**THE HEFFERNAN GROUP**

**P.O. Box 1180**
**Surrey Hills North VIC 3127**
**Phone 03 9836 5021**
**Fax 03 9836 5025**
**info@theheffernangroup.com.au**
**www.theheffernangroup.com.au**

# MATHS METHODS (CAS) 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2012

## SECTION 1 – Multiple-choice answers

| | | |
|---|---|---|
| **1. A** | **9. E** | **17. D** |
| **2. A** | **10. E** | **18. B** |
| **3. D** | **11. D** | **19. D** |
| **4. E** | **12. B** | **20. C** |
| **5. D** | **13. C** | **21. C** |
| **6. B** | **14. D** | **22. C** |
| **7. B** | **15. E** | |
| **8. B** | **16. C** | |

## SECTION 1 – Multiple-choice solutions

### Question 1

The line passes through $(0,3)$ and $(-6,0)$

$$\begin{aligned}\text{gradient} &= \frac{3-0}{0-{}^{-}6}\\ &= \frac{3}{6}\\ &= \frac{1}{2}\end{aligned}$$

The gradient of the line that is perpendicular to this line is – 2 because $-2\times\frac{1}{2}=-1$.

The answer is A.

### Question 2

If the function $y=\frac{1}{\sqrt{4-x}}$ is to exist then

$$\begin{aligned}4-x &> 0\\ -x &> -4\\ x &< 4\end{aligned}$$

So $x\in(-\infty,4)$

The answer is A.

**Question 3**

$$f(x) = e^{2x} \text{ and } g(x) = \frac{2}{x}$$

$$f(g(x)) = e^{2\times\left(\frac{2}{x}\right)}$$

$$= e^{\frac{4}{x}}$$

$$d_{f\circ g} = d_g = R\setminus\{0\}$$

The answer is D.

**Question 4**

<u>Method 1</u> – using CAS
Let $y = (x-1)^3 + 2$
Swap $x$ and $y$ for inverse.
Solve $x = (y-1)^3 + 2$ for $y$.

$$y = 1 + \sqrt[3]{x-2}$$

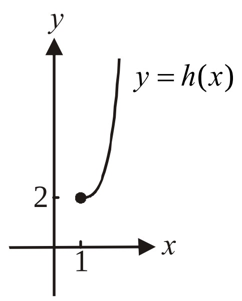
$$h^{-1}(x) = 1 + \sqrt[3]{x-2}$$

$$d_h = [1,\infty)$$

$$r_h = [2,\infty)$$

So $d_{h^{-1}} = r_h = [2,\infty)$

$$h^{-1}:[2,\infty)\to R, \quad h^{-1}(x) = 1 + \sqrt[3]{x-2}$$




The answer is E.

<u>Method 2</u> – by hand
Let $y = (x-1)^3 + 2$
Swap $x$ and $y$ for inverse.

$$x = (y-1)^3 + 2$$

$$x - 2 = (y-1)^3$$

$$\sqrt[3]{x-2} = y - 1$$

$$y = 1 + \sqrt[3]{x-2}$$

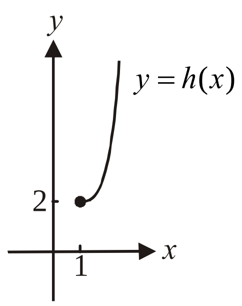
$$h^{-1}(x) = 1 + \sqrt[3]{x-2}$$

$$d_h = [1,\infty)$$

$$r_h = [2,\infty)$$

So $d_{h^{-1}} = r_h = [2,\infty)$

$$h^{-1}:[2,\infty)\to R, \quad h^{-1}(x) = 1 + \sqrt[3]{x-2}$$




The answer is E.

**Question 5**

$f(x)=e^{3g(x)}$
<u>Method 1</u>
$f'(x)=3g'(x)e^{3g(x)}$
$\quad\quad =3g'(x)f(x)$
The answer is D.

<u>Method 2</u>

$f(x)=e^{3g(x)}$ $\quad\quad$ Let $u=3g(x)$

Let $y=e^{3g(x)}$ $\quad\quad$ $\dfrac{du}{dx}=3g'(x)$

$$\begin{aligned}&=e^{u}\\ \frac{dy}{dx}&=\frac{dy}{du}\cdot\frac{du}{dx} \text{ (Chain rule)}\\ &=e^{u}\cdot 3g'(x)\\ &=e^{3g(x)}\cdot 3g'(x)\\ &=f(x)\cdot 3g'(x)\end{aligned}$$

So $f'(x)=3g'(x)f(x)$
The answer is D.

**Question 6**

$f(x)=e^{x}$ so $f'(x)=e^{x}$
$f(1+h)\approx f(1)+hf'(1)$
So, an approximation for $e^{1.3}$ is given by
$$\begin{aligned}f(1+0.3)&\approx f(1)+0.3f'(1)\\ &=e^{1}+0.3\times e^{1}\\ &=e^{1}(1+0.3)\\ &=1.3e\end{aligned}$$
The answer is B.

**Question 7**

The period of the graph is $\dfrac{2\pi}{8}=\dfrac{\pi}{4}$.

The period of the graph of $y=\tan(nx)$ is $\dfrac{\pi}{n}$.

Let $\dfrac{\pi}{n}=\dfrac{\pi}{4}$ so $n=4$.
The graph of $y=\tan(4x)$ has been reflected in the $x$-axis to obtain the graph shown.
The rule $-y=\tan(4x)$ would define this reflection. So the graph shown could have the rule $y=-\tan(4x)$
The answer is B.

