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Engage Education Foundation

Units 3 and 4 Maths Methods (CAS): Exam 2

Technology-enabled Practice Exam Question and Answer Booklet

Duration: 15 minutes reading time, 2 hours writing time

Structure of book:

Section	Number of questions	Number of questions to be answered	Number of marks
A	22	22	22
B	4	4	58
		Total	80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers and rulers, one bound reference, one approved CAS calculator (memory does not need to be cleared) and, if desired, one scientific calculator.
- Students are not permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied:

- This question and answer booklet of 17 pages, including a formula sheet on the last page.

Instructions:

- You must complete all questions of the examination.
- Write all your answers in the spaces provided in this booklet.

Section A – Multiple-choice questions

Instructions

Answer all questions by circling your choice.
Choose the response that is correct or that best answers the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Questions

Question 1

The equation of the normal to the curve $y = \sqrt{2x}$ at $x = 2$ is given by:

A. $y - 2 = -\sqrt{2}(x - 2)$
B. $y - 2 = 2(x - 2)$
C. $y = 6 - 2x$
D. $y = \frac{x}{2} + 1$
E. $y - 2 = \frac{1}{2}(x - 2)$

Question 2

If a 6 sided die is weighted such that the probability of rolling any integer value x from 1 to 6 is given by $\Pr(X = x) = \frac{x}{21}$, the value of σ_X is:

A. $\frac{21}{6}$
B. $\sqrt{\frac{35}{12}}$
C. $\frac{13}{3}$
D. $\frac{20}{9}$
E. $\frac{2\sqrt{5}}{3}$

Question 3

The average value of the function $f(x) = \frac{1}{x}$ over $[1,5]$ is:

A. $\frac{\log_e(5)-1}{4}$
B. $\frac{\log_e(5)}{4}$
C. $\log_e(5)$
D. $-\frac{1}{5}$
E. $-\frac{2}{5}$

Question 4

The general solution to the equation $2\cos(x) = \sqrt{3}$ is:

A. $x = 2k\pi \pm \frac{\pi}{6}, k \in \mathbb{Z}$
B. $x = 2k\pi \pm \frac{\pi}{3}, k \in \mathbb{Z}$
C. $x = \frac{\pi}{3} + 2k\pi \text{ } or \text{ } \frac{2\pi}{3} + 2k\pi, k \in \mathbb{Z}$
D. $x = \frac{\pi}{6} + 2k\pi \text{ } or \text{ } \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$
E. $x = \frac{\pi}{6} + 2k\pi \text{ } or \text{ } \frac{13\pi}{6} + 2k\pi, k \in \mathbb{Z}$

Question 5

If the range of $f: D \to \mathbb{R}: f(x) = x^{1.5}$ is $[1,8)$, D is given by:

A. $[1,2]$
B. $(1,2)$
C. $\{x: -1 \leq x < 2\}$
D. $\{x: -1 \leq x \leq 4\}$
E. $\{x: x \geq 1\} \cap \{x: x < 4\}$

Question 6

A matrix equation equivalent to the system

$$x + 2y = 3$$
$$y - 4z = a$$
$$z - 2x = 6$$

is given by:

A. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -4 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \\ a \\ 6 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -4 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ a \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

C. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ a \\ 6 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -4 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ a \\ 6 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -4 & 0 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ a \\ 6 \end{bmatrix}$

Question 7

5 cards are drawn at random from a 52 card deck containing equal numbers of each suit (spades, clubs, hearts, and diamonds) without replacement. The probability that all the cards drawn are of the same suit is:

A. C_5^{13}
B. $\frac{4 \times 13! \times 47!}{8! \times 52!}$
C. $C_5^{52} \left(\frac{1}{13}\right)^5 \left(\frac{12}{13}\right)^{47}$
D. $C_5^{5} \left(\frac{1}{13}\right)^5 \left(\frac{12}{13}\right)^{0}$
E. $\left(\frac{1}{13}\right)^5$

Question 8

The equations of the asymptote(s) of $y = x^2 + \frac{1}{x}$ are given by:

A. $x = 0$ and $y = 0$
B. $y = x^2$ and $y = 0$
C. $y = x^2$ and $x = 0$
D. $x = 0$ only
E. $y = 0$ only

Question 9

$\int 2^x dx$ is equivalent to:

