Neap:

Trial Examination 2011

VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 1

Suggested Solutions

Question 1

a.

$$\begin{aligned}(f-g)(1) &= f(1)-g(1)\\ &= 2-6\\ &= -4\end{aligned}$$

A1

b. As $g(2)=4$, we have $g^{-1}(4)=2$. Thus $f(g^{-1}(4))=f(2)=0$. A1

Question 2

a. The simultaneous equations can be represented in matrix form as $\begin{bmatrix}2 & p\\5 & q\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\6\end{bmatrix}$. These equations will have a unique solution if $\det\begin{bmatrix}2 & p\\5 & q\end{bmatrix}\neq 0$.

$\therefore$ The relationship required is $2q-5p\neq 0$ or $p\neq \frac{2}{5}q$. A1

b. Rearranging the simultaneous equations and representing them in matrix form:

$$\begin{aligned}-mx+y&=n \;\ldots(1)\\ 3x-7y&=2 \;\ldots(2)\end{aligned} \qquad \begin{bmatrix}-m & 1\\3 & -7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}n\\2\end{bmatrix}$$

It is not a unique solution if $\det\begin{bmatrix}-m & 1\\3 & -7\end{bmatrix}=0$,

$$\begin{aligned}\therefore (7m-3)&=0\\ m&=\frac{3}{7}\end{aligned}$$

A1

And if equation (1) is identical to equation (2), an infinite set of solutions will occur.

equation (1) $y=\frac{3}{7}x+n$

equation (2) $y=\frac{3}{7}x-\frac{2}{7}$

$\therefore n=-\frac{2}{7}$ A1

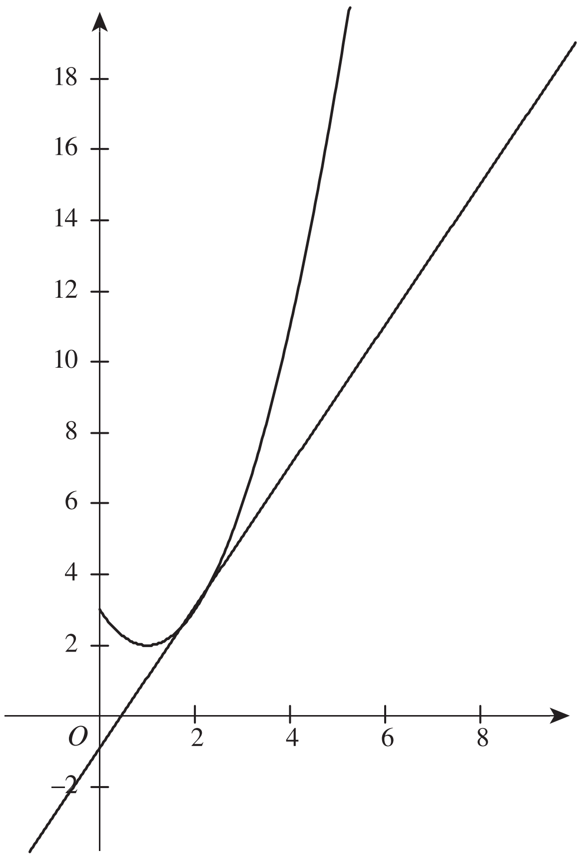
Question 3

a. $f(x)=x^2-2x+3 \Rightarrow f'(x)=2x-2$

Thus $f'(2)=4-2=2$ and $f(2)=4-4+3=3$.

Equation of the tangent at $x=2$ is given by $y-3=2(x-2)\Rightarrow y=2x-1$. A1

b.



Parabola over domain with tangent shown A1

c. **i.** The size of the error is given by the vertical distance between the graphs.

$\text{Error} = x^2 - 2x + 3 - (2x - 1) = (x - 2)^2$ M1

$x = 2.5$ gives error $= \frac{1}{4}$. A1

ii. Now $(x - 2)^2 \le 0.4$, giving $-\sqrt{0.4} \le (x - 2) \le \sqrt{0.4}$. M1

Thus the largest value of x will be $\sqrt{0.4} + 2 = \sqrt{\frac{4}{10}} + 2$

$$= \frac{2}{\sqrt{10}} + 2$$

$$= \frac{\sqrt{10}}{5} + 2$$

$$= \frac{10 + \sqrt{10}}{5}$$ A1

Question 4

a. $\Pr(X > 8) = \Pr\left(Z < \frac{8-8}{4}\right)$

$= \Pr(Z > 0)$

$= 0.5$ A1

b. $\Pr(X < 0) = \Pr\left(Z < \frac{0-8}{4}\right)$ M1

$= \Pr(Z < -2)$

$= \Pr(Z > 2)$

$\therefore k = 2$ A1

Question 5

a. Area below the curve must equal 1.

$\therefore \text{Area} = \frac{1}{2}(5)a + 5a = 1$ M1

$\frac{15}{2}a = 1$

$\therefore a = \frac{2}{15}$ A1

b. m is such that $\Pr(X < m) = \frac{1}{2}$.

