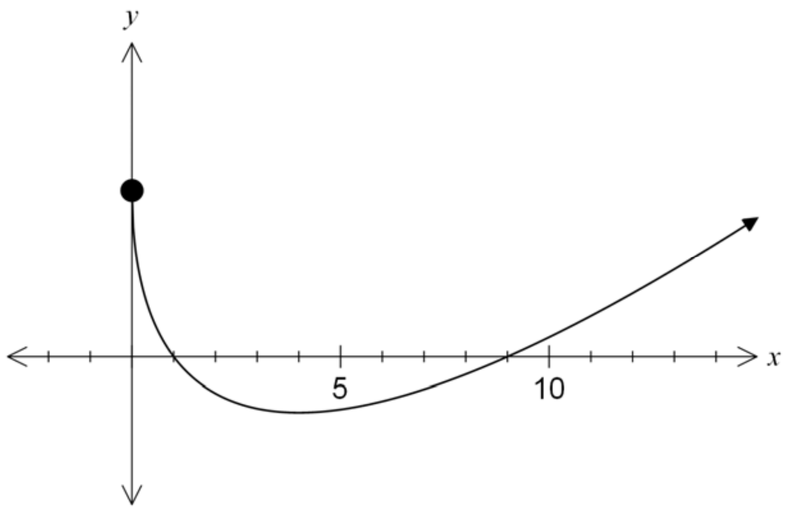
g. Find the value of *k*, where $k < 8$, such that the probability of each flower type lasting longer than *k* days is the same, correct to 2 decimal places. State the value of *k*, correct to 2 decimal places.

2 marks

Question 2 (16 marks)

Let $f:[0,\infty)\to R,\ f(x)=x-4\sqrt{x}+3$.

The graph of $y=f(x)$ is shown below.



a. State the interval for which the graph of f is strictly increasing.

2 marks

____

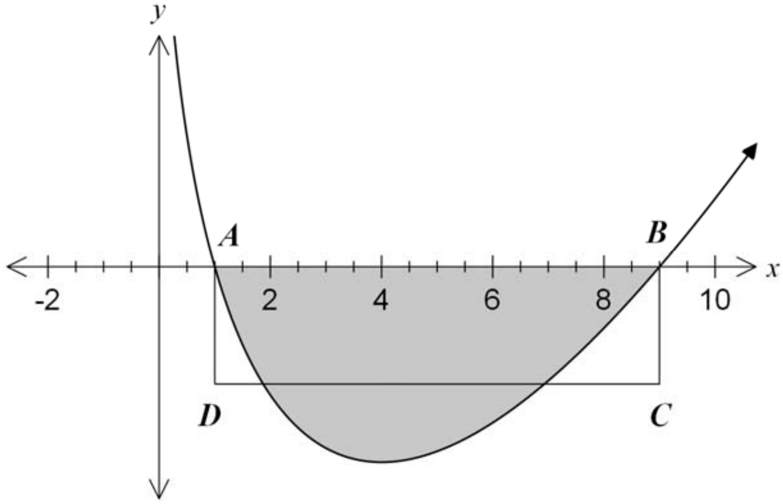
____

b. On the axes above, sketch the graph of $y=\left|f(2x)\right|$. Label the axes intercepts and turning point(s) with co-ordinates.

2 marks

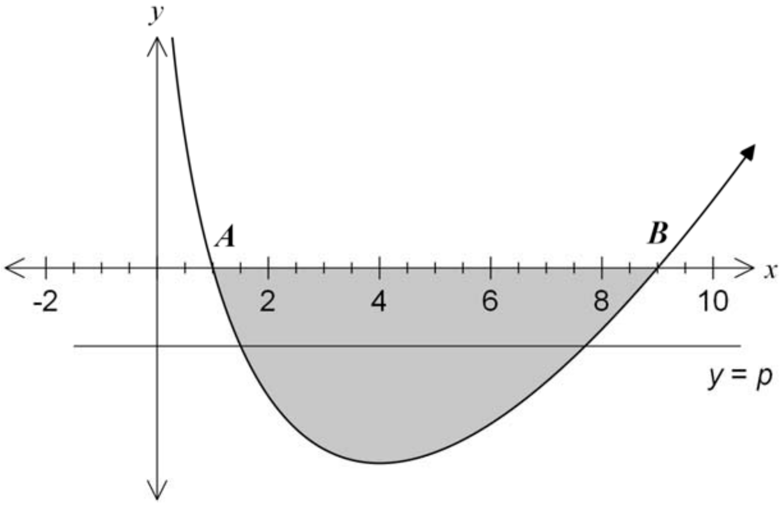
c. Points A and B are the x-intercepts of the graph $y = f(x)$ and have the co-ordinates $(1, 0)$ and $(9, 0)$, respectively. A rectangle, $ABCD$, is drawn over the curve.

Find the co-ordinates of D such that the area of the rectangle is equal to the shaded area bounded by the curve $y = f(x)$ and the x-axis.



