2012 Mathematical Methods (CAS) Trial Exam 1 Solutions

Q1a $x^2+x+1=x^2$, $x+1=0$, $x=-1$

Q1b $y=\dfrac{x^2+x+1}{x^2}=\dfrac{x^2}{x^2}+\dfrac{x+1}{x^2}=1+\dfrac{x+1}{x^2}$

As $x\to-\infty$, $y\to 1+\dfrac{x}{x^2}=1+\dfrac{1}{x}\to 1^-$

As $x\to\infty$, $y\to 1+\dfrac{x}{x^2}=1+\dfrac{1}{x}\to 1^+$

As $x\to 0^-$, $y\to\dfrac{x+1}{x^2}\to\dfrac{1}{x^2}\to\infty$

As $x\to 0^+$, $y\to\dfrac{x+1}{x^2}\to\dfrac{1}{x^2}\to\infty$

Q1c $f(g(x))=(g(x))^2+g(x)+1=x^4+x^2+1$

$(f(x))^2=(x^2+x+1)^2=x^4+2x^3+3x^2+2x+1$

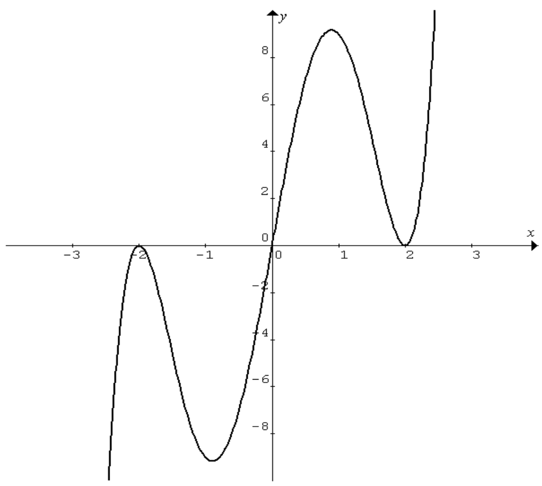
$=x^4+x^2+1+2x^3+2x^2+2x=f(g(x))+2xf(x)$

.: $f(g(x))=(f(x))^2-2xf(x)=f(x)(f(x)-2x)$

Q2a All odd polynomials pass through the origin, .: $x=0$ is a root of $P(x)=0$, .: x is a factor of $P(x)$. Since $x=-2$ is a double root, .: $(x+2)$ is a repeated factor of $P(x)$. Since $P(x)$ is an odd polynomial, .: $(x-2)$ is also a repeated factor of $P(x)$.

.: $P(x)=x(x+2)^2(x-2)^2$

Q2b



Q3a

$f(x)+f(-x)=\dfrac{2}{e^x-e^{-x}}+\dfrac{2}{e^{-x}-e^x}=\dfrac{2}{e^x-e^{-x}}-\dfrac{2}{e^x-e^{-x}}=0$

Q3b $y=\dfrac{2}{e^x-e^{-x}}$, equation of inverse function is $x=\dfrac{2}{e^y-e^{-y}}$

$x=\dfrac{2e^y}{(e^y)^2-1}$, $x(e^y)^2-2e^y-x=0$,

$e^y=\dfrac{2\pm\sqrt{4+4x^2}}{2x}=\dfrac{1\pm\sqrt{1+x^2}}{x}$, $y=\log_e\dfrac{1\pm\sqrt{1+x^2}}{x}$

.: $f^{-1}(x)=\begin{cases}\log_e\dfrac{1-\sqrt{1+x^2}}{x} & for\quad x<0\\ \log_e\dfrac{1+\sqrt{1+x^2}}{x} & for\quad x>0\end{cases}$

Q4 $x^2+y^2+z^2+10x+20y-30z+350=0$

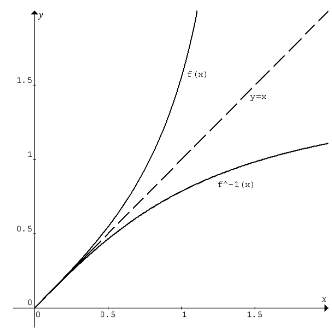
$x^2+10x+25+y^2+20y+100+z^2-30z+225=0$

$(x+5)^2+(y+10)^2+(z-15)^2=0$

Since $square\ge 0$, .: $(x+5)^2=0$, $(y+10)^2=0$ and $(z-15)^2=0$, .: $x=-5$, $y=-10$ and $z=15$

Hence $(x-y-z)^2=(-5+10-15)^2=100$

Q5a $y=f^{-1}(x)$ is the reflection of $y=f(x)$ in the line $y=x$.



