2011 Mathematical Methods (CAS) Trial Exam 1 Solutions

Q1a $a = x - 2y$, .: $2y = x - a$; $b = z - 2x$, .: $z = 2x + b$

$-3x - 2y + 2z = 3$, .: $a + 2b = 3$ ...... (1)

$4x - 4y - z = 1$, .: $2a - b = 1$ .... (2)

Q1b (1) + 2×(2): $5a = 5$, .: $a = 1$, and from (1), $b = 1$

From $b = z - 2x$, $x = \frac{z-1}{2}$

From $a = x - 2y$, $y = \frac{x-1}{2} = \frac{z-3}{4}$

Alternatively,

$-3x - 2y + 2z = 3$ ...... (1)

$4x - 4y - z = 1$ .... (2)

(2) □ 2×(1): $10x - 5z = -5$, .: $x = \frac{z-1}{2}$

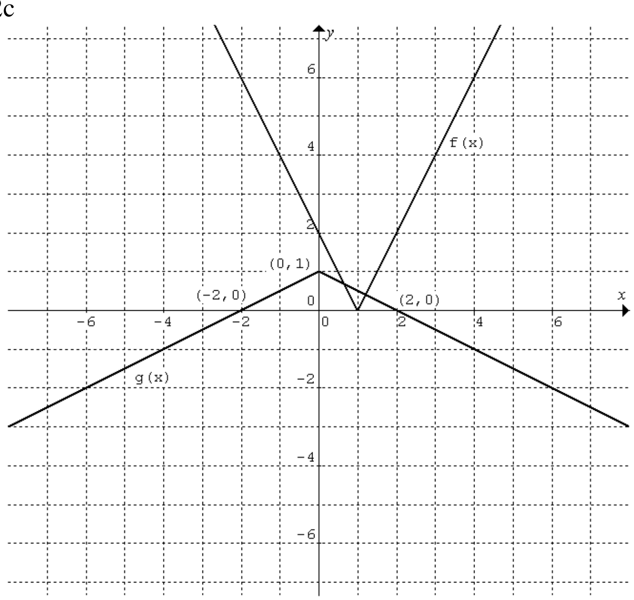
From (2): $y = x - \frac{z+1}{4} = \frac{z-1}{2} - \frac{z+1}{4} = \frac{z-3}{4}$

Q2a $f(x) = |2 - 2x|$,

$$g(x) = -\frac{1}{2}\left|2 - 2\left(1 - \frac{x}{2}\right)\right| + 1 = -\frac{1}{2}|x| + 1 = \begin{cases} \frac{1}{2}x + 1, & x < 0 \\ -\frac{1}{2}x + 1, & x \geq 0 \end{cases}$$

Q2b $g(x)$ is the image of $f(x)$ under the following transformations:
Reflection in the *x*-axis; reflection in the *y*-axis; vertical dilation by a factor of 1/2; horizontal dilation by a factor of 2; right translation by 2 units and upward translation by a unit.

Q2c



Q3a $f(x) = 1 + \frac{1}{e^x}$, $f(-x) = 1 + e^x$

.: $f(x) + f(-x) = 2 + e^x + \frac{1}{e^x}$ and

$f(x) \times f(-x) = \left(1 + \frac{1}{e^x}\right)(1 + e^x) = 2 + e^x + \frac{1}{e^x}$

.: $f(x) + f(-x) = f(x) \times f(-x)$

Q3b $f(x) = 1 + \frac{1}{e^x}$, $f'(x) = -\frac{1}{e^x}$, .: $f'(-x) = -\frac{1}{e^{-x}} = -e^x$

$f'(x) \times f'(-x) = 1$.

Q4a(i) $P(x) = x^4 + 2x^3$

$P(-2) = (-2)^4 + 2(-2)^3 = 0$, .: $x + 2$ is a factor of $P(x)$.

(ii) $P(1) = 1^4 + 2 \times 1^3 = 3$, .: the remainder is 3 when $P(x)$ is divided by $x - 1$.

Q4b(i) $P(x) = (x-1)(x+2)Q(x) + ax + b$

$P(-2) = (-2-1)(-2+2)Q(-2) - 2a + b = 0$

.: $-2a + b = 0$ ...... (1)

$P(1) = (1-1)(1+2)Q(1) + a + b = 3$

.: $a + b = 3$ .... (2)

(2) □ (1): $3a = 3$, .: $a = 1$, and from (2), $b = 2$

Q4b(ii) $P(x) = (x-1)(x+2)Q(x) + ax + b$

.: $P(x) = (x^2 + x - 2)Q(x) + ax + b$

When $P(x)$ is divided by $x^2 + x - 2$, the remainder is $ax + b$, i.e. $x + 2$.

Q4c $P(x) = (x-1)(x+2)Q(x) + x + 2 = x^4 + 2x^3$

.: $(x-1)(x+2)Q(x) + x + 2 = x^3(x+2)$

.: $(x-1)Q(x) + 1 = x^3$, $(x-1)Q(x) = x^3 - 1$

.: $(x-1)Q(x) = (x-1)(x^2 + x + 1)$

.: $Q(x) = x^2 + x + 1$

Q5 $x^a = y^b$, .: $x = y^{\frac{b}{a}}$

$y^b = \left(\frac{y}{x}\right)^c$, .: $x^c = \frac{y^c}{y^b} = y^{c-b}$, $x = y^{\frac{c-b}{c}}$

.: $y^{\frac{b}{a}} = y^{\frac{c-b}{c}}$, $\frac{b}{a} = \frac{c-b}{c}$, $bc = ac - ab$, $ac - bc = ab$

.: $(a-b)c = ab$, $c = \frac{ab}{a-b}$

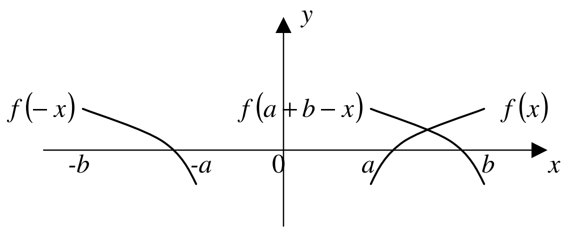
Alternative method:

Let $x^a = y^b = \left(\frac{y}{x}\right)^c = p$, $\therefore\ \frac{1}{a} = \frac{\log x}{\log p}$, $\frac{1}{b} = \frac{\log y}{\log p}$ and

$$\frac{1}{c} = \frac{\log\left(\frac{y}{x}\right)}{\log p} = \frac{\log y - \log x}{\log p} = \frac{\log y}{\log p} - \frac{\log x}{\log p} = \frac{1}{b} - \frac{1}{a}$$

$\therefore\ \frac{1}{c} = \frac{a-b}{ab}$, $c = \frac{ab}{a-b}$

Q6 $f(-x)$ is the reflection of $f(x)$ in the y-axis, $f(a+b-x)$ is the translation of $f(-x)$ to the right by $a+b$ units.
The relationship between $f(x)$, $x \in [a,b]$ and $f(a+b-x)$, $x \in [a,b]$ is shown below, where $f(x)$ is an arbitrary function.



$\therefore\ \int_a^b f(a+b-x)dx = \int_a^b f(x)dx = ab$

Alternative method for SM students:

Let $u = a+b-x$, $\frac{du}{dx} = -1$

$\therefore$

$$\int_a^b f(a+b-x)dx = -\int_a^b f(u)\frac{du}{dx}dx = -\int_b^a f(u)du = \int_a^b f(u)du = ab$$

Q7 $e^{2\sin 2x} + e^{\sin 2x + 1} - e^{\sin 2x} - e = 0, x \in [0,\pi]$

$\left(e^{\sin 2x}\right)^2 + e.e^{\sin 2x} - e^{\sin 2x} - e = 0$, $e^{\sin 2x}\left(e^{\sin 2x} + e\right) - \left(e^{\sin 2x} + e\right) = 0$

$\therefore\ \left(e^{\sin 2x} - 1\right)\left(e^{\sin 2x} + e\right) = 0$

Since $e^{\sin 2x} + e \neq 0$, $\therefore\ e^{\sin 2x} - 1 = 0$, $\therefore\ \sin 2x = 0$

$\therefore\ x = 0$, $\frac{\pi}{2}$ or π

Q8a $f(x) = (\cos 2x)^{-1}$, $x \in \left[0, \frac{\pi}{4}\right)$

$$f'(x) = -(\cos 2x)^{-2}(-2\sin 2x) = \frac{2\sin 2x}{\cos 2x \cos 2x} = 2\sec 2x \tan 2x$$

Q8b $\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x dx = \int_0^{\frac{\pi}{8}} \frac{1}{2} f'(x)dx$

$$= \frac{1}{2}\left[\frac{1}{\cos 2x}\right]_0^{\frac{\pi}{8}} = \frac{1}{2}\left(\frac{1}{\cos\frac{\pi}{4}} - \frac{1}{\cos 0}\right)$$

$$= \frac{\sqrt{2}-1}{2}$$

Q9a $p^2 = 0.16$, $\therefore\ p = 0.4$ and $q = 0.6$

$\therefore\ \Pr(X=0) = q^2 = 0.36$ and $\Pr(X=1) = pq + qp = 0.48$

X	0	1	2
$\Pr(X = x)$	0.36	0.48	0.16

Q9b $E(X) = np = 2 \times 0.4 = 0.8$

Alternatively, $E(X) = 0 \times 0.36 + 1 \times 0.48 + 2 \times 0.16 = 0.8$

Q9c $\Pr(X > 0 \mid X < 2) = \frac{\Pr(X=1)}{\Pr(X<2)} = \frac{0.48}{0.36+0.48} = \frac{4}{7}$

Q10a $\int_{-\infty}^{\infty} f(x)dx = \int_{\pi}^{2\pi} a\sin^2 x dx = 1$

$\therefore\ \frac{a}{2}\int_{\pi}^{2\pi}(1-\cos 2x)dx = 1$, $\frac{a}{2}\left[x - \frac{\sin 2x}{2}\right]_{\pi}^{2\pi} = 1$

$\therefore\ \frac{\pi a}{2} = 1$, $a = \frac{2}{\pi}$

Q10b $\frac{2}{\pi}\sin^2 x$ for $x \in [\pi, 2\pi]$ is symmetrical about $x = \frac{3\pi}{2}$

$\therefore\ \overline{X} = \frac{3\pi}{2}$

Q10c $\int_{\overline{X}-b}^{\overline{X}+b} f(x)dx = \frac{1}{\pi}$, $\int_{\frac{3\pi}{2}-b}^{\frac{3\pi}{2}+b} \frac{2}{\pi}\sin^2 x dx = \frac{1}{\pi}$

$\therefore\ \int_{\frac{3\pi}{2}-b}^{\frac{3\pi}{2}+b}(1-\cos 2x)dx = 1$, $\left[x - \frac{\sin 2x}{2}\right]_{\frac{3\pi}{2}-b}^{\frac{3\pi}{2}+b} = 1$

$$\therefore\ \left(\frac{3\pi}{2} + b - \frac{\sin(3\pi+2b)}{2}\right) - \left(\frac{3\pi}{2} - b - \frac{\sin(3\pi-2b)}{2}\right) = 1$$

$$\therefore\ 2b - \frac{\sin(3\pi+2b)}{2} + \frac{\sin(3\pi-2b)}{2} = 1$$

$$\therefore\ 2b - \frac{-\sin 2b}{2} + \frac{\sin 2b}{2} = 1$$

$\therefore\ 2b + \sin 2b = 1$

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