2 marks

d. Consider the shaded area bounded by the graph and the x-axis. A horizontal line with the equation $y = p$ is drawn through the graph, as shown below. Find the value of p, correct to 3 decimal places, such that the shaded area is divided into two equal parts.



3 marks

e. Let $g: R \rightarrow R$, where $g(x) = x^2$.

i. Find the rule for $f(g(x))$.

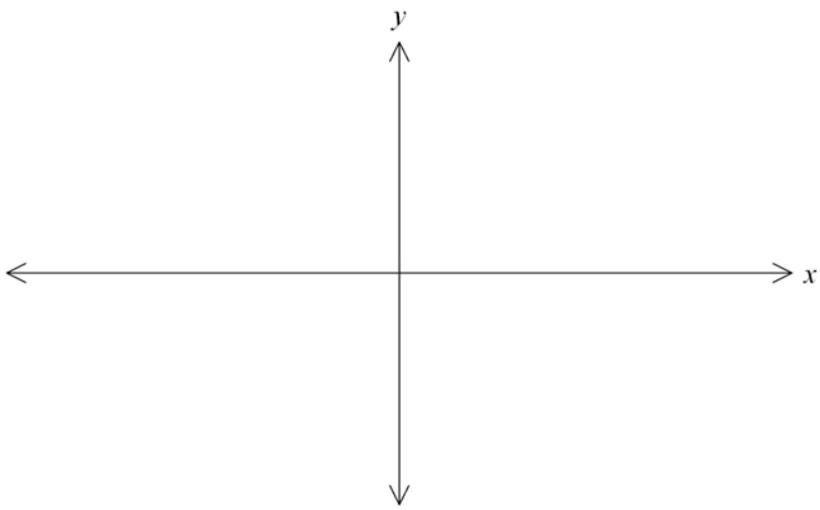
2 marks

Let $h(x) = f(g(x))$.

ii. Find the derivative of $h(x)$ with respect to x.

2 marks

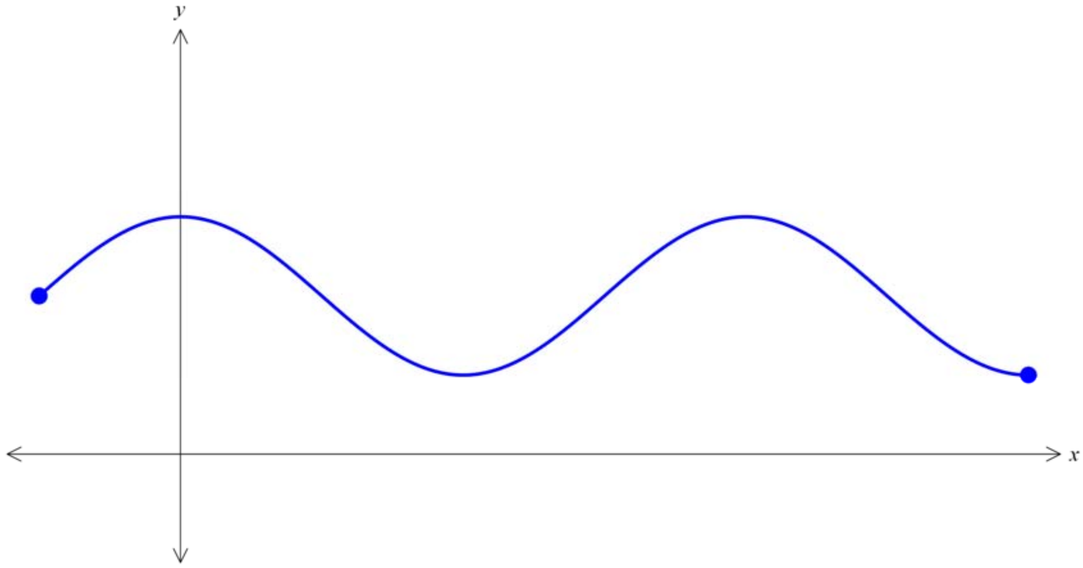
iii. Sketch the graph of the derivative function $h'(x)$ on the axes below.



3 marks

Question 3 (16 marks)

The graph of $f:[-\pi, 6\pi] \rightarrow R,\ f(x) = 2\cos^2\left(\dfrac{x}{4}\right) + 1$ is shown below.



a. Find $f'(\pi)$ and $f'(3\pi)$.

2 marks

b. **i.** Find the gradient of the tangent to the curve at any point *x*.

1 mark

ii. Find exact co-ordinates of the points on the curve $y = f(x)$ where the gradient of the *normal* to the curve is equal to 2.

2 marks

c. Find the exact equations of the *tangents* to the curve at the points where $x = 3\pi$ and $x = 5\pi$.

2 marks

d. The graph of $y = f(x)$ is transformed to give the graph of $y = f(x) + b$. Find the exact value of b, such that the graph of the tangent at $x = 3\pi$ and the graph of the tangent at $x = 5\pi$ intersect on the x-axis.

2 marks

e. The equation for $y = f(x)$ can be written in the form $y = \cos(a\,x) + b$. Find the values of a and b. Hence, show that the period is 4π.