Here, $\Pr(X < 5) = \frac{5}{2}a = \frac{5}{2} \times \frac{2}{15} = \frac{1}{3}$ M1

$\therefore$ require:

$\Pr(5 < X < m) = \frac{1}{6}$ M1

$\Rightarrow \frac{2}{15}(m - 5) = \frac{1}{6}$

$m - 5 = \frac{5}{4}$

$m = 6\frac{1}{4}$ or 6.25 A1

Question 6

$$E(X) = 1$$

$$\Sigma xp(x) = 1$$

$$(-1)p^2 + 0 + (1)\left(\frac{1+3p}{4}\right) + 2\left(\frac{1}{8}\right) + 3\left(\frac{p}{2}\right) = 1 \qquad \text{M1}$$

$$-p^2 + \frac{1+3p}{4} + \frac{1}{4} + \frac{3p}{2} = 1$$

$$-4p^2 + 1 + 3p + 1 + 6p = 4$$

$$-4p^2 + 9p - 2 = 0 \qquad \text{A1}$$

$$4p^2 - 9p + 2 = 0$$

$$(4p-1)(p-2) = 0$$

$$p = \frac{1}{4} \text{ or } p = 2 \qquad \text{A1}$$

$$\text{Given } \Pr(X = -1) = p^2,\ \therefore\ 0 < p < 1,\ \therefore p = \frac{1}{4} \qquad \text{A1}$$

Question 7

a. $$\frac{d}{dx}\left(x\log_e\left(\frac{x^2}{4}\right)\right) = 1 \cdot \log_e\left(\frac{x^2}{4}\right) + x \cdot \frac{\frac{x}{2}}{\frac{x^2}{4}} \qquad \text{M1}$$

$$= \log_e\left(\frac{x^2}{4}\right) + 2 \qquad \text{A1}$$

b. $$\text{Area} = \left|\int_{-2}^{-1} \log_e\left(\frac{x^2}{4}\right)dx\right|$$

$$\text{Using } \int\left(\log_e\left(\frac{x^2}{4}\right) + 2\right)dx = x\log_e\left(\frac{x^2}{4}\right) + c \qquad \text{M1}$$

$$\int \log_e\left(\frac{x^2}{4}\right)dx = x\log_e\left(\frac{x^2}{4}\right) - \int 2\,dx + c$$

$$= x\log_e\left(\frac{x^2}{4}\right) - 2x + c \qquad \text{A1}$$

$$\therefore \text{Area} = \left|\left[x\log_e\left(\frac{x^2}{4}\right) - 2x\right]_{-2}^{-1}\right|$$

$$= \left|\left(-\log_e\left(\frac{1}{4}\right) + 2\right) - (-2\log_e(1) + 4)\right|$$

$$= \left|-\log_e\left(\frac{1}{4}\right) - 2\right|$$

$$= 2 + \log_e\left(\frac{1}{4}\right)$$

$$= 2 - \log_e(4) \qquad \text{A1}$$

Question 8

At $x = b$, $y = \sqrt{a - b}$

$$\frac{dy}{dx} = \frac{1}{2}(-1)(a - x)^{-\frac{1}{2}}$$ M1

$$= \frac{-1}{2\sqrt{a - x}}$$

$\therefore$ gradient of normal = $2\sqrt{a - x}$

Gradient of normal at $x = b$ is $2\sqrt{a - b}$. A1

Gradient of line through (0, 0) and $(b, \sqrt{a - b})$ is $\frac{\sqrt{a - b}}{b}$. A1

$$\sqrt{a - b} = \frac{\sqrt{a - b}}{b}$$

$$2 = \frac{1}{b} \qquad \text{as } a \neq b$$

$$b = \frac{1}{2}$$ A1

Question 9

a. $x = \sin(\pi t^2)$ so we have $v = \frac{dx}{dt} = 2\pi t \cos(\pi t^2)$ A1

$$a = \frac{dv}{dt} = 2\pi(t \cdot (-2\pi t)\sin(\pi t^2) + 1 \cdot \cos(\pi t^2))$$

$$= -4\pi^2 t^2 \sin(\pi t^2) + 2\pi\cos(\pi t^2)$$ A1

b. $v = 0$ which occurs at: M1

$t = 0$ and $\cos(\pi t^2) = 0 \Rightarrow \pi t^2 = \frac{\pi}{2} + k\pi,\ k \in Z$

$$t^2 = \frac{1}{2} + k \Rightarrow t = \sqrt{\frac{1}{2} + k}$$ M1

Velocity is zero at $t = 0$ and $t = \sqrt{\frac{1}{2} + k},\ k \in Z^+ \cup \{0\}$. A1

Question 10

a. Condition II states that $g'(1) = \tan(45°) = 1$.

Using condition III, we have $\frac{d}{dx}(g(2x)) = g'(x) \Rightarrow 2g'(2x) = g'(x)$ by the Chain Rule. M1

So $g'(1) = 2g'(2)$, giving $1 = 2g'(2) \Rightarrow g'(2) = \frac{1}{2}$. A1

b. Using condition III, we have $\frac{d}{dx}(g(2x)) = g'(x)$. Integrating both sides with respect to x:

$g'(2x) = g'(x) \Rightarrow g(2x) = g(x) + c$ M1

Let $x = 1$, $g(2) = g(1) + c$.

Using condition I, $g(1) = 0$ and $c = g(2)$, giving $g(2x) = g(x) + g(2)$. A1

