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Unit 3 and 4 Mathematical Methods (CAS): Exam 1

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Found a mistake?

Check the Engage Education website for updated solutions, and then email practiceexams@ee.org.au.

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

Question 1a

$f'(x) = 3\cos(3x)\, e^{\sin(3x)}$ [2]

Question 1b

$f'(x) = 2\cos(2x)\, e^{3x} + 3\sin(2x)\, e^{3x}$ [1]

$f'\left(\frac{\pi}{2}\right) = -5e^{\left(\frac{3\pi}{2}\right)}$ [1]

Question 2a

$\int \frac{1}{x} dx = \log_e|x| + c$ [1]

$\log_e 3 + c = 10$

$c = 10 - \log_e 3$

Therefore the antiderivative is $\log_e\left(\frac{x}{3}\right) + 10$ [1]

Question 2b

$\int_1^c 3x^2 dx = [x^3]_1^c = c^3 - 1$ [1]

$c^3 - 1 > 2$

$c^3 > 3$

$c > \sqrt[3]{3}$ [1]

Question 3a

$f^{-1}(x)$ is formed by swapping x and y [1 mark for statement or use of this fact]

$x = \log_e\frac{(y-3)}{2} + 5$ [1]

$y = 2e^{(x-5)} + 3 = f^{-1}(x)$ [1]

Question 3b

$g(f(x)) = 2e^{\left(\log_e\frac{(x-3)}{2} + 5 - 5\right)} + 3$ [1]

$= 2e^{\left(\log_e\left(\frac{x-3}{2}\right)\right)} + 3$

$= 2 * \frac{x-3}{2} + 3$

$= x$ [1]

Question 4

First deduce that the transformations necessary are $(x, y) \rightarrow (\frac{x}{2} - \frac{\pi}{4}, \pi y)$ [2]

Now we have $\begin{bmatrix} \frac{x}{2} - \frac{\pi}{4} \\ \pi y \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} + \boldsymbol{b}$

Hence $\mathbf{A}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\frac{x}{2} - \frac{\pi}{4}\\ \pi y\end{bmatrix} - \boldsymbol{b}$ and so $\boldsymbol{b} = \begin{bmatrix}-\frac{\pi}{4}\\ 0\end{bmatrix}$ [1]

Then $\mathbf{A}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\frac{x}{2}\\ \pi y\end{bmatrix}$ and so $\boldsymbol{A} = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \pi\end{bmatrix}$ [1]

Question 5

$3x + 3\pi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + n\pi$, where $n \in \mathbb{Z}$ [1]

$$3x + 3\pi = \frac{\pi}{6} + n\pi$$

$$x + \pi = \frac{(1+6n)\pi}{18}$$

$x = \frac{(6n-17)\pi}{18}$ [1]

$x = -\frac{17\pi}{18}, -\frac{11\pi}{18}, -\frac{5\pi}{18}, \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$ [1]

Question 6a

Let μ' be the mean of aX, and σ' be the standard deviation of aX

$\mu' = \mu a = 27.2a$ by the properties of the normal distribution.

$\text{Var}(aX) = a^2\,\text{Var}(X) = a^2 \times 27.2^2 = (\sigma')^2$

$\therefore \sigma' = 27.2a$ [1 for giving the correct expressions for the mean and standard deviation for aX]

As $\Pr(aX \geq 108.8) = 0.025$, 108.8 must be 2 standard deviations away from the mean. [1]

Therefore, $\frac{108.8-\mu'}{\sigma'} = 2$

$$\frac{108.8 - 27.2a}{27.2a} = 2$$

$$108 - 27.2a = 54.4a$$

$$108.8 = 81.6a$$

$a = \frac{108.8}{81.6} = \frac{4}{3}$ [1]

Question 6b

$\frac{0-27.2}{27.2} = \frac{b-27.2a}{27.2a}$ [1]

$$-27.2a = b - 27.2a$$

$b = 0$ [1]

Question 7a

$f(x) \geq 0$ for all x [1]

$$\int_0^\infty \frac{1}{\theta} e^{\left(\frac{-x}{\theta}\right)} dx$$

$= \lim_{t\to\infty}\left[-e^{\left(\frac{-x}{\theta}\right)}\right]_0^t$

$= \lim_{t\to\infty}\left(-e^{\left(\frac{-t}{\theta}\right)} + e^0\right)$

$= 1$ [1]

Question 7b

$\Pr(X \le \theta \mid X \le 3\theta) = \frac{\int_0^{\theta}\frac{1}{\theta}e^{\left(\frac{-x}{\theta}\right)}dx}{\int_0^{3\theta}\frac{1}{\theta}e^{\left(\frac{-x}{\theta}\right)}dx}$ [1]

$= \frac{\left[-e^{\left(\frac{-x}{\theta}\right)}\right]_0^{\theta}}{\left[-e^{\left(\frac{-x}{\theta}\right)}\right]_0^{3\theta}} = \frac{-e^{-1}+1}{-e^{-3}+1}$ [1]

Question 8a

$E(X) = 0$ [1]

Question 8b

$E(X^2) = 1.2$ [1]

Question 8c

$\text{Var}(X) = E(X^2) - \left(E(X)\right)^2 = 1.2 - 0 = 1.2$

$\text{Var}(2X) = 4\,\text{Var}(X) = 4.8$ [1]

Question 9a

$f'(x) = 2x$

$f'(3) = 6$ [1]

Hence the tangent is given by:

$y = 6(x-3) + f(3)$

$y = 6x - 18 + 5$

$y = 6x - 13$ [1]

Question 9b

$f(3.02) = f(3 + 0.02)$ [1]

$f(3 + 0.02) \approx f(3) + 0.02f'(3)$

$= 5 + 0.02 \times 6 = 5.12$ [1]

Question 10a

$\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$ [1]

$\frac{dS}{dr} = 8\pi r$ and $\frac{dr}{dt} = 3 \times 10^8$ [1]

$\frac{dS}{dt} = 8\pi r \times 3 \times 10^8 = 24\pi r \times 10^8$ [1]

Question 10b

$\frac{dS}{dt} = \frac{8}{3}\pi r \times 10^8 = 10^{19}$ [1]

$r = \frac{10^{19}}{\frac{8}{3}\pi \times 10^8} = \frac{3}{8\pi} \times 10^{11}$ m [2]

Question 10c

The ratio is given by $\frac{S(10^{10})}{S(10^{12})}$ [1]

$= \frac{4\pi(10^{10})^2}{4\pi(10^{12})^2}$

$= \frac{1}{10^4}$ [1]