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Unit 3 and 4 Mathematical Methods (CAS): Exam 2

Practice Exam Question and Answer Booklet

Duration: 15 minutes reading time, 2 hours writing time

Structure of book:

Section	Number of questions	Number of questions to be answered	Number of marks
A	22	22	22
B	5	5	58
		Total	80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory does not need to be cleared) and, if desired, one scientific calculator.
- Students are not permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied:
- This question and answer booklet of 16 pages, including a detachable formula sheet.

Instructions:
- You must complete all questions of the examination.
- Write all your answers in the spaces provided in this booklet.

Section A – Multiple-choice questions

Instructions

Answer all questions by circling your choice.
Choose the response that is correct or that best answers the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Questions

Question 1

The linear function $f: D \to \mathbb{R}, f(x) = 4x - 1$ has range $(-1, 11]$. The domain D is:

A. $[0,3)$
B. $(-3, 0]$
C. $(0, 3]$
D. $[3, 0]$
E. $\mathbb{R}$

Question 2

The graph of $y = kx - 4$ intersects the graph of $y = x^2 + 4x$ at one distinct point for:

A. $0 < k < 1$
B. $k = 0$
C. $k = 1$
D. $k > 0$
E. $k < 0$

Question 3

The solution set of the equation $e^{6x} - 6e^{3x} + 5 = 0$ is:

A. $\left\{-\frac{1}{3}\log_e 5, 0, \frac{1}{3}\log_e 5\right\}$
B. $\emptyset$
C. $\left\{0, \frac{1}{3}\log_e 5\right\}$
D. $\{1, 5\}$
E. $\{0\}$

Question 4

The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ with rule

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

maps the curve with equation $y = 4x^3$ to the curve with equation:

A. $y = 2 - \frac{1}{2}(x - 1)^{\frac{1}{3}}$
B. $y = 4x^3$
C. $x = 2 - \frac{1}{2}(x - 1)^{\frac{1}{3}}$
D. $x = 4y^3$
E. $y = \frac{1}{2}(x - 1)^3 + 2$

Question 5

The average rate of change of the function with rule $f(x) = \sin(x)$ between $x = 0$ and $x = \pi$ is:

A. $\frac{1}{\pi}$
B. π
C. 1
D. -1
E. 0

Question 6

The graph of the function $f: [0, \infty) \to \mathbb{R}$ where $f(x) = 2\sqrt{x}$ is translated 2 units down and 3 units to the right. The equation of the new graph is:

A. $y = 2\sqrt{x+2} + 3$
B. $y = 2\sqrt{x-2} + 3$
C. $y = 2\sqrt{x+3} - 2$
D. $y = 2\sqrt{x-3} - 2$
E. $y = 2\sqrt{x-3} + 2$

Question 7

$\{x: \sin^2 x + \sqrt{3}\sin x = 0\} =$

A. $\{x: \sin x = 0\} \cup \{x: \sin x = -\sqrt{3}\}$
B. $\{x: \sin x = 0\}$
C. $\left\{x: \sin x = \frac{1}{\sqrt{3}}\right\}$
D. $\{x: \sin x = \sqrt{3}\} \cup \{x: \sin x = -\sqrt{3}\}$
E. $\left\{x: \sin x = \frac{1}{\sqrt{3}}\right\} \cup \{x: \sin x = 0\} \cup \left\{x: \sin x = -\frac{1}{\sqrt{3}}\right\}$

Question 8

Let $f: [0, \infty) \to \mathbb{R}, f(x) = \log_e x$. For all $u \in [0, \infty), f(u) + f(2u)$ is equal to:

A. $f\left(\frac{1}{2}\right)$
B. $f(2u^2)$
C. $3f(u)$
D. $f(3u)$
E. $f\left(\frac{1}{3}u\right)$

Question 9

The graph of the function $f: D \to \mathbb{R}, f(x) = \frac{x-6}{x+2}$, where D is the maximal domain, has asymptotes:

A. $x = 6, y = -2$
B. $x = -2, y = 6$
C. $x = -2, y = 1$
D. $x = 2, y = 6$
E. $x = 2, y = -6$

Question 10
The graph of $y = -x(x-3)(x-4)$ has x-axis intercepts at 0, 3 and 4.
The graph of $y = |-x(x-3)(x-4)|$ is strictly positive for:

A. $x \in \mathbb{R}$
B. $x \in \{x: x < 0\} \cup \{x: 3 < x < 4\}$
C. $x \in \{x: x \leq 0\} \cup \{x: 3 \leq x \leq 4\}$
D. $x \in \{x: x < 0\} \cup \{x: 0 < x < 3\} \cup \{x: 3 < x < 4\} \cup \{x: x > 4\}$
E. $x \in \{x: 3 \leq x \leq 4\}$

Question 11
For $y = e^{4x}\sin(3x)$, the rate of change of y with respect to x when $x = 0$ is:

A. 0
B. 1
C. -1
D. 3
E. -3

Question 12
At the point $x = -1$ on the graph of the function with rule $y = \log_e x$:

A. there is a local maximum
B. there is a local minimum
C. there is a stationary point of inflection
D. the function is not defined
E. there is a point of discontinuity

Question 13
For $y = \sin\big(f(x)\big)$, $\frac{dy}{dx}$ is equal to:

A. $\cos\big(f(x)\big)$
B. $\cos\big(f'(x)\big)$
C. $f(x)\cos\big(f(x)\big)$
D. $f'(x)\cos\big(f(x)\big)$
E. $f'(x)\cos\big(f'(x)\big)$

Question 14
If $f'(x) = x^2 + 2x + 1$, then $f(x)$ has:

A. one local maximum
B. two local maxima
C. one local maximum and one local minimum
D. one local minimum and one stationary point of inflection
E. one stationary point of inflection

Question 15

If $\int_1^4 f(x)dx = 2$, then $\int_1^4 (2f(x) - 4)dx$ is equal to:

A. -8
B. 1
C. 2
D. 4
E. 8

Question 16

The average value of the function $f(x) = 1 - x^2$ for $-1 \leq x \leq 1$ is closest to:

A. 0
B. $^1/_3$
C. $^2/_3$
D. 1
E. 2

Question 17

The function f has rule $f(x) = \log_e(2x)$. If $f(10x) = \log_e y$, then y is equal to:

A. $\log_e 20x$
B. $20x$
C. $\log_e 10x$
D. e^{10x}
E. x

Question 18

The probability density function for the continuous random variable X is given by:

$$f(x) = \begin{cases} 0.5x & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

The probability that $X < 1.5$ is equal to:

A. 0
B. $^1/_2$
C. $^9/_{16}$
D. $^3/_4$
E. 1

Question 19

A fair coin is tossed 11 times. How many heads would you expect to obtain?

A. 4.5
B. 5
C. 5.5
D. 6.5
E. 11

Question 20

A fair die, with each of the six outcomes being equally likely, is rolled multiple times. Which of the following pairs of events are **not** mutually exclusive?

A. $\{1, 2, 3\}$ and $\{3, 3, 6\}$
B. $\{1\}$ and $\{1\}$
C. $\{1\}$ and $\{3\}$
D. $\{1, 2, 3\}$ and $\{1, 2, 4\}$
E. $\{1, 2, 3\}$ and $\{4, 5, 6\}$

Question 21

The random variable X has a normal distribution with mean 10 and standard deviation 3. If the random variable Z has the standard normal distribution, then the probability that X is less than 11.5 is equal to:

A. $\Pr(Z > -0.5)$
B. $\Pr(Z > 0.5)$
C. $\Pr(Z > 0.85)$
D. $\Pr(Z > -0.85)$
E. $\Pr(Z > 1.5)$

Question 22

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(5) = 0$ and $f'(x) > 0$ when $x < 5$ and $f'(x) < 0$ when $x > 5$.

When $x = 5$, the graph of f has a:

A. local minimum
B. local maximum
C. stationary point of inflection
D. point of discontinuity
E. gradient of 5

Section B – Short-answer questions

Instructions

Answer all questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Questions

Question 1

Consider the function $f: \mathbb{R} \to \mathbb{R}, f(x) = a\cos(x+b) + c$, where $0 \le b < \pi$.

a. Find $f'(x)$ in terms of a, b and c.

2 marks

f has a turning point at $\left(\frac{\pi}{2}, 2-\sqrt{2}\right)$ and also passes through the point $\left(\frac{\pi}{4}, 4-2\sqrt{2}\right)$.

b. Use this information to write three simultaneous equations in a, b and c.

3 marks

c. Solve for a, b and c, giving your answers in exact form.

3 marks

d. Find $f^{-1}(x)$.

2 marks

e. Find $\left(f(x)\right)^{-1}$, and state its domain.

3 marks

f. Solve $f(x) = 4 - \sqrt{2}$ for $x \in \mathbb{R}$. Leave your solution in general form.

3 marks

Total 16 marks

Question 2

Hamish is planning on taking his dog, Marty, for a walk from his cave to the shops. Hamish walks across his field to the road, and then walks along the road to the shops. His field is 5 km by 20 km, with the road running along the longer side. His cave is diagonally opposite the shops, as shown in Figure 1 below:

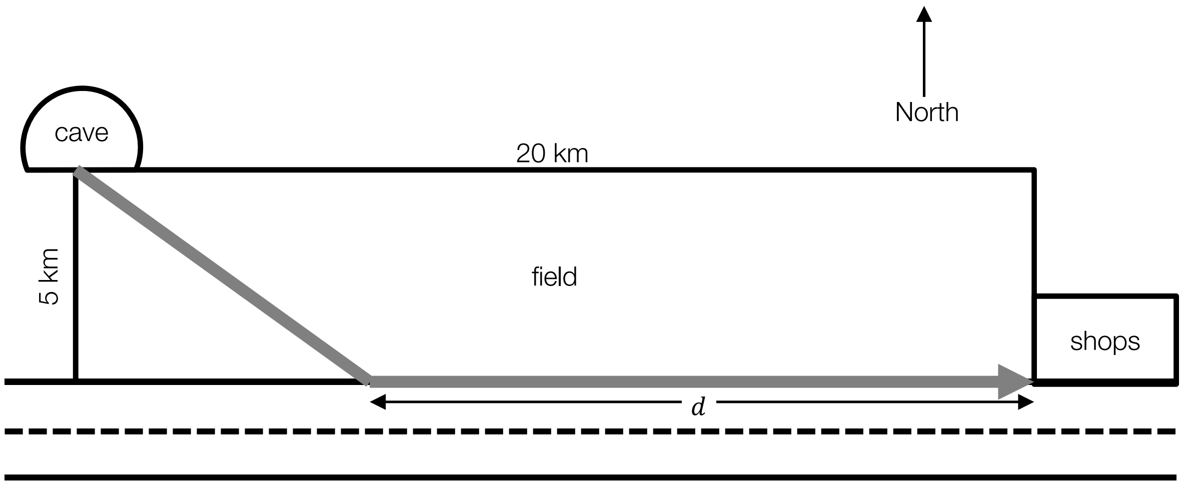


Figure 1

For the following questions, assume that the only way to access the shops is through their south-west corner (ie. where the grey arrow ends in Figure 1 above).

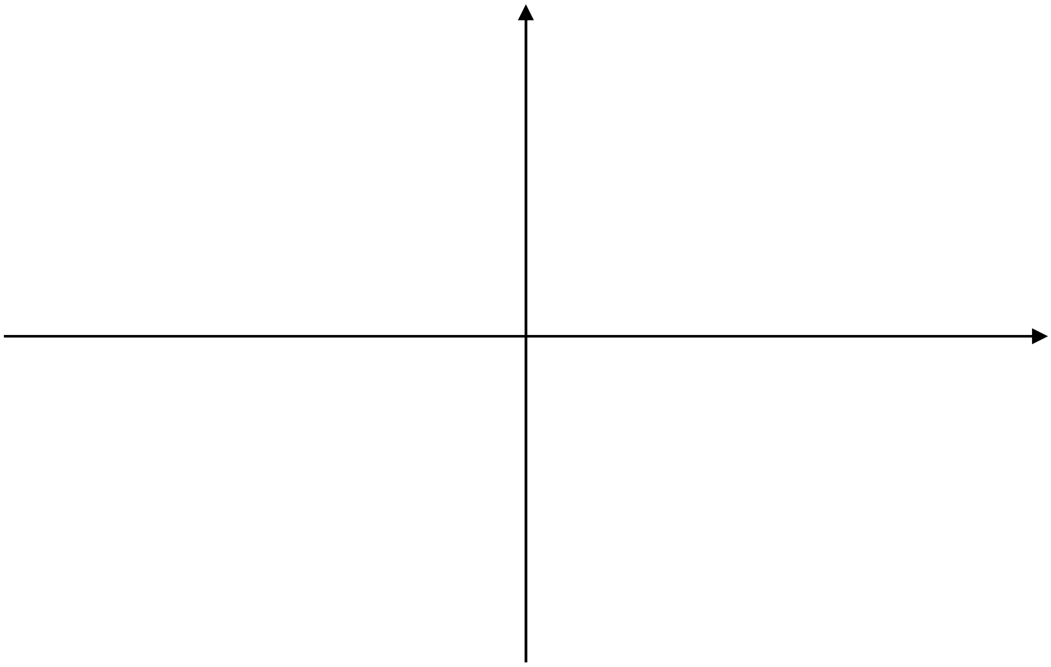
a. If Hamish walked directly from his cave to the shops, how far would he walk?

1 mark

b. Suppose Hamish decides to walk in a straight line from his cave to the road, and then walks the remaining d km along the road, as shown by the grey arrow in Figure 1 above. Write an expression for L, the distance Hamish walks, in terms of d.

2 marks

c. Draw L on the axes below, for a suitable range of d. Include all points of intersection, maxima and minima.



3 marks

d. Marty has bad knees, and can only walk 45 km in a day. If Hamish wants to have the longest walk to the shops and back without injuring Marty, how far should he walk along the road on each trip? Assume he follows the same path in both directions.

2 marks

e. Hamish wants to meet his friend Dave on his way to the shops at the location 2 km due north of the field's boundary with the road, and 5 km due east of his cave. How far will he walk on his trip to the shops? Give your answer to 2 decimal places.

4 marks

Total 12 marks

Question 3
Callum and Eloise are painting the office floor. Their office has an unusual shape, so they are trying to find a suitable model for the area of the floor, in order to calculate how much paint they need to buy. All dimensions are in metres unless specified otherwise.

a. They start by modelling it as a simple rectangle, with dimensions 2 m x 11.5 m. If the floor actually has area 16.95 m^2, what percentage error does this model have? Give your answer to two significant figures.

2 marks

b. The second model they try is the area bounded by the x-axis and the curve $y = -\frac{3}{10}(x - 11.5)$. What is the area given by this model?

2 marks

c. They finally settle on a model that represents the area of the floor as the area between the two curves:

$$y = \frac{4}{5}\left(x^{\frac{5}{3}}\right)$$
$$y = 3\left(x^{\frac{3}{5}}\right)$$

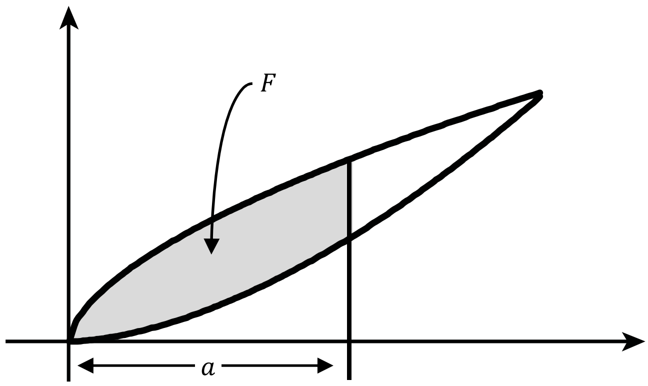
i. Find all points of intersection. Simplify your answers as much as possible, and leave them in exact form.

3 marks

ii. Write a definite integral, that, when evaluated, gives the area of the floor predicted by this model, and evaluate it to two decimal places.

2 marks

Callum and Eloise realise that they can only afford 5 m^2 worth of paint. They decide to paint as much as they can, as shown below:



d. Write an expression for F, the area bounded by $x = 0$, $x = a$ and the two curves given in part c. above, as a definite integral in terms of a.

2 marks

e. Hence, find the value of a if they paint 5 m^2 of the floor. Give your answer to two decimal places.

1 mark

Total 12 marks

Question 4

Clare and Sally play badminton against one another frequently and know that, on average, Clare wins 52% of points played.

a. What is the probability, correct to three decimal places, that Clare wins 7 points in a row?

1 mark

b. What is the probability, correct to three decimal places, that Clare wins at least 7 points out of 10?

2 marks

However, Clare and Sally have noticed that recently, if Sally wins a point, the probability of her then winning the next point is 0.61, whereas if Clare wins a point, the probability of her then winning the next point is 0.49. Assume that Sally has just won a point.

c. What is the probability of Sally winning the next two points?

1 mark

d. What is the probability of Sally winning two of the next three points?

3 marks

e. What is the probability of Clare winning the 12^{th} point?

3 marks

f. In the long term, what percentage of points will Clare win?

1 mark

Clare and Sally come home to discover that their mother has baked them a cake. Unfortunately, the cake has become infested with weevils. Each slice contains N weevils, where N is a random variable with the probability density function:

$$f(n) = \begin{cases} ae^{-\frac{n}{100}} & 0 \leq n < \infty \text{ and } a \text{ is a positive real constant} \\ 0 & \text{otherwise} \end{cases}$$

g. Find the value of a.

2 marks

h. On average, how many weevils are in a single slice of cake?

2 marks

i. What proportion of cake slices have more than 50 weevils?

3 marks

Total 18 marks

Formula sheet

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin A$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$\int \frac{1}{x} dx = \log_e|x| + c$$

$$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$$

$$\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$$

$$\int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$$

product rule $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)}{v^2}$

chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation $f(x+h) = f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices $S_n = T^n \times S_0$

mean $\mu = E(X)$

variance $var(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma xp(x)$	$\sigma^2 = \Sigma(x-\mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} xf(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$