**Question 8**

$$(a+1)x+y=1$$
$$5x+(a-3)y=2.5a$$
$$\begin{bmatrix} a+1 & 1 \\ 5 & a-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 2.5a \end{bmatrix}$$

$(a+1)(a-3)-5\times 1=0$ for no solution or infinitely many solutions

$$a^2-2a-3-5=0$$
$$a^2-2a-8=0$$
$$(a-4)(a+2)=0$$
$$a=4 \text{ or } a=-2$$

If $a=4$, we have

$$5x+y=1$$
$$5x+y=10$$

Therefore there are no solutions since these equations represent parallel lines.
If $a=-2$, we have

$$-x+y=1 \quad -(A)$$
$$5x-5y=-5 \quad -(B)$$
$$(A)\times -5 \qquad 5x-5y=-5 \quad -(C)$$

$(A)$ and $(C)$ represent the same line so there are infinitely many solutions.
So for no solutions $a\in\{4\}$.
The answer is B.

**Question 9**

$$\text{Let } T=\begin{array}{c} \text{one day} \\ \begin{array}{cc} HW & \text{no } HW \end{array} \\ \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \end{array}\begin{array}{l} HW \\ \text{no } HW \end{array} \text{ next day}$$

be the transition matrix representing this situation.

Let $S_1=\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{array}{l} HW \\ \text{no } HW \end{array}$ be the initial state matrix.

Note that if you use $\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{array}{l} HW \\ \text{no } HW \end{array}$ as your initial state matrix you will get the same result.

Now, using CAS,

$$T^{20}\times S_1=\begin{bmatrix} 0.636364 \\ 0.363636 \end{bmatrix}$$
$$T^{21}\times S_1=\begin{bmatrix} 0.636364 \\ 0.363636 \end{bmatrix}$$

So we have found a steady state.
So over the long term, the probability that George does his homework is 0.636364.
The closest answer is 0.64.
The answer is E.

**Question 10**

$$\begin{aligned}\text{average value} &= \frac{1}{2-{}^{-}1} \times \int_{-1}^{2} e^{2x}(x^3-2)dx\\ &= \frac{1}{3}\int_{-1}^{2} e^{2x}(x^3-2)dx\\ &= \frac{3(e^6+3)e^{-2}}{8} \quad \text{(using CAS)}\\ &= \frac{3(e^6+3)}{8e^2}\end{aligned}$$

The answer is E.

**Question 11**

This is a related rates question.
Let $h$ be the height of the liquid at any time $t$.

$$\begin{aligned}\text{Volume of liquid} &= \frac{1}{3}Ah \quad \text{(volume of a pyramid from formula sheet)}\\ &= \frac{1}{3} \times 2h \times h \times h\end{aligned}$$

$$\text{So } V = \frac{2}{3}h^3$$

$$\frac{dV}{dh} = 2h^2$$

$$\text{Also, } \frac{dV}{dt} = 0.1 \quad \text{(given)}$$

$$\begin{aligned}\text{So, } \frac{dh}{dt} &= \frac{dh}{dV} \cdot \frac{dV}{dt} \quad \text{(Chain rule)}\\ &= \frac{1}{2h^2} \cdot 0.1\\ &= \frac{1}{20h^2}\end{aligned}$$

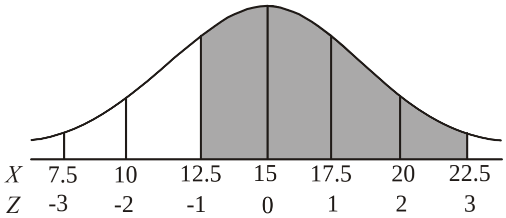
$$\text{When } h = 0.2,$$

$$\begin{aligned}\frac{dh}{dt} &= \frac{1}{20 \times 0.04}\\ &= 1.25\end{aligned}$$

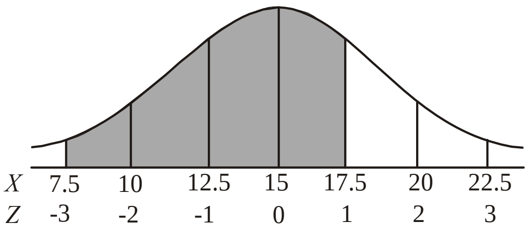
The answer is D.

**Question 12**

Draw a diagram.




The area representing $\Pr(-1 < Z < 3)$ is shaded.
This area could also be expressed as $\Pr(12.5 < X < 22.5)$ however this is not offered as an alternative.
Because of the symmetry of the normal curve, an equal area could be given by the shaded area below.



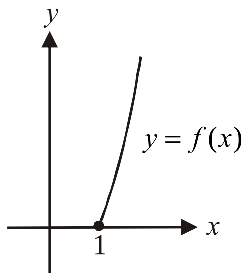

This area is given by $\Pr(7.5 < X < 17.5)$.
The answer is B.

**Question 13**

Since $f = g \times h$

$$\begin{aligned} d_f &= d_g \cap d_h \\ &= [1, \infty) \cap R \\ &= [1, \infty) \end{aligned}$$

Use CAS to sketch the graph of $f$
that is, $f(x) = \log_e(x) \times |x|$
Since $d_f = [1, \infty)$, from the graph, $r_f = [0, \infty)$.
So $d_{f^{-1}} = r_f = [0, \infty)$
The answer is C.



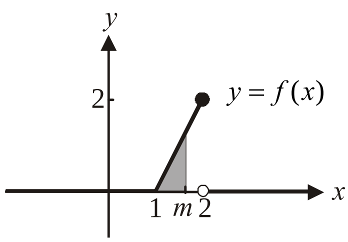

**Question 14**

Let $m$ be the median we want to find so that the shaded area $= 0.5$.

$$\int_1^m (|2x| - 2)\,dx = 0.5$$

$$m = 1.70711$$

The closest answer is 1.7.
The answer is D.



**Question 15**

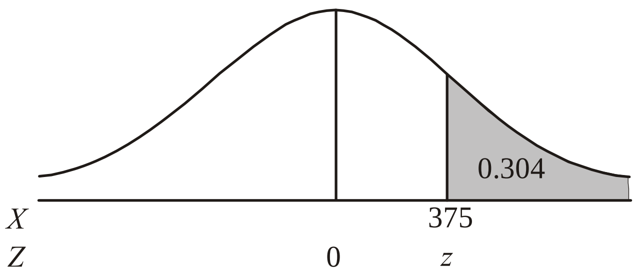
$y = \log_e(\sqrt{x})$
At $x = 4, \frac{dy}{dx} = \frac{1}{8}$ (using CAS).
We are looking for a line with a gradient of $\frac{1}{8}$.
The line $y = \frac{x}{4}$ has a gradient of $\frac{1}{4}$.
The line $y = 8x$ has a gradient of 8.
The line $y - 8x = 1$ becomes $y = 8x + 1$ and has a gradient of 8.
The line $8x - 8y = 1$ becomes $8y = 8x - 1$.

$y = x - \frac{1}{8}$ and has a gradient of $1$.

The line $x = 8y$ becomes $y = \frac{x}{8}$ and has a gradient of $\frac{1}{8}$.
The answer is E.

**Question 16**

$\left(\frac{1520}{5000} \times \frac{100}{1}\right)\% = 30.4\%$ of packets weigh more than 375g.
Draw a diagram.




So $\Pr(X < 375) = 1 - 0.304$

$= 0.696$

Using CAS and Inverse Normal find $\Pr(Z < z) = 0.696$ where $Z$ is the standard normal distribution (i.e. with mean of zero and $\sigma = 1$)

So $z = 0.51293$

Now, $z = \frac{x - \mu}{\sigma}$

$0.51293 = \frac{375 - \mu}{2}$

$\mu = 373.974...$

The closest answer is 373.97.
The answer is C.

**Question 17**

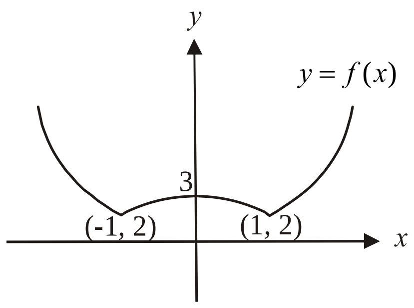
Option A describes the shaded area as does option B.

The graph of the function $y = \frac{4}{x+1} - 1$ is the graph of $y = \frac{4}{x+1}$ which has been translated 1 unit down. The integral $\int_0^3 \left(\frac{4}{x+1} - 1\right)dx$ gives the area which is equivalent to that part of the shaded area in the question above the rectangle with corner points $(0,0)$, $(3,0)$, $(3,1)$ and $((0,1)$. Since the area of the rectangle is 3 units, then option C does equal the area of the shaded region.

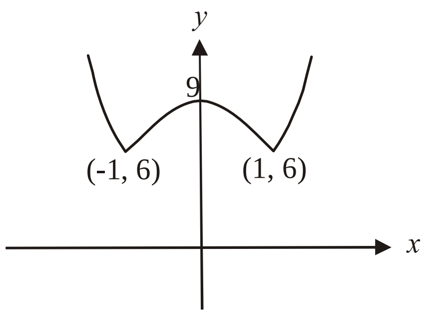
Option D does not equal the area of the shaded region.
The answer is D.

**Question 18**

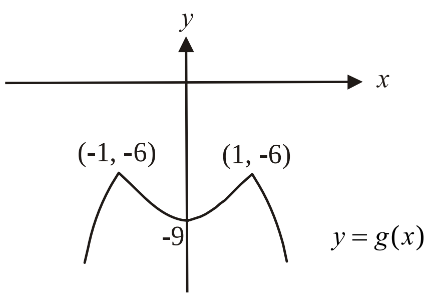
The graph below shows the graph of $y = f(x)$.




The graph below shows the graph of $y = f(x)$ after being dilated by a factor of 3 from $x$-axis.

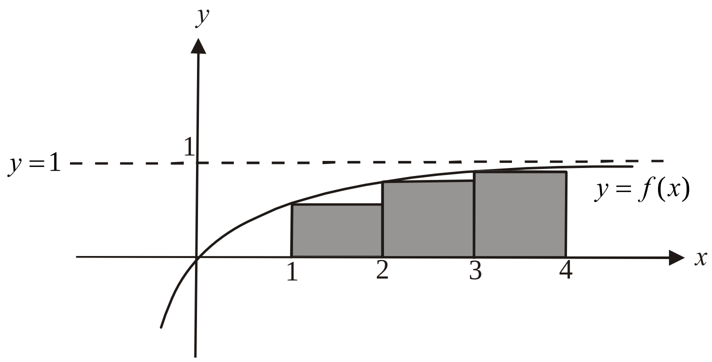



The graph below shows the graph of $y = f(x)$ after being dilated by a factor of 3 from $x$-axis and then reflected in the $x$-axis.




The maximum value of $g$ is $-6$.
The answer is B.