2 marks

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f. The graph of $f:[-\pi,\ 6\pi]\to R,\ f(x)=2\cos^2\left(\dfrac{x}{4}\right)+1$ is transformed to give $y=f(kx)$.

i. Find the value of *k* such that the equation $f(kx)=1$ has only *one* solution in the domain $[-\pi,\ 6\pi]$ and that the solution is at $x=6\pi$.

1 mark

ii. Find the values of *k* such that the equation $f(kx)=1$ has only *one* solution in the domain $[-\pi,\ 6\pi]$.

2 marks

iii. Find the values of *k* such that the equation $f(kx)=1$ has exactly *three* solutions in the domain $[-\pi,\ 6\pi]$.

2 marks

Question 4 (12 marks)

Let $f : R \to R,\ f(x) = e^{ax}$, where $a \in R$.

a. **i.** Show that $f(u)f(-u) = 1$, where u is a real number.

1 mark

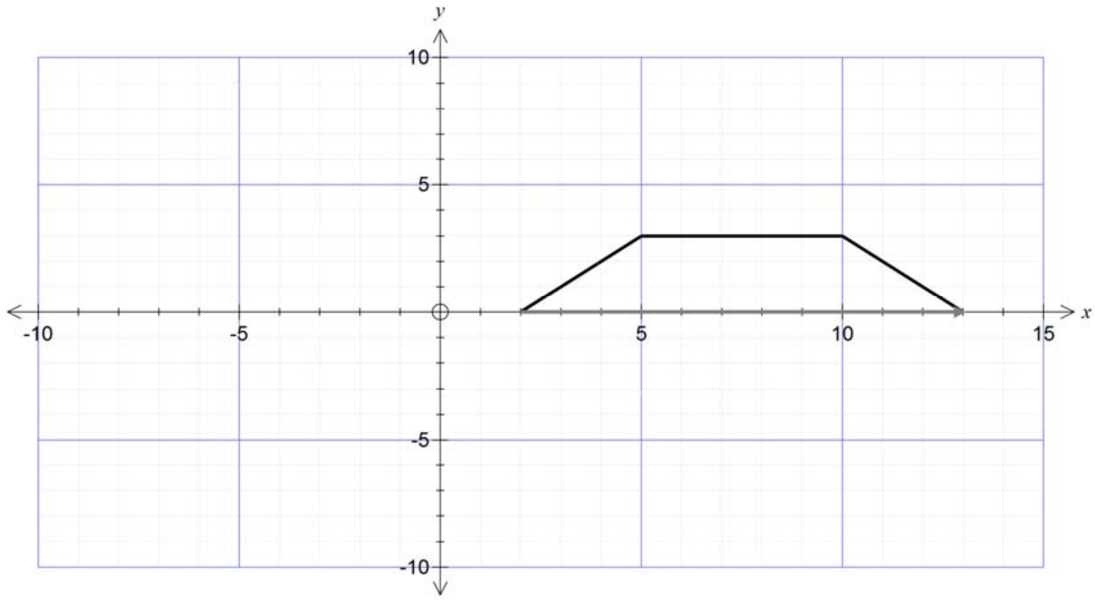
ii. Show that $f(u+v) = f(u)f(v)$, where u and v are real numbers.

2 marks

SECTION 2 – Question 4 – continued

A particular computer game features a character called Super Marius. The objective of the game involves jumping Super Marius over obstacles.

During a stage in the computer game, the player must make Super Marius jump over a trapezium obstacle of vertical height 3 units, as shown below. The base of the obstacle extends from $x = 2$ to $x = 13$. Super Marius jumps from the point $(0, 8)$ according to the equation $y = 9 - e^{0.2x}$, $x \ge 0$, but fails to clear the obstacle.



b. The horizontal top section extends from $x = 5$ to $x = 10$. Find the exact co-ordinates of the point where Super Marius lands on the obstacle.

2 marks

When the random button is pressed, Super Marius jumps according to the equation $y = 9 - e^{ax}$.

c. Find the values of a so that Super Marius lands on the horizontal section of the obstacle.

2 marks

d. Find the values of a so that Super Marius clears the obstacle.

1 mark

At a particular stage in the game, the equation of the curve that defines Super Marius' jump at that moment is given by $y = 9 - e^{0.2x}$. When the random button is pressed on this occasion, a randomly assigned matrix, *M*, transforms the jumping equation such that

$$T : R^2 \to R^2, \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M\begin{bmatrix} x \\ y \end{bmatrix}$$

e. If the matrix on this occasion is $M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, find the rule for the transformed jumping equation.

2 marks

At another point in the game, Super Marius is able to jump according to the fixed equation $y = 9 - e^{0.1x}$; however, during this stage the horizontal top section of the original trapezium obstacle is now moving up and down and is represented by the rule $y = p,\ p \geq 0$. This changes the shape of the obstacle by altering the height of the trapezium.

f. Find the values of *p*, correct to 2 decimal places, for which Super Marius is able to clear the obstacle.

2 marks

END OF QUESTION AND ANSWER BOOK