A. $\frac{2^x}{\log_e(2)}$
B. $\frac{2x}{\log_e(2)}$
C. $2^x + c$
D. $\frac{2x}{\log_e(2)} + c$
E. $\frac{2^x}{\log_e(2)} + c$

Question 10

If $\Pr(A) = 0.3$, $\Pr(B) = 0.6$, and $\Pr(A^c \cap B) = 0.4$, $\Pr((A \cap B)^c)$ is:

A. 0.8
B. 0.3
C. 0.1
D. 0.2
E. 0.7

Question 11

Let $f(x) = \sqrt[3]{x}$. $\sqrt[3]{345}$ may be approximated linearly by:

A. $f(343) + 2f'(345)$
B. $7 + 2f'(343)$
C. $7 + 2f'(345)$
D. $f(345) + 2f'(343)$
E. $7 + 2f(343)$

Question 12

The function $f(x) = \frac{x^2}{log_e x}$ has:

A. a local maximum at $(0,0)$
B. a local maximum at $\left(\sqrt{e}, \frac{e}{2}\right)$
C. a local minimum at $(0, 0)$
D. a local minimum at $\left(\sqrt{e}, \frac{e}{2}\right)$
E. a local minimum at $\left(\sqrt{e}, 2e\right)$

Question 13

The number of solutions to the equation $-3\tan(ax) + 5 = k$ over the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$, with $a \in \mathbb{Z}$ and $k \in \mathbb{R}$ is:

A. $2a$
B. $\frac{a}{2}$
C. a
D. $|a|$
E. not enough information is given

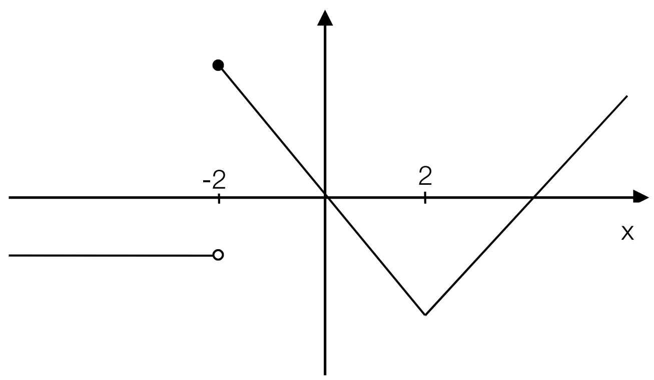
Question 14

In a random experiment, a fair six sided die is rolled and the upward facing number recorded. Two independent events could be:

A. $\{1,2,3\}$ and $\{4,5,6\}$
B. $\{1,3\}$ and $\{2,3,6\}$
C. $\{1,5\}$ and $\{1,3\}$
D. $\{3,5\}$ and $\{1,3,5\}$
E. $\{1\}$ and $\{2\}$

Question 15

Consider the function f shown below



A. f is undefined at $x = -2$
B. f is discontinuous at $x = 2$
C. f is not differentiable at $x = 2$
D. f has a positive gradient for $x < -2$
E. f has a negative gradient for $x < -2$

Question 16

A coin is weighted such that when tossed, heads is shown with probability $\frac{4}{7}$. When this coin is tossed 4 times, the probability that we obtain exactly 3 heads is:

A. $C_4^7\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^3$
B. $C_3^4\left(\frac{4}{7}\right)^3\left(\frac{3}{7}\right)^1$
C. $C_3^4\left(\frac{4}{7}\right)^1\left(\frac{3}{7}\right)^4$
D. $\left(\frac{4}{7}\right)^4\left(\frac{3}{7}\right)^1$
E. $\left(\frac{4}{7}\right)^4\left(\frac{3}{7}\right)^3$

Question 17

The area enclosed by the graphs of $y = 3x$ and $y = x(x^2 - 1)$ is equal to:

A. $\int_0^2[-x^3 + 4x]dx + \int_0^{-2}[x^3 - 4x]dx$
B. $2\int_0^4[-x^3 + 4x]dx$
C. $\int_0^4[-x^3 + 4x]dx + \int_{-4}^0[x^3 - 4x]dx$
D. $\int_0^2[-x^3 + 4x]dx - \int_{-2}^0[x^3 - 4x]dx$
E. $2\int_0^2[-x^3 + 4x]dx$

Question 18

The smallest value of a such that the inverse of $f: [a, \infty) \to \mathbb{R}, f(x) = 3(x - 1)^2 - 1$ exists, and the rule for f^{-1} are:

A. $a = 1 \text{ and } f^{-1}(x) = 1 + \sqrt{\frac{x+1}{3}}$
B. $a = 1 \text{ and } f^{-1}(x) = 1 \pm \sqrt{\frac{x+1}{3}}$
C. $a = 1 \text{ and } f^{-1}(x) = 1 - \sqrt{\frac{x+1}{3}}$
D. $a = -1 \text{ and } f^{-1}(x) = 1 + \sqrt{\frac{x+1}{3}}$
E. $a = -1 \text{ and } f^{-1}(x) = 1 \pm \sqrt{\frac{x+1}{3}}$

Question 19

The range of the function $f: [0, \pi] \to \mathbb{R}, f(x) = \sin(x) + \frac{x}{2}$ is:

A. $[0,1]$
B. $\left[0, \frac{\pi}{2}\right]$
C. $\left[0, \frac{2\sqrt{3}+3\pi}{6}\right]$
D. $\left[0, \frac{3\sqrt{3}+2\pi}{6}\right]$
E. $\left[0, \frac{\sqrt{3}+\pi}{6}\right]$

Question 20
The average rate of change of $f(x)$ over $[a, b]$ is given by:

A. $\frac{f'(b)-f'(a)}{b-a}$
B. $\frac{f'(b)-f'(a)}{2}$
C. $\frac{f'(b)+f'(a)}{2}$
D. $\frac{f(b)-f(a)}{b+a}$
E. $\frac{f(b)-f(a)}{b-a}$

Question 21
A random variable X is normally distributed with mean 80 and variance 16. To three decimal places, $\Pr(68 \leq X \leq 84)$ is:

A. 0.372
B. 0.599
C. 0.840
D. 0.841
E. 0.950

Question 22
The largest possible domain of $f\left(g(x)\right)$, where $f(x) = log_e x$ and $g(x) = \sqrt{x+3} - 2$ is:

A. $\mathbb{R}$
B. $\mathbb{R}^+ \cup \{0\}$
C. $(1, \infty)$
D. $[-3, \infty)$
E. $(0, \infty)$

Section B – Short-answer questions

Instructions

Answer all questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Questions

Question 1

Tiny Tony is building an open rectangular container with a square base so he can play with his rubber ducky. He wants the container to hold as much water as possible, but he only has $108cm^2$ of material to work with.

Let the dimensions of the container be $x\ cm \times x\ cm \times y\ cm$, and the volume and surface area of the box be V and S respectively.

a. State any physical constraints on the values of x and y.

1 marks

b. Show that $V = 27x - \frac{1}{4}x^3$.

2 marks

c. Find $\frac{dV}{dx}$ and hence the maximum possible volume of the container, stating its corresponding dimensions.

4 marks

d. Tony then fills the container with water up to a depth of $1cm$ and installs a wave machine along its base. He notices that the height of the water in the very center of the container, h, t seconds after turning on the wave machine appears to follow the curve:

$$h = \frac{\cos(\pi(t-3))}{5} + 1$$

Sketch the graph of h against t on the axes below for $t \in [0,6]$. Label any stationary points.

4 marks

e. Tony wishes to drop his rubber ducky in to the center of the container, but he is afraid that if he does so while the water level is below $0.9cm$ he may hurt him. In first 6 seconds after turning on the wave machine, how much time is there during which it is not safe for Tony to drop his duck?

4 marks

Total: 15 marks

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Question 2

Consider the functions $f(x) = 2log_e(x)$ and $g(x) = x + 1$.

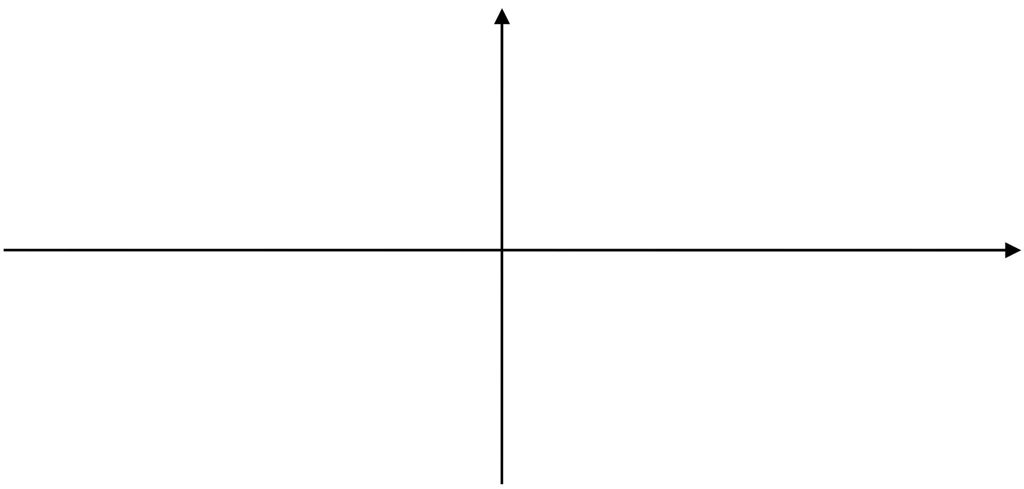
a. State the maximal domain and range of f and g.

1 marks

b. If we restrict the domain of g to (a, ∞), find the smallest value of a such that $f[g(x)]$ exists, and find a rule for $f[g(x)]$.

2 marks

c. For this value of a, sketch the function $f[g(x)]$ on the axes below. Label any asymptotes and intercepts.



2 marks

d. Find the rule of $(f \circ g)^{-1}(x)$, and state its domain and range.

2 marks

e. Evaluate $\int_0^{2\pi}(f \ o \ g)(x)dx$, and shade an equivalent area on your graph from part c.

2 marks

f. Hence find $\int_0^{e^{\pi}-1}(f \circ g)(x)dx$ both in exact terms and correct to 3 decimal places.

4 marks

Total: 13 marks

Question 3

a. State the equation of a circle of radius 3 centred at the origin, as well as separate equations for the top and bottom halves of this circle.

2 marks

b. Find the equations of all tangents to this circle passing through the point $(5,0)$.

8 marks

c. Write down an expression for the area bounded by these two tangents and the circle, and use your calculator to find this area correct to 3 decimal places.

3 marks

Total: 13 marks

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Question 4

Callum and Charles meet every week for a game of ping pong. Disappointed with his recent losing streak, Callum tries to identify the cause of his problems. Being an emotional player, Callum feels that the result of his previous game affects his performance. He decides that his probability of winning given that he won the last game is 0.6, while his probability of winning given that he lost the last game is only 0.3.

a. Write down a transition matrix T describing Callum's wins and losses, and use it to calculate the probability that he wins his fifth game correct to 3 decimal places, given that he has an equal chance of winning or losing his first game.

2 marks

b. Draw a tree diagram to show all possible outcomes of the first 3 games, assigning exact probabilities to each element of the sample space.

4 marks

c. Use your tree diagram to find the probability that Callum won the first game given that he lost the third correct to 3 decimal places.

3 marks

d. Find the exact proportion of games Callum is expected to win in the long run.

1 marks

Following further investigation, Callum decides that he needs help controlling the power of his serve. He uses a device which measures his serve power P: if P is below 30, his serve hits the net, and if P is above 80, his serve goes long. Statistical analysis shows that P is a normally distributed random variable, and that 5% of Callum's serves hit the net while 10% go long.

e. Find the mean and standard deviation of P, correct to 3 decimal places.

3 marks

Still unsatisfied with these figures, Callum discovers a performance enhancing drug which claims to give users greater control over their strength. Under the influence of this drug, Callum's new serve power follows another normal distribution P_d, such that $P_d = \frac{7}{8}P + 10$.

f. Find the mean and standard deviation of $\mathrm{P_d}$, and hence the probabilities of Callum's serve being too weak or too strong while using the drug correct to 3 decimal places. Should Callum take the drug?

4 marks

Total: 17 marks

Formula sheet

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin A$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\frac{d}{dx}(e^{ax}) = ae^{ax}$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}(\log_e x) = \frac{1}{x}$

$\int \frac{1}{x} dx = \log_e|x| + c$

$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$

$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$

$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$

product rule $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)}{v^2}$

chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation $f(x+h) = f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices $S_n = T^n \times S_0$

mean $\mu = E(X)$

variance $var(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma xp(x)$	$\sigma^2 = \Sigma(x-\mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} xf(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

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