**Question 19**




$$\begin{aligned}\text{approximate area} &= 1\times f(1) + 1\times f(2) + 1\times f(3)\\ &= 1 - e^{-1} + 1 - e^{-2} + 1 - e^{-3}\\ &= 2.447...\\ \text{actual area} &= \int_0^4 (1 - e^{-x})dx\\ &= 3.01832\\ 3.01832... - 2.447... &= 0.571316...\\ \left(\frac{0.571316}{3.01832}\times\frac{100}{1}\right)\% &= 18.9283...\%\end{aligned}$$

The closest answer is 19%.
The answer is D.

**Question 20**

<u>Method 1</u> – by hand
$\log_a(b)\times\log_b(c)$

$=\log_a(b)\times\dfrac{\log_a(c)}{\log_a(b)}$ (change of base rule)

$=\log_a(c)$
The answer is C.

<u>Method 2</u> – using CAS
Key in the various alternatives and check whether you get a true or a false.
The answer is C.

**Question 21**

We are looking for the antiderivative of *g*. Another way of thinking of this is that the graph of *g* is the gradient function. From the graph of *g* we see therefore that the antiderivative graph must have

- a gradient that is negative for $x < 0$
- a gradient equal to zero at $x = 0$
- a gradient that is positive for $x > 0$

Option A does not have a gradient equal to zero at $x = 0$.
Option B has a positive gradient for $x < 0$ and a negative gradient for $x > 0$.
Option D has the same.
Option E has a sharp point at $x = 0$ and therefore the gradient at $x = 0$ is not defined.
The answer is C.

**Question 22**

This is a binomial distribution with $n = n$ and $p = 0.4$.

$$\begin{aligned}
\Pr(X \geq 2) &> 0.9 \\
1 - (\Pr(x = 0) + \Pr(x = 1)) &> 0.9 \\
-(\Pr(x = 0) + \Pr(x = 1)) &> -0.1 \\
\Pr(x = 0) + \Pr(x = 1) &< 0.1 \\
{}^nC_0(0.4)^0(0.6)^n + {}^nC_1(0.4)^1(0.6)^{n-1} &< 0.1 \\
1 \times 1 \times 0.6^n + n \times 0.4(0.6)^{n-1} &< 0.1 \\
0.6^n + 0.4n(0.6)^{n-1} &< 0.1 \quad (1)
\end{aligned}$$

Solve $0.6^n + 0.4n(0.6)^{n-1} = 0.1$ for $n$
$n = -1.42766$ or $n = 8.15209$

Reject the negative answer.

Test $n = 8$ in equation (1) $\text{Left side} = 0.106376 > 0.1$
Test $n = 9$ in equation (1) $\text{Left side} = 0.070544 < 0.1$

So Nick must have played at least 9 games.
The answer is C.

## SECTION 2

### Question 1

**a.** **i.** $f(x)=e^{ax}-k$

$f'(x)=ae^{ax}$

**(1 mark)**

**ii.** Stationary points occur when $f'(x)=0$ . Since $ae^{ax}=0$ has no solutions, $f$ has no stationary points.

**(1 mark)**

**b.** **i.** <u>$x$-intercepts</u> occur when $y=0$

$y=e^{ax}-k$

$0=e^{ax}-k$

Rearrange to read $x$ equals.

<u>Method 1</u> – using CAS

$x=\frac{1}{a}\ln(k)$ **(1 mark)**

<u>Method 2</u> – by hand

$$0=e^{ax}-k$$
$$k=e^{ax}$$
$$\log_e(k)=ax$$
$$x=\frac{1}{a}\log_e(k)$$

The $x$-intercept is $x=\frac{1}{a}\log_e(k)$.

**(1 mark)**

**ii.** We require $\frac{1}{a}\log_e(k)>0$ from part **b. i.**

$\log_e(k)>0$ since $a>0$ (given)

Using CAS $k>1$

**(1 mark)**

**c.** <u>$y$-intercept</u> occurs when $x=0$

$y=e^{0}-k$

$y=1-k$

**(1 mark)**

The graph of $y=f(x)$ has a positive $y$-intercept when

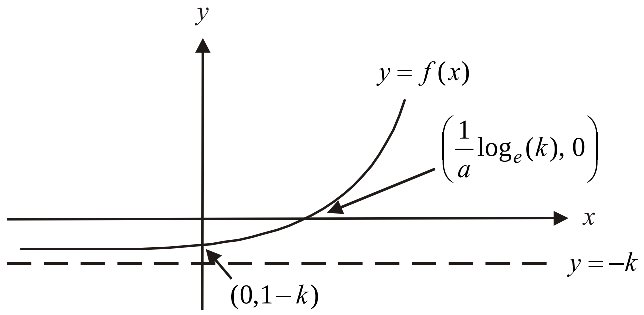
$1-k>0$

$-k>-1$

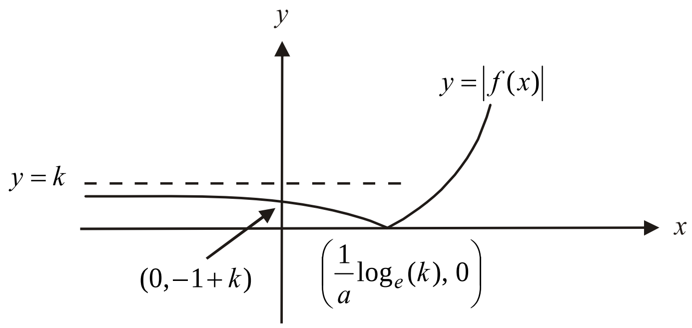
$k<1$

**(1 mark)**

**d.** **i.** $k > 1$ so from part **b.**, the graph of $y = f(x)$ has a positive $x$-intercept. The graph of $y = f(x)$ is shown below.

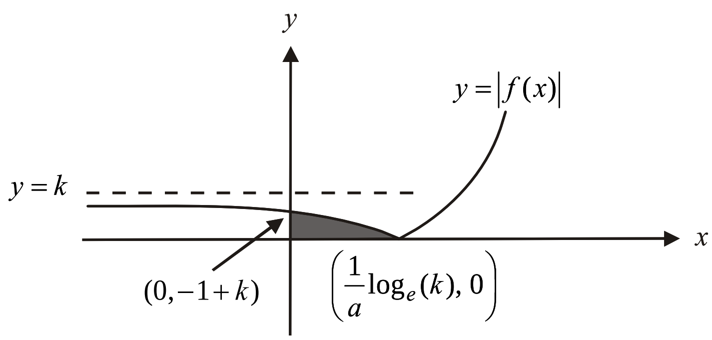



Therefore the graph of $y = |f(x)|$ will be given by




**(1 mark)** – correct $x$ and $y$ intercepts
**(1 mark)** – correct asymptote
**(1 mark)** – correct shape including sharp point and graph approaching asymptote

**ii.**




The area required is shaded in the diagram above.

$$\text{area required} = -1 \times \int_0^{\frac{1}{a}\log_e(k)} (e^{ax} - k)\,dx$$

**(1 mark)** – correct integrands
**(1 mark)** – correct terminals

**iii.** $\text{area} = \dfrac{k\log_e(k) - k + 1}{a}$ square units

**(1 mark)**
**Total 12 marks**

**Question 2**

**a.** $Y$ is normally distributed with mean 2 and standard deviation 0.6.
Using CAS, $\Pr(Y \le 1) = 0.0478$ (correct to 4 decimal places).

**(1 mark)**

**b.** $\Pr(Y < p) = 0.6$
Using CAS and the inverse normal function $p = 2.1520$ (correct to 4 decimal places).

**(1 mark)**

**c.** $\Pr(2 \le X \le 4)$

Method 1 – using CAS
Define $f(x)$.

**(1 mark)**

Evaluate $\int_2^4 f(x)dx$ **(1 mark)**

$= 0.3746$ (correct to 4 decimal places)

**(1 mark)**

The key to this method is to be able to define a hybrid function (a function with different rules over different parts of the domain and sometimes called a piecewise function) on your CAS.

Method 2 – using CAS

$$\begin{aligned}\Pr(2 \le X \le 4) &= \int_2^3 0.3125dx + \int_3^4 \frac{5}{16}e^{-5(x-3)}dx \\ &= 0.3125 + 0.062079... \\ &= 0.3746 \text{ (correct to 4 decimal places)}\end{aligned}$$

**(1 mark)** – first integral
**(1 mark)** – second integral
**(1 mark)** – correct answer

**d.** For a normal distribution, the mode is the mean because with the bell-shaped curve the mean is the *x*-value at the highest point on the curve.
The mode of $Y$ is 2.

**(1 mark)**

**e.** **i.** 
$$\begin{aligned}\text{var}(Y) &= (\text{standard deviation of } Y)^2\\ &= 0.6^2\\ &= 0.36\end{aligned}$$

**(1 mark)**

**ii.**
$$\begin{aligned}\text{var}(X) &= E(X^2) - \mu^2\\ &= \int_{-\infty}^{\infty} x^2 f(x)dx - 1.60625^2\\ &= 3.455 - 1.60625^2\\ &= 0.8750 \text{ (correct to 4 decimal places)}\end{aligned}$$

**(1 mark)** – integral
**(1 mark)** - $\mu^2$
**(1 mark)** – correct answer

**f.** We have a binomial distribution where $n = 6$ and $p = 0.5$ since $\Pr(Y < 2) = 0.5$ since $\mu = 2$.

**i.** Method 1 – using CAS

$\text{binomPdf}(6, 0.5, 1) = 0.0938$

**(1 mark)**

Method 2 – by hand

Let the binomial variable be *w*.
$$\begin{aligned}\Pr(w = 1) &= {}^6C_1 (0.5)^1 (0.5)^5\\ &= 0.0938\end{aligned}$$

**(1 mark)**

**ii.** Method 1 – using CAS

$\text{binomcdf}(6, 0.5, 1, 6) = 0.9844$ (correct to 4 decimal places)

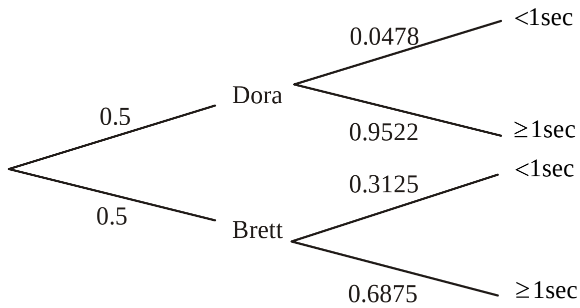
**(1 mark)**

Method 2 – by hand

$$\begin{aligned}\Pr(W \geq 1) &= 1 - \Pr(W = 0)\\ &= 1 - {}^6C_0 (0.5)^0 (0.5)^6\\ &= 0.9844 \text{ (correct to 4 decimal places)}\end{aligned}$$

**(1 mark)**

**g.** Method 1




**(1 mark)**

For Dora, $\Pr(Y < 1) = 0.0478$ from part **a.**

For Brett, $\Pr(X < 1) = \int_0^1 0.3125\, dx = 0.3125$

$$\begin{aligned}&\Pr(\text{Dora's log-in} \mid < 1 \text{ second})\\ &= \frac{0.5 \times 0.0478}{0.5 \times 0.0478 + 0.5 \times 0.3125}\\ &= 0.133 \text{ (correct to 3 decimal places)}\end{aligned}$$

**(1 mark)** - correct numerator
**(1 mark)** – correct denominator
**(1 mark)** – correct answer

Method 2 – conditional probability formula

$$\begin{aligned}&\Pr(\text{Dora's log-in} \mid < 1 \text{ second})\\ &= \frac{\Pr(\text{Dora's log-in} \cap < 1 \text{ second})}{\Pr(< 1 \text{ second})}\\ &= \frac{0.0478}{\Pr(X < 1) + \Pr(Y < 1)}\\ &= \frac{0.0478}{0.3125 + 0.0478}\\ &= 0.133 \text{ (correct to 3 decimal places)}\end{aligned}$$

**(1 mark)** – use of conditional probability formula
**(1 mark)** - correct numerator
**(1 mark)** – correct denominator
**(1 mark)** – correct answer
**Total 16 marks**

**Question 3**

**a.** $f(x)=2\sin\left(\dfrac{\pi x}{3}\right)+1$

$$\begin{aligned}\text{period}&=\frac{2\pi}{n} \text{ where } n=\frac{\pi}{3}\\&=2\pi\div\frac{\pi}{3}\\&=2\pi\times\frac{3}{\pi}\\&=6 \text{ as required}\end{aligned}$$

**(1 mark)**

**b.** The maximum occurs at $(1.5,3)$ and the minimum occurs at $(4.5,-1)$. The function $f$ is strictly decreasing for $x\in[1.5,4.5]$.

**(1 mark)**

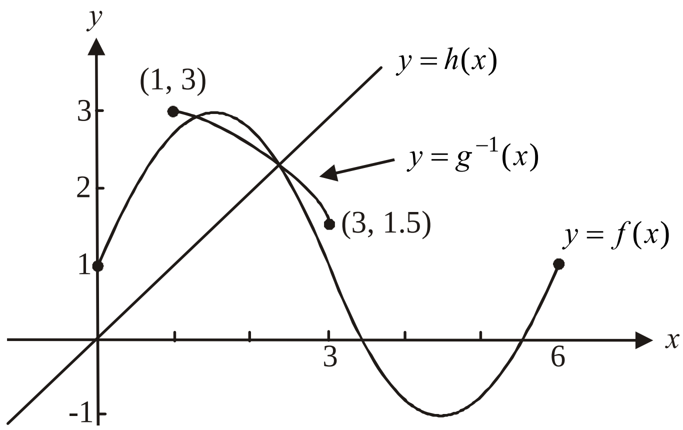
**c.** The inverse function $f^{-1}$ does not exist because $f$ is not a $1:1$ function.

**(1 mark)**

**d. i.** The function $g$ has the same rule as the function $f$ but a smaller domain. If $g^{-1}$ is to exist then $g$ must be $1:1$. The domain of $g$ is $[a,3]$. If $g$ is to be $1:1$ then the least value that $a$ can take is 1.5. So $a=1.5$.

**(1 mark)**

**ii.**




**(1 mark)** – correct endpoints
**(1 mark)** – correct intersection with graph of $y=f(x)$ and $y=h(x)$

**iii.** Method 1 – using CAS

Solve $x = 2\sin\left(\frac{\pi y}{3}\right) + 1$ for $y$

$$y = \frac{3}{\pi}\sin^{-1}\left(\frac{x-1}{2}\right) \qquad \text{so, } g^{-1}(x) = \frac{3}{\pi}\sin^{-1}\left(\frac{x-1}{2}\right)$$

**(1 mark)**

Method 2 – by hand

Let $y = 2\sin\left(\frac{\pi x}{3}\right) + 1$

Swap $x$ and $y$ for inverse.

$$x = 2\sin\left(\frac{\pi y}{3}\right) + 1$$

$$x - 1 = 2\sin\left(\frac{\pi y}{3}\right)$$

$$\frac{x-1}{2} = \sin\left(\frac{\pi y}{3}\right)$$

$$\frac{\pi y}{3} = \sin^{-1}\left(\frac{x-1}{2}\right)$$

$$y = \frac{3}{\pi}\sin^{-1}\left(\frac{x-1}{2}\right) \qquad \text{so } g^{-1}(x) = \frac{3}{\pi}\sin^{-1}\left(\frac{x-1}{2}\right)$$

**(1 mark)**

**e. i.** Solve $g(x) = h(x)$ for $x$

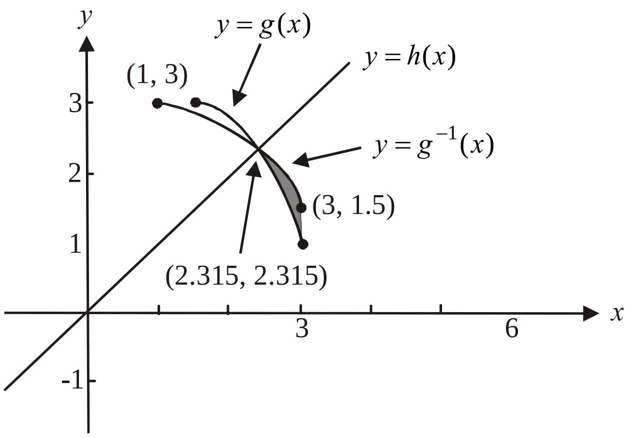
i.e. $2\sin\left(\frac{\pi x}{3}\right) + 1 = x$ for $x$

$x = 2.3149...$

So $x = 2.315$ correct to 3 decimal places

**(1 mark)**

**ii.** The area required is shaded in the diagram below.

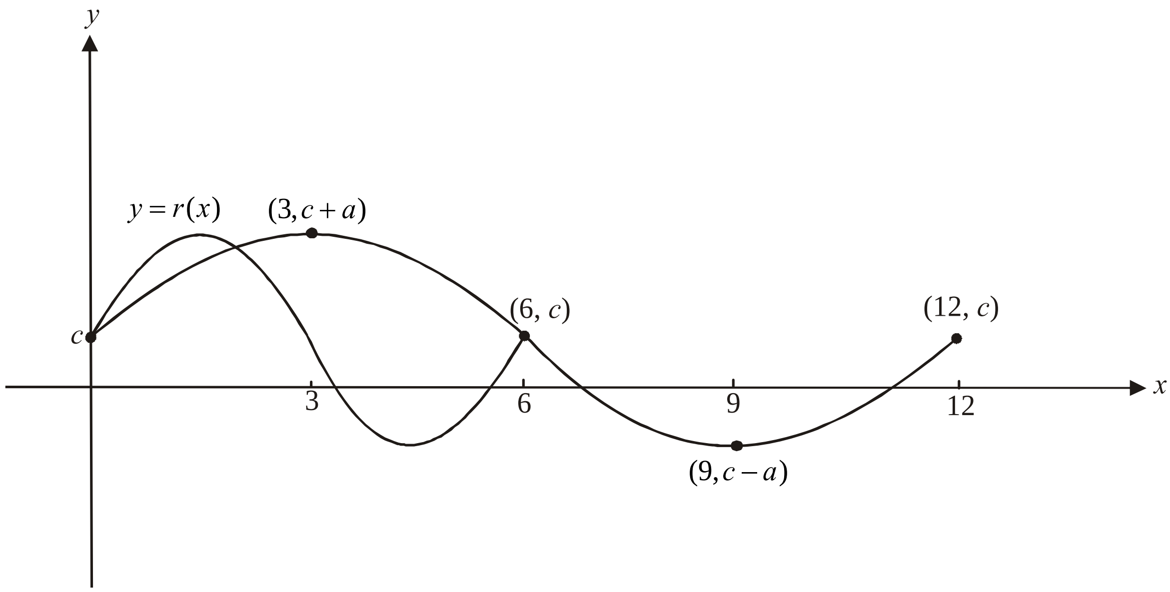



The definite integral that describes this area is

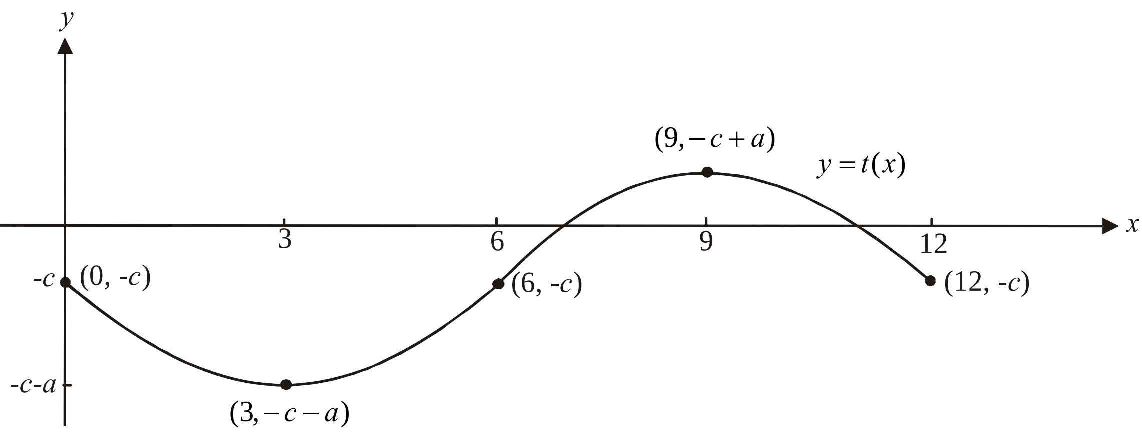
$$\text{area} = \int_{2.315}^{3} (g^{-1}(x) - g(x))\, dx$$

**(1 mark)** – correct integrand
**(1 mark)** – correct terminals

**f.** Step 1 – this shows the dilation




Step 2 – this shows the dilation and the reflection




**(1 mark)** – correct endpoints
**(1 mark)** – correct max/min points
**(1 mark)** – correct shape
**Total 13 marks**

**Question 4**

**a.** From the graph, the $x$-coordinates of the point $V$ is 0.

Since $y=-\sqrt{2-x}$

When $x=0$,

$$y=-\sqrt{2-0}$$
$$=-\sqrt{2}$$

The point $V$ is $\left(0,-\sqrt{2}\right)$

**(1 mark)**

**b.**

$$y=-\sqrt{2-x}$$
$$\frac{dy}{dx}=\frac{1}{2\sqrt{2-x}}$$

**(1 mark)**

Solve $\frac{1}{2\sqrt{2-x}}\le 2$ for $x$

$$x\le\frac{31}{16}$$ **(1 mark)**

When $x=\frac{31}{16}$, $y=-\sqrt{2-\frac{31}{16}}$

$$=-\frac{1}{4}$$

The required point is $\left(\frac{31}{16},-\frac{1}{4}\right)$.

**(1 mark)**

**c.** **i.** The rescue tunnel is a straight line that runs between the points $P(x,y)$ and $D(2+k,0)$ .

Since $P$ lies on the line representing the shaft with equation $y=-\sqrt{2-x}$, we can express $P$ as $\left(x,-\sqrt{2-x}\right)$. **(1mark)**

So the gradient of the straight line joining points $P$ and $D$ is given by

$$\frac{y_2-y_1}{x_2-x_1}$$
$$=\frac{-\sqrt{2-x}-0}{x-(2+k)}$$
$$=\frac{-\sqrt{2-x}}{x-2-k}$$
$$=\frac{\sqrt{2-x}}{2+k-x}$$

**(1 mark)**

**ii.** Since the gradient of the shaft is given by $\frac{dy}{dx} = \frac{1}{2\sqrt{2-x}}$ from part **b.** and since the gradient of the shaft equals the gradient of the tunnel at $P$, we have $\frac{1}{2\sqrt{2-x}} = \frac{\sqrt{2-x}}{2+k-x}$.

Solve this for $x$ to find the $x$-coordinate of $P$. **(1 mark)**

Using CAS, $x = -(k-2)$

$= 2-k$ **(1 mark)**

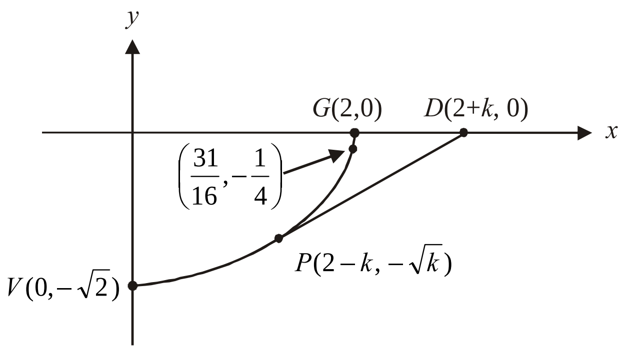
Substitute this into $y = -\sqrt{2-x}$

$y = -\sqrt{2-(2-k)}$

$y = -\sqrt{k}$

The coordinates of $P$ are $\left(2-k, -\sqrt{k}\right)$ as required. **(1 mark)**

**d.**




Method 1

At point $P$, $x = 2-k$ so, $k = 2-x$

Now, the domain of the shaft is $x \in [0,2]$.

When $x = 0$, $k = 2$.

Victoria can't get to point $G(2,0)$ because of the steepness of the tunnel. The last point she can get to is $\left(\frac{31}{16}, -\frac{1}{4}\right)$ so this is the last point that the rescuers should be aiming their tunnel to end at.

When $x = \frac{31}{16}$, $k = 2 - \frac{31}{16}$

$= \frac{32}{16} - \frac{31}{16}$

$= \frac{1}{16}$

So $k \in \left[\frac{1}{16}, 2\right]$

Note that the endpoints are included because if the tunnel reaches the shaft at $x = 0$ i.e. $k = 2$ or at $x = \frac{31}{16}$ i.e. $k = \frac{1}{16}$, then Victoria can escape through it.

**(1 mark)** – one correct endpoint **(1 mark)** – a second correct endpoint

Method 2

At point $P$, $x = 2-k$ and we know that $0 \le x \le \frac{31}{16}$.

Solve $0 \le 2-k \le \frac{31}{16}$ for $k$. So $k \in \left[\frac{1}{16}, 2\right]$.

**(1 mark)** – one correct endpoint **(1 mark)** – a second correct endpoint

**e.** Since the rescue tunnel is a straight line joining the points $P\left(2-k,-\sqrt{k}\right)$ and $D(2+k,0)$,

the distance $PD=\sqrt{((2-k)-(2+k))^2+\left(-\sqrt{k}-0\right)^2}$

$$=\sqrt{(2-k-2-k)^2+\left(-\sqrt{k}\right)^2}$$

$$=\sqrt{(-2k)^2+k}$$

$$=\sqrt{4k^2+k}\text{ km}$$

**(1 mark)**

Victoria can move at 0.5 $\frac{\text{km}}{\text{hr}}$ through the rescue tunnel so the time taken to move through it is

$\sqrt{4k^2+k}\text{ km}\div 0.5\frac{\text{km}}{\text{hr}}$

$=\sqrt{4k^2+k}\text{ km}\times\frac{\text{hr}}{0.5\text{km}}$

$=\sqrt{4k^2+k}\times 2\text{ hr}$

$=2\sqrt{4k^2+k}\text{ hr}$

Total time taken to move from $V$ to $D$ via $P$ is given by $T=6\left(\sqrt{2}-\sqrt{k}\right)+2\sqrt{4k^2+k}$ as required.

**(1 mark)**

**f.** **i.** Using CAS

$$\frac{dT}{dk}=\frac{8k+1}{\sqrt{k(4k+1)}}-\frac{3}{\sqrt{k}}$$

**(1 mark)**

**ii.** Solve $\frac{dT}{dk}=0$ for $k$.

$$k=\frac{3\sqrt{17}+5}{32}$$

**(1 mark)**

Since $P=\left(2-k,-\sqrt{k}\right)$

$$=\left(\frac{59-3\sqrt{17}}{32},\frac{-\sqrt{6\sqrt{17}+10}}{8}\right)$$

**(1 mark)**

$$=(1.46,-0.74)$$

where coordinates are expressed correct to 2 decimal places.

**iii.** $T\left(\frac{3\sqrt{17}+5}{32}\right)=6.68877...$

It takes 6.69 hours (correct to 2 decimal places) for Victoria to escape.

**(1 mark)**

**Total 17 marks**