Q5b The tangent to the curve $y=f^{-1}(x)$ at $x=1$ is the reflection (in the line $y=x$) of the tangent to the curve $y=f(x)$ at $x=\dfrac{\pi}{4}$.

Tangent to the curve $y=f(x)$ at $x=\dfrac{\pi}{4}$, i.e. at $\left(\dfrac{\pi}{4},1\right)$:

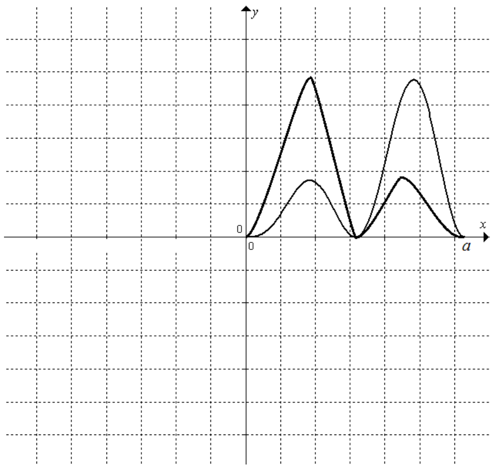
$f'(x)=\sec^2 x$, $f'\left(\dfrac{\pi}{4}\right)=\sec^2\dfrac{\pi}{4}=2$

$y-1=2\left(x-\dfrac{\pi}{4}\right)$, $y=2x+1-\dfrac{\pi}{2}$

.: the tangent to the curve $y=f^{-1}(x)$ at $x=1$ is

$x=2y+1-\dfrac{\pi}{2}$, $y=\dfrac{1}{2}x-\dfrac{1}{2}+\dfrac{\pi}{4}$

Q6a



Q6b $f(x)=\dfrac{\cos x}{\cos x+\sin x}$, $x\in\left[0,\dfrac{\pi}{2}\right]$

$$f\left(\frac{\pi}{2}-x\right)=\frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}=\frac{\sin x}{\sin x+\cos x}=\frac{\sin x}{\cos x+\sin x}$$

Q6c From part 6a, same area under $y=f(x)$ and $y=f\left(\dfrac{\pi}{2}-x\right)$

for $x\in\left[0,\dfrac{\pi}{2}\right]$, i.e. $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x+\sin x}dx$

$$\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx+\int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x+\sin x}dx=\int_0^{\frac{\pi}{2}}\frac{\cos x+\sin x}{\cos x+\sin x}dx=\int_0^{\frac{\pi}{2}}1dx$$

$$.\because\ 2\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx=[x]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$

$$.\because\ \int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx=\frac{\pi}{4}$$

Q7 $k\left(e^x\right)^2-2e^x+k=0$

Since e^x is a one to one function, two solutions for $x\rightarrow$ two solutions for e^x in the equation.

$.\because\ \Delta=(-2)^2-4(k)(k)>0$ and $k\neq 0$, $.\because\ 1-k^2>0$ and $k\neq 0$

$.\because\ \{k:-1<k<0\}\cup\{k:0<k<1\}$

Q8a $f(x)=\dfrac{\log_e x}{x}$,

$$f'(x)=\frac{(x)\left(\frac{1}{x}\right)-(\log_e x)(1)}{x^2}=\frac{1-\log_e x}{x^2}=\frac{1}{x^2}-\frac{\log_e x}{x^2}$$

Q8b $\displaystyle\int_1^e f'(x)dx=\int_1^e\frac{1}{x^2}dx-\int_1^e\frac{\log_e x}{x^2}dx$

$$.\because\ \int_1^e\frac{\log_e x}{x^2}dx=\int_1^e\frac{1}{x^2}dx-\int_1^e f'(x)dx=\left[-\frac{1}{x}\right]_1^e-\left[\frac{\log_e x}{x}\right]_1^e$$

$$=\left(-\frac{1}{e}+1\right)-\left(\frac{1}{e}\right)=1-\frac{2}{e}$$

Q9 $\Pr(ttc)+\Pr(tct)+\Pr(ctc)+\Pr(cct)$

$=0.5\times0.6\times0.4+0.5\times0.4\times0.7+0.5\times0.7\times0.4+0.5\times0.3\times0.7$

$=0.505$

Q10a $\displaystyle\sum_i p_i=1$, $0.1a+0.3(5-a)=1$, $-0.2a+1.5=1$, $a=2.5$

Q10b Let m be the median. $(5-m)0.3=0.5$, $m=\dfrac{10}{3}$

Q10c $\displaystyle\overline{X}=\int_0^{2.5}0.1x dx+\int_{2.5}^{5}0.3x dx=\left[\frac{0.1x^2}{2}\right]_0^{2.5}+\left[\frac{0.3x^2}{2}\right]_{2.5}^{5}$

$=3.125$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors