QATs
Quality Assessment Tasks

UNIT 3 ■ OUTCOMES 1, 2 & 3

VCE Mathematical Methods (CAS)

SCHOOL-ASSESSED COURSEWORK

Introduction

Outcome 1
Define and explain key concepts as specified in the content from the areas of study, and apply a range of related mathematical routines and procedures.

Outcome 2
Apply mathematical processes in non-routine contexts, and analyse and discuss these applications of mathematics.

Outcome 3
Select and appropriately use technology to develop mathematical ideas, produce results and carry out analysis in situations requiring problem-solving, modeling or investigative techniques or approaches.

Task
Application Task
This task will contribute 40 of the total marks (60) allocated for SAC in Unit 3.This task will be marked out of 90 and then converted to a mark out of 40.

The task has been designed to allow achievement up to and including the highest level in the Performance Descriptors and meets a broad range of **key knowledge** and **key skills** related to each outcome.

The marks for each part are indicated in brackets at the beginning of each part.

This task is to be done in class over a period of approximately two weeks. You may use a summary book and an approved CAS calculator.

CAS TECHNOLOGY Indicates where use of the technology is specifically required in order to answer the question.

Answer in spaces provided or as indicated.

Your teacher will advise you of any variation to these conditions.

NAME:

Task

Give all answers to 3 decimal places unless otherwise indicated.

Part A: Coffee Cultivation

Worldwide, an estimated 15 billion coffee trees are grown on 100,000 km^2 of land.

1 sq kilometre = 100 hectares. **1 hectare = 10 000 m^2** **15 Billion = 15 000 000 000**

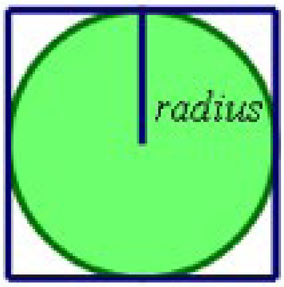
a) How many plants per hectare?

(1 mark)

b) What is the area needed for each plant?

(1 mark)

c) If each tree is planted in the middle of a square what is the largest radius a mature plant can have before it outgrows its plot of land? Give your answer as an exact value.

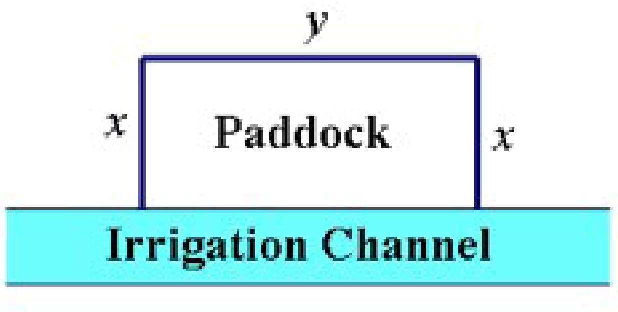


(2 marks)

Task

Gloria and Jean are farmers who want to fence off a paddock to grow coffee plants. They have **4 km of fencing materials**. If one side of the paddock will border an irrigation channel and the other three sides are to be fenced:

d) Write an equation that connects the length (x) and width (y) of the paddock to the amount of fencing material they can use.



(1 mark)

e) Show that the area of the paddock is defined by the equation:

$$A = 2(2x - x^2)$$

(2 marks)

f) What is the maximum area of the paddock? **Briefly explain or demonstrate how you found this maximum**.

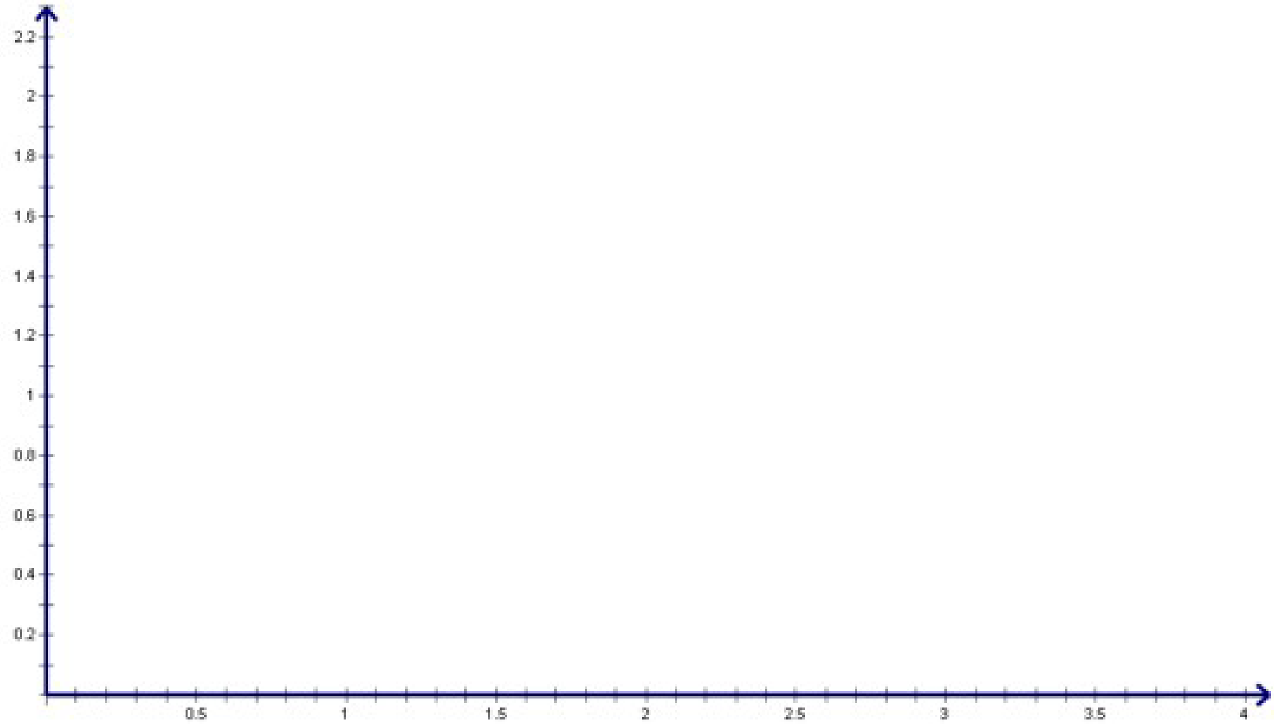
(2 marks)

Task

g) What are the dimensions of the paddock with this maximum area?

(2 marks)

h) Sketch a graph of the area function. Include domain and range. Also label the axes appropriately.



(7 marks)

Task

i) Under ideal conditions, how many plants can Jean and Gloria grow?

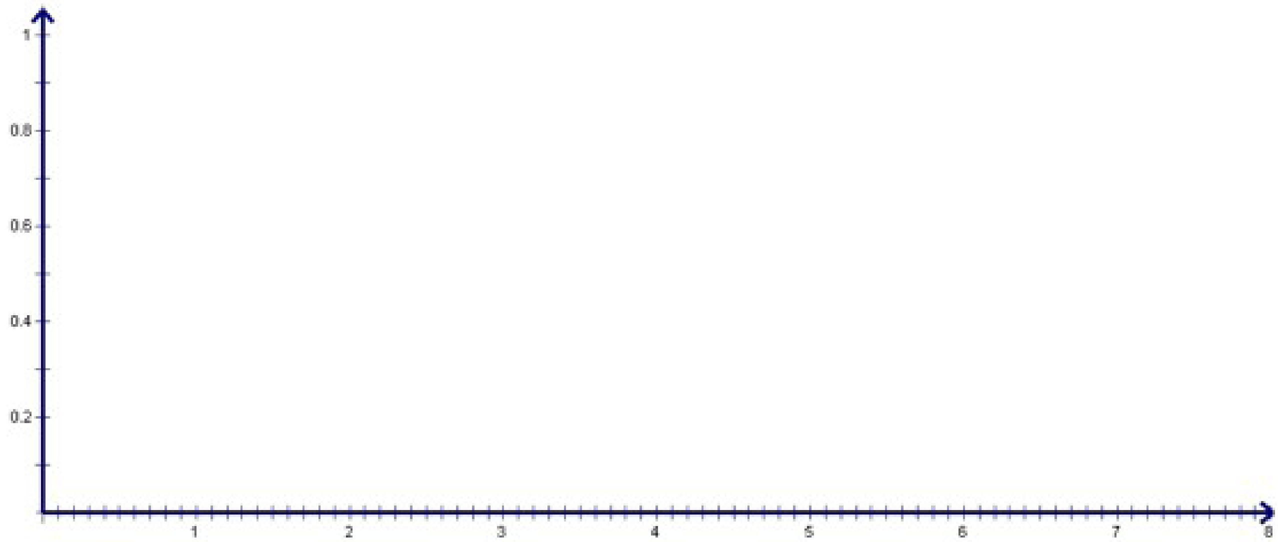
(2 marks)

Jean has conducted an experiment controlling the amount of water given to coffee plants using a drip system. The yield of coffee beans in kilograms for a single coffee plant was found to follow the equation

$$Y = -0.0072\,L^3 - 0.0036\,L^2 + 0.4068\,L - 0.2016$$

Where L is the number of litres per day given to the plant and Y is the yield in kilograms of coffee.

j) Sketch the graph of yield vs water volume on the axes provided over an appropriate domain.



(4 marks)

Task

k) *Find* the derivative of Y and hence find the **maximum** yield and the volume of water per day needed to achieve this yield.

(3 marks)

l) What is the **minimum** yield of coffee over the given domain? Explain your reasoning in one or two sentences.

(3 marks)

Task

Part B: Packaging

Kara Starbuck buys coffee beans from Gloria and Jean's farm. She roasts it, grinds it and packs it under the label "*Galactica*" and boasts that it is the best coffee in the galaxy!

When vacuum packed, **500 grams** of ground coffee takes up a volume of **990 cm³**.

Starbuck is designing a new style of packaging for her line of 500 gram packs. The pack is to be a **rectangular prism**.

a) The rectangular prism has a base whose dimensions are to be a golden rectangle: this means that the width is *x* cm and the length is *p.x* cm where

$$p = \frac{(1 + \sqrt{5})}{2}$$

Save the value of p in your calculator and show your teacher before you proceed.

Teacher signature:________________________________

(1 mark)

b) Show algebraically that the height of the prism is given by the expression:

$$h = \frac{990}{p.x^2}$$

(2 marks)

c) Now list the expressions for all three dimensions of the prism in terms of x.

$L =$ $W =$ $H =$

(3 marks)

Task

d) The general formula for the Surface Area of a rectangular prism is:

$$SA = 2LW + 2LH + 2WH$$

Use this formula to determine an equation for the Surface Area of Starbuck's rectangular package in terms of **x** and **p** only.

(2 marks)

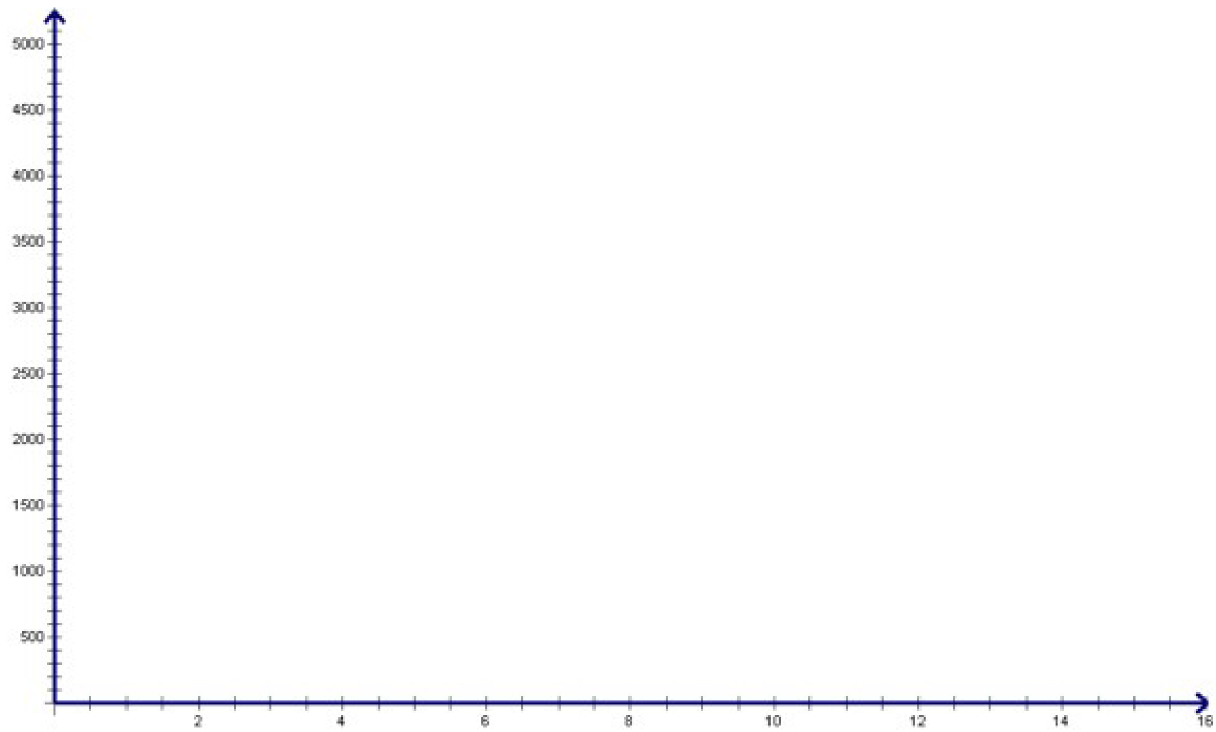
e) **Show using algebra** that this Surface Area formula is the same as:

$$S.A. = \frac{1980(1+p)}{px} + 2px^2$$

(2 marks)

Task

f) Sketch the Surface Area function on the axes provided for the domain (0, 15].



(4 marks)

CAS TECHNOLOGY

g) Use your calculator to differentiate the function for surface area with respect to x.

(2 marks)

Task

h) Determine the ***minimum*** Surface Area of the prism.

(1 mark)

i) What are the **linear dimensions** of the prism with this minimum Surface Area?
Express your answer in exact form.

(3 marks)

Task

The other packaging shape is a cylinder. Again the volume is 990 cm^3.

The volume of a cylinder is given by $V = \pi r^2 H$

j) Find and expression for the height of the cylinder in terms of radius.

(1 mark)

The surface area of a cylinder is given by:

$$S.A = 2\pi r(r+h)$$

k) Rewrite the equation for surface area just in terms of r.

(1 mark)

l) Differentiate the equation for surface area.

(2 marks)

Task

m) What is the exact radius needed to obtain the minimum surface area for this container?

(2 marks)

CAS TECHNOLOGY

n) What is the exact value of the height of this container, again when surface area is a minimum?

(2 marks)

Task

o) What is the minimum surface area for this style of package? Quote your answer to 3 decimal places.

(1 mark)

p) Which of the two package styles uses the least amount of material to make?

(1 mark)

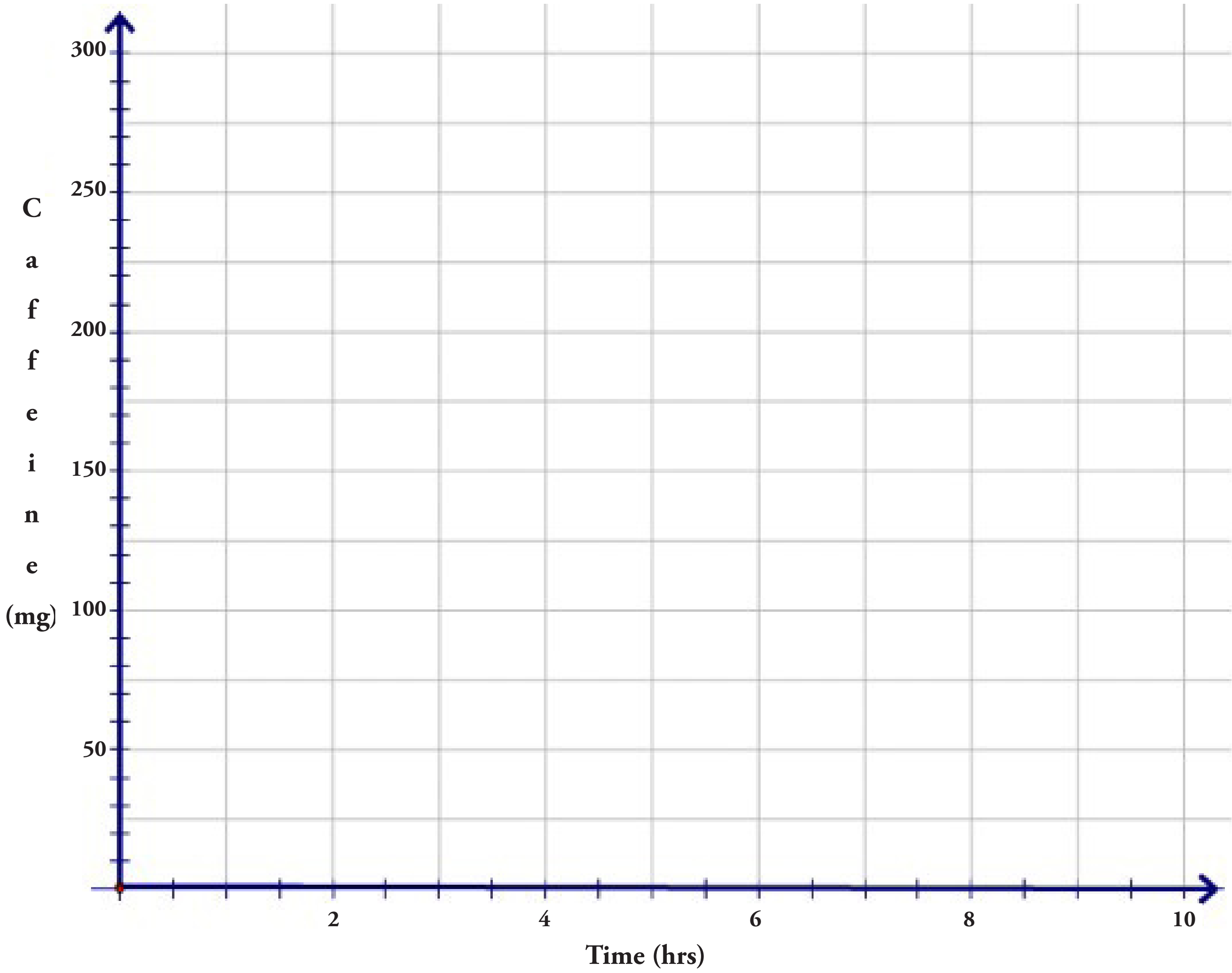
Task

Part C: Caffeine Concentration
Caffeine levels in the body drop off in such a way that no matter what amount you start with it takes the same amount of time to go from 300 mg to 150 mg as it does to go from 150 mg to 75 mg. Each time the amount left is half of what there was at the beginning. *This is called half life*.

A standard cup of coffee contains about 100 mg of caffeine. After 3 cups of coffee the amount of caffeine in a person's blood can reach 300mg. After 4 hours this drops to 150 mg.

a) Sketch a graph of the decay curve of caffeine in the body over the domain given in the axes below. Start your graph with 300mg of caffeine in the bloodstream. Show the coordinates of 3 points on your curve.

(3 marks)



Task

b) After how long has the concentration of caffeine in the brain dropped to 25% of its starting value?

(1 mark)

c) The equation for this graph is of the form:

$$C = Ae^{nt}$$

Show that $A = 300$.

(1 mark)

d) Now show algebraically that $n = \frac{\ln(\frac{1}{2})}{4}$

(3 marks)

Task

e) What is the amount of caffeine, in milligrams (mg), after 6 hours? Express your answer to three decimal places.

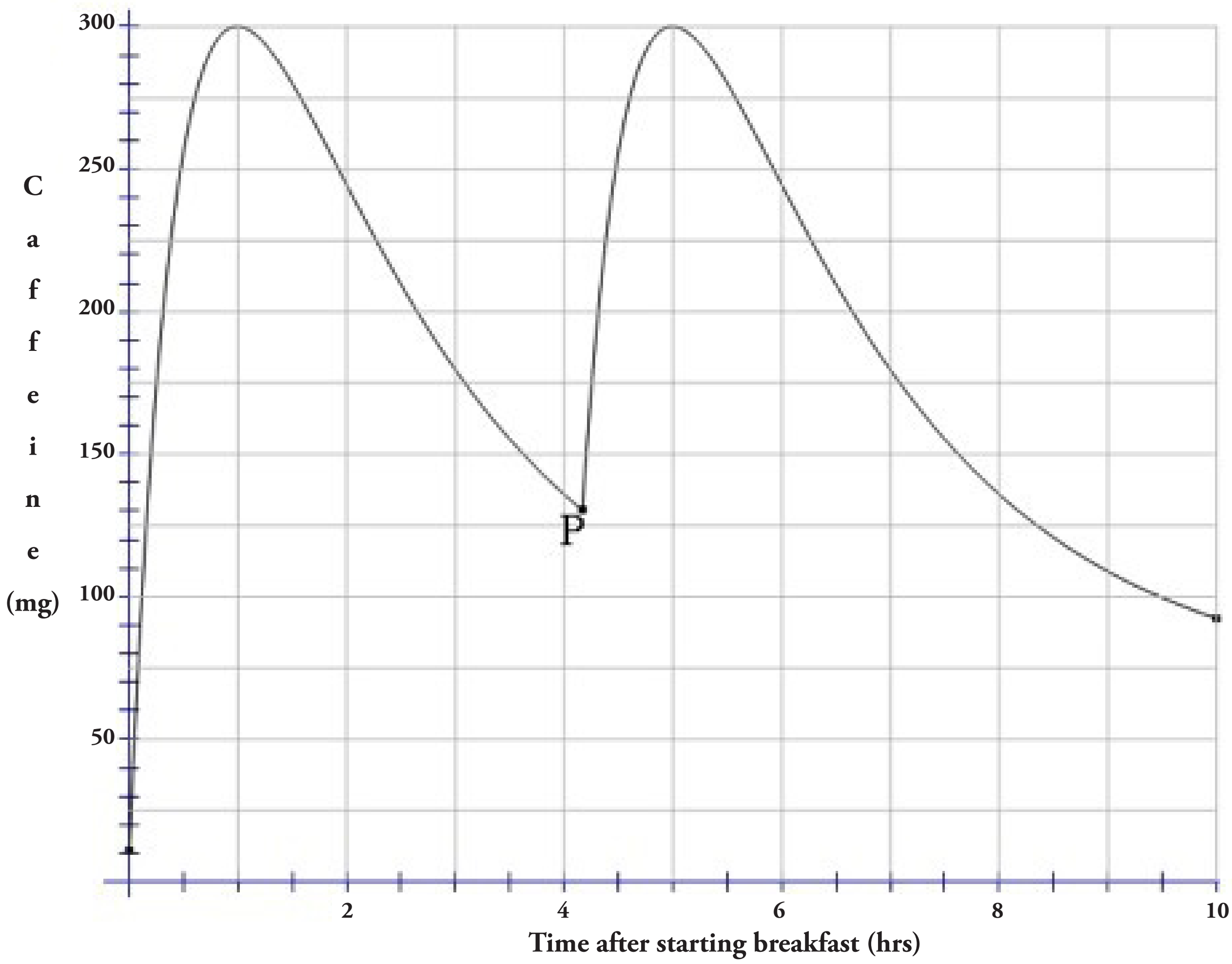
(2 marks)

f) If the concentration of caffeine in the blood is to be kept above 100 mg, how long after $t = 0$ should the person have another cup of coffee? (Express your answer to the nearest minute.)

(3 marks)

Task

Not only does caffeine take time to break down in the body, it also takes time for it to enter the bloodstream through the stomach. The graph below shows the rise and fall of caffeine over a ten hour period. **The two peaks represent the caffeine peaks after breakfast and morning tea respectively**.



g) Reading from the graph, approximately how long does it take for the caffeine concentration to reach the first maximum?

(1 mark)

h) From the graph, if morning tea is four hours after breakfast, how long after starting breakfast does it take for the caffeine level to reach 300 mg for the second time?

(1 mark)

Task

i) The first section of the graph above can be modelled by the sum of the functions:

$$f(x) = 3.9e^{-0.5(x-9.75)} + 33 \text{ and } g(x) = -e^{-2(x-3.17)} + 34$$

Write down the equation, *s(x)*, that is the *sum of the functions f(x) and g(x)*.

(3 marks)

CAS TECHNOLOGY

j) *Use your calculator* to find the derivative of s(x) with respect to x.

i) Show your calculator screen answer to your teacher.

Teacher signature:______________________________

(1 mark)

ii) Write down the derivative as it is on your screen.

(2 marks)

k) From this function, use your calculator to find the maximum concentration of caffeine in the blood. Give your answer to 3 decimal places.

(2 marks)

Task

l) At what time does this maximum occur?

(1 mark)

m) If the second section of the graph is given by

$$t(x) = s(x - 4)$$

after what time is the second maximum achieved?

(1 mark)

n) Describe how to use your calculator, to find the point of intersection, marked as P, between the first and second sections of the graph.

(2 marks)

o) Now use the calculator to find the coordinates of this intersection point.
   i) Briefly explain your calculator method. Writing out the input line is sufficient.

(1 mark)

Task

ii) Write down the coordinates of this point of intersection to 3 decimal places.

(2 marks)

Teacher Advice

This is the Application Task, as suggested to be undertaken during Weeks 12 and 13 in the sample teaching sequence on page 193 of the VCAA Study Design.

This task contributes 40 of the 60 SAC marks in Unit 3.
The coursework scores for this task are:
Outcome 1 7.5 marks 37.5%
Outcome 2 10 marks 50%
Outcome 3 2.5 marks 12.5%
TOTAL 40 marks
This weighting can be used in the conversion of their mark out of 90.
For example, a score of 80 results in:

OUTCOME 1	OUTCOME 2	OUTCOME 3
80/90*40*0.37	80/90*40*0.5	80/90*40*0.125
= 13.3	= 17.8	= 4.4

Rounding gives

= 13	= 18	= 5

The above can be established in an Excel file.

This QAT has been designed to meet the highest level in the performance descriptors provided by VCAA for each outcome in unit 3 in the VCAA Mathematics Study Design February 2010.

Solution Pathway

Part A: Coffee Cultivation

a) $\frac{15000000000}{100000 \times 100} = 1500$ *(1 mark)*

b) $\frac{10000}{1500} = \frac{20}{3}\ m^2$ *(1 mark)*

c) $Length = \sqrt{\frac{20}{3}} = \sqrt{\frac{60}{9}} = \frac{2\sqrt{15}}{3}$ *(2 marks)*

$Radius = \frac{L}{2}$

$\frac{\frac{2\sqrt{15}}{3}}{2} = \frac{\sqrt{15}}{3}$

d) *4 = 2x + y.*

(2 marks)

e)
$$\begin{aligned} y &= 4 - 2x \\ A &= x \times y \\ &= x(4-2x) \\ &= 4x - 2x^2 \\ &= 2(2x - x^2) \end{aligned}$$

(2 marks)

f) Various methods appropriate.

x = 1, Area = 2 km^2.

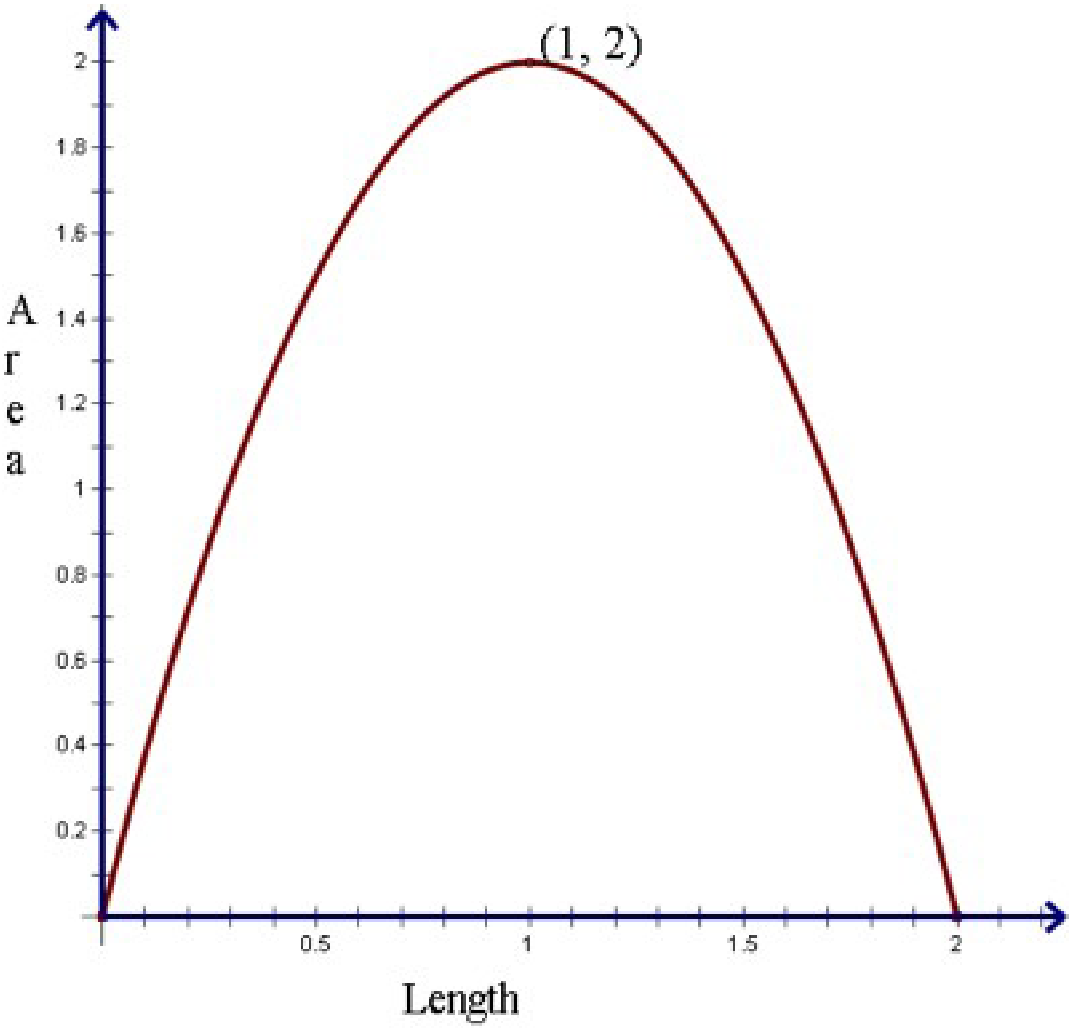
(3 marks)

g) *Length = x = 1 km*
Width = y = 2 km

(3 marks)

Solution Pathway

h) Shape – 1, Axial intercepts – 2, Domain – 2, Range - 2



Domain = [0, 2] Range = [0, 2] (Allow (0, 2) and (0, 2])

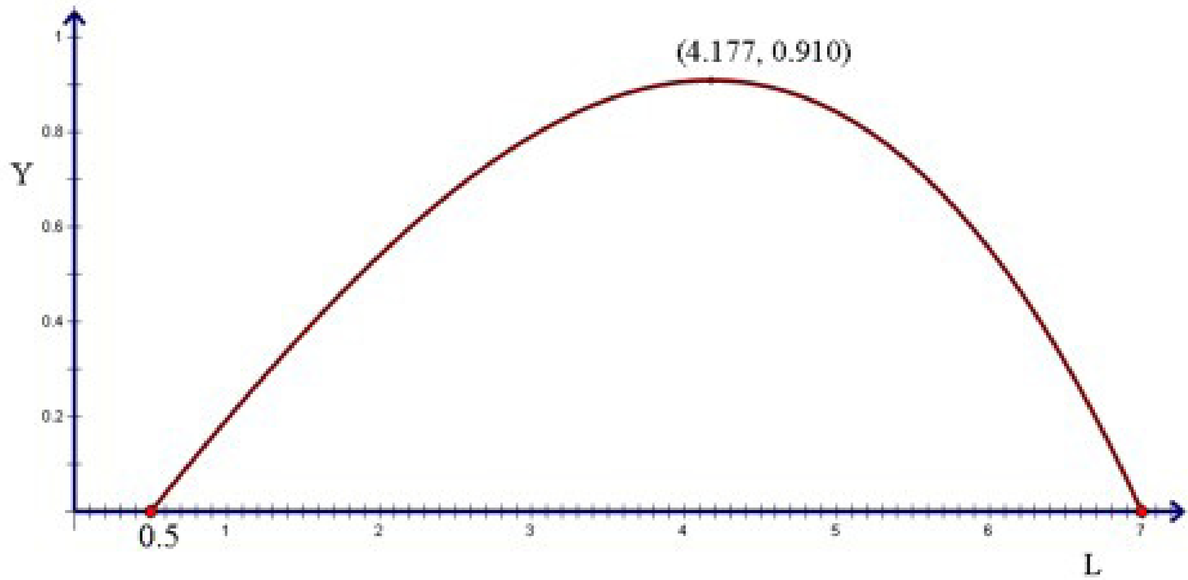
Solution Pathway

i) $2\ km^2 = 1000 \times 2000 = 2\ 000\ 000\ m^2$

$$\frac{2000000}{\frac{20}{3}} = 300000\ plants$$

(2 marks)

j)



Shape – 1, endpoints – 2, turning point – 1

k) $$\frac{d}{dL}\left(-0.0072L^3 - 0.0036L^3 + 0.4068L - 0.216\right) = -0.0216L^2 - 0.0072L + 0.4068$$

Let derivative = 0
Solve (0= –0.0216L^2=0.0072L+0.4068,L)
L=4.17627 or L= –4.5096) *(1 mark)*
Only L = 4.17627 is within the domain. *(1 mark)*

Therefore maximum yield = 0.910 kg and water needed is 4.176 L *(2 mark)*

l) Minimum yield is 0 kg *(1 mark)*
This occurs when L = 0.5 l or 7 l. *(1 mark)*
Too little or too much water causes the plants to produce no beans. *(1 mark)*

Solution Pathway

Part B: Packaging

a) This check of the students saved value is to ensure as much as possible that the student begins this section on the right track.

(1 mark)

b) $990 = L.WH$

$990 = x.pxh$

$990 = p.x^2h$

$\therefore h = \dfrac{990}{px^2}$

(3 mark)

c) $L = x$ $\quad W = p.x$ $\quad H = \dfrac{990}{px^2}$

(3 marks)

d) $$S.A. = 2\left(x.p.x + \frac{x.(990)}{px^2} + \frac{px(990)}{px^2}\right) = \frac{2(p^2x^3 + 990(p+1))}{px}$$ or equivalent

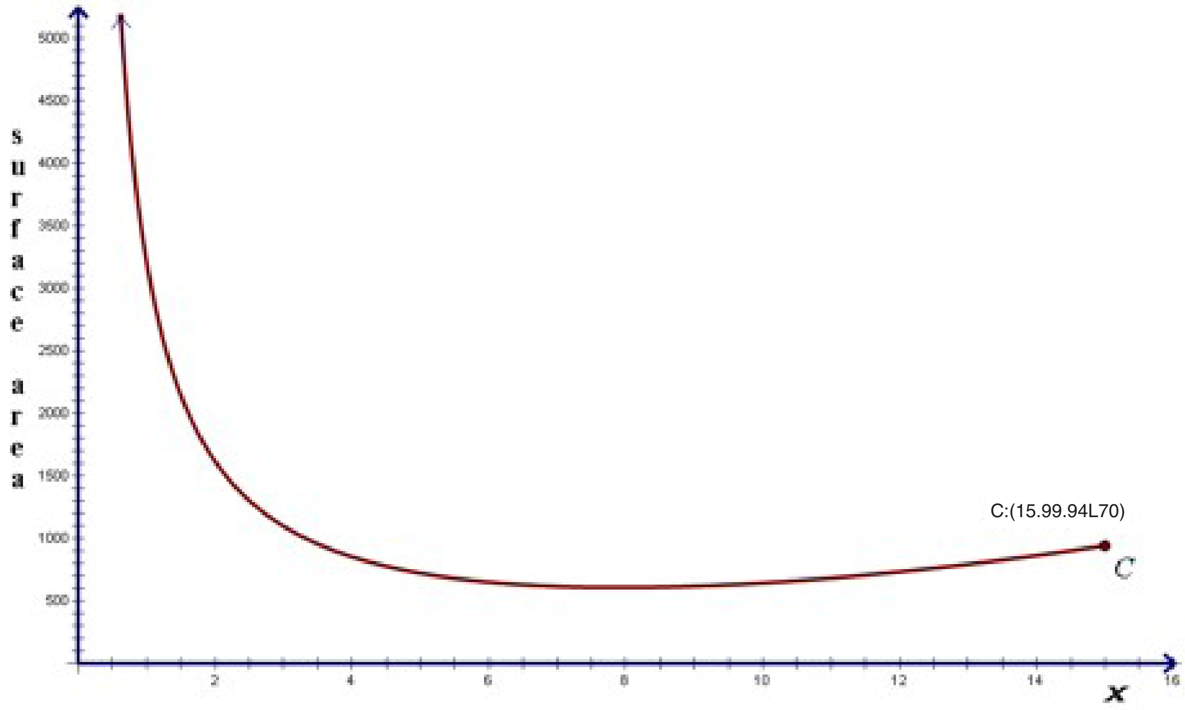
(2 marks)

e) Expand previous expression. Answers in this section will depend on how far students have gone in part d.

(3 marks)

Solution Pathway

f) Shape – 1, asymptote – 1, endpoint – 1, domain correct - 1.



g) $$\frac{dA}{dx} = \frac{d}{dx}\left(\frac{1980(1+p)}{px} + 2px^2\right)$$

$$= 4px = \frac{1980(p+1)}{px^2}$$

$$= 2(\sqrt{2}+1).x = \frac{990(\sqrt{5}+1)}{x^2}$$

(2 marks)

h) The *minimum* Surface Area of the prism is 607.495 cm^2

i) Substitute values into the calculator: Exact value = $x = \frac{\sqrt[3]{55(p+1)}3^{2/3}}{p^{2/3}}$
Approximate value = 7.910

(1 mark)

Solution Pathway

$x = 55^{1/3}.3^{2/3} \approx 7.91046$

$p.x = \dfrac{(\sqrt{5}+1).55^{1/3}.3^{2/3}}{2} \approx 12.7994$

$\dfrac{990}{p.x^2} = (\sqrt{5}-1)^3.55^{1/3}.3^{2/3}$

(3 marks)

j)
$$990 = \pi r^2 h$$
$$\frac{990}{\pi r^2} = h$$

(1 mark)

k)
$$S.A = 2\pi r\left(r + \frac{990}{\pi r^2}\right)$$
$$= 2\pi r^2 + \frac{1980}{r}$$

(1 mark)

l)
$$\frac{dA}{dr} = 4\pi r - \frac{1980}{r^2}$$

(2 marks)

Solution Pathway

m) Below is the by hand technique. Allow a CAS answer full marks as well.

$$0 = 4\pi r - \frac{1980}{r^2}$$

$$\frac{1980}{r^2} = 4\pi r$$

$$1980 = 4\pi r^3$$

$$\frac{1980}{4\pi} = r^3$$

$$r = \sqrt[3]{\frac{495}{\pi}}$$

$$r = \left(\frac{495}{\pi}\right)^{\frac{1}{3}} = \frac{\sqrt[3]{55}.3^{\frac{2}{3}}}{\sqrt[3]{\pi}}$$

(2 marks)

n)

$$h = \frac{990}{\pi r^2}$$

$$= \frac{990}{\pi\left(\frac{495}{\pi}\right)^{\frac{2}{3}}}$$

$$= \frac{990\pi^{\frac{2}{3}}}{495^{\frac{2}{3}}\pi}$$

$$= \frac{990}{495^{\frac{2}{3}}\pi^{\frac{1}{3}}} = \frac{2.\sqrt[3]{55}\,.3^{\frac{2}{3}}}{\sqrt[3]{\pi}}$$

(2 marks)

Solution Pathway

o) Minimum surface area (cylinder)= 549.884 cm^2

(1 marks)

p) Minimum surface area prism = 607.495 cm^2

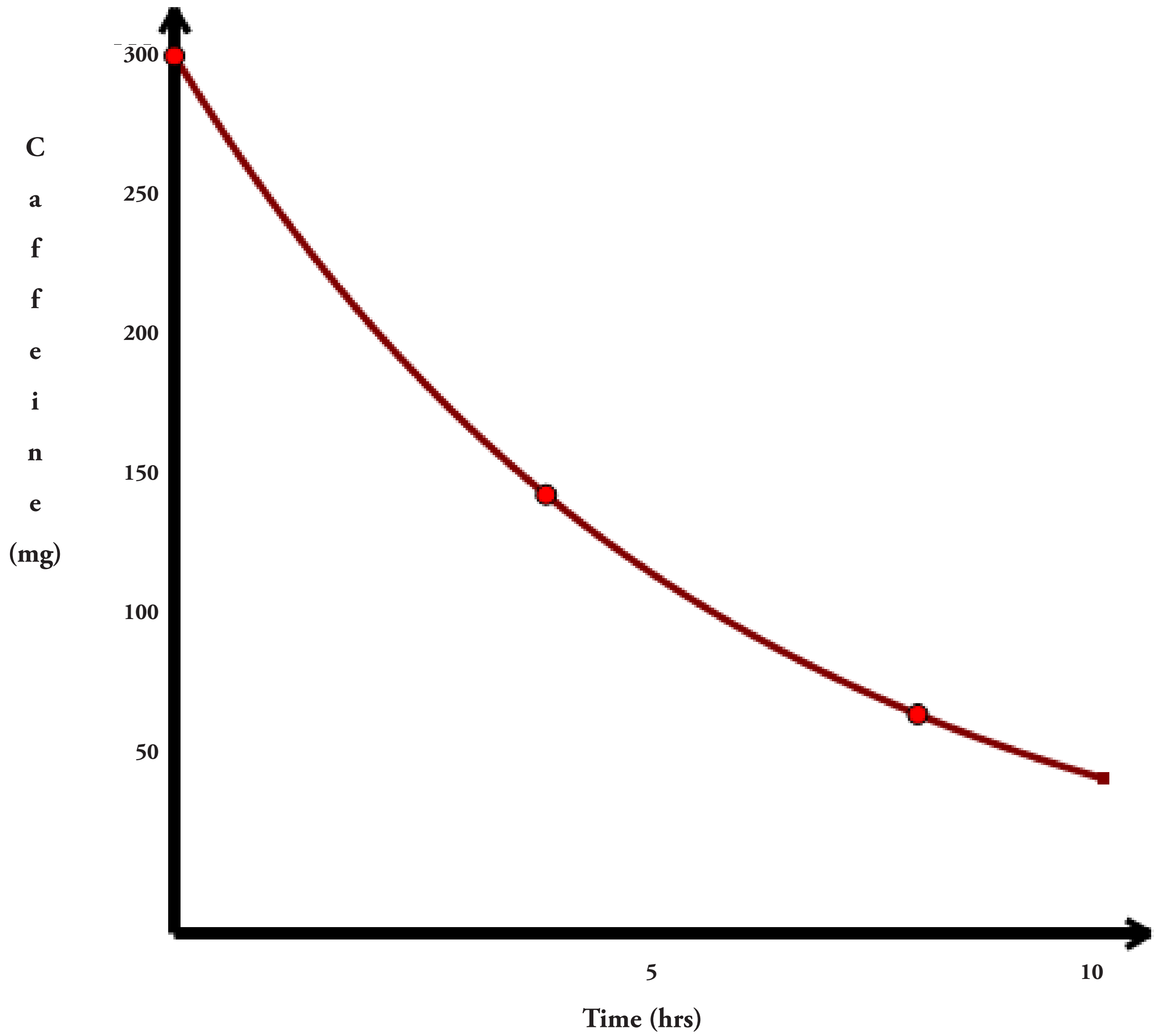
(1 mark)

The cylinder uses the least amount of surface area.

Solution Pathway

Part C: Caffeine Concentration

a) Endpoints – 1, half life points – 1, shape - 1



(4 marks)

Solution Pathway

b) 8 hrs

(1 mark)

c) When $t = 0$, $C = 300$. Therefore $A = 300$

(1 mark)

$$150 = 300e^{4n}$$

$$\frac{1}{2} = e^{4n}$$

$$\ln\left(\frac{1}{2}\right) = 4n$$

$$\therefore n = \frac{\ln\left(\frac{1}{2}\right)}{4}$$

d) When t = 4, C = 150

(3 marks)

e) Let t = 6. C = 106.066 mg

approx(solve(100=300.$e^{(t.(\ln(½)/4}$,*t*) = 6.33985 hrs)

(2 marks)

f) Using the calculator is appropriate: entry line is

(2 marks)

6 hrs 20 min

(1 mark)

(4 marks)

g) 1hr

(1 mark)

h) 5hrs

(1 mark)

i) $s(x) = 3.9e^{-0.5(x-9.75)} + 33 + (-e^{-2(x-3.17)} + 34)$

$= 3.9e^{-0.5(x-9.75)} + e^{-2(x-3.17)} + 67$

(3 marks)

Solution Pathway

j) i) This mark is just to ascertain that the students can use the calculator to differentiate a function on the calculator. It is not suggesting that the derivative is correct.

(1 mark)

ii) $2e^{(6.34 - 2x)} \ln(e) - 1.95e^{(4.875 - 0.5x)} \ln(e)$

(2 marks)

k) 300.113 mg.

(2 marks)

l) At 59.61 min = 59 min, 36.76 sec. Accept 60 min.

(1 mark)

m) Add 4 hours, so 4 hrs, 59.61 min. Accept 5 hrs.

(1 mark)

n) Let $f1(x) = = 3.9e^{-0.5(x-9.75)} + e^{-2(x-3.17)} + 67$ and $f2(x) = f1(x-4)$

Solve $(f1(x) = f2(x), x)$ and substitute the answer into $f1(x)$

(2 marks)

o) i) Solve $(f1(x) = f2(x), x)$ and substitute the answer into $f1(x)$ or equivalent

ii) (4.166, 130.485)

(1 mark)