INSIGHT

YEAR 12 ***Trial Exam Paper***

2011

MATHEMATICAL METHODS (CAS)

Unit 3

Written examination 1

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details

Question 1

Let $f(x) = x^2 - 3$ and $g(x) = \cos(x)$. Write down the rule for $(g(f(x))$.

1 mark

Solution

$g(f(x)) = \cos(x^2 - 3)$

Mark allocation

- 1 mark for the correct answer.

Question 2

For the function $f : R^+ \to R,\ f(x) = 2e^{3x} - 1$, find

a. the rule for the inverse function f^{-1}.

2 marks

Solution

$$x \Leftrightarrow y$$

$$x = 2e^{3y} - 1$$

$$\frac{x+1}{2} = e^{3y}$$

$$y = \frac{1}{3}\log_e(\frac{x+1}{2})$$

$$f^{-1}(x) = \frac{1}{3}\log_e(\frac{x+1}{2})$$

Mark allocation

- 1 mark for swapping *x* and *y*.
- 1 mark for the correct answer.

b. the domain of the inverse function f^{-1}.

1 mark

Solution

Domain of the inverse equals the range of the original.

So if range of $f(x)$ is $(+1, \infty)$, then the domain of the inverse is $(+1, \infty)$.

Mark allocation

- 1 mark for the correct answer.

[Total: 2 + 1 = 3 marks]

Question 3

a. Let $f(x)=e^{\sin(2x)}$. Find $f'(x)$.

1 mark

Solution

Using the chain rule gives:

$f'(x)=2\cos(2x)e^{\sin(2x)}$

Mark allocation

- 1 mark for the correct answer.

b. Let $y=x^2\tan(x)$. Evaluate $\dfrac{dy}{dx}$ when $x=\dfrac{\pi}{4}$.

2 marks

Solution

Product rule gives $\dfrac{dy}{dx}=2x\tan(x)+x^2\sec^2(x)$.

Into which we substitute $x=\dfrac{\pi}{4}$, giving $\dfrac{dy}{dx}=\dfrac{2\pi}{4}\times 1+\dfrac{\pi^2}{16}\times 2 = \dfrac{\pi}{2}+\dfrac{\pi^2}{8}$.

Mark allocation

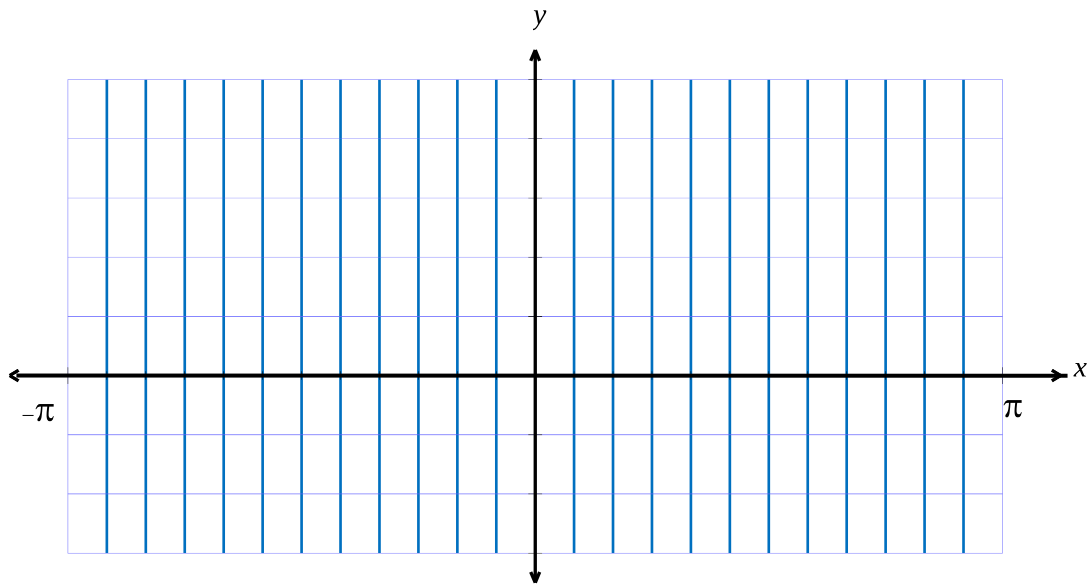
- 1 mark for using product rule correctly.
- 1 mark for the correct answer.

[Total: 1 + 2 = 3 marks]

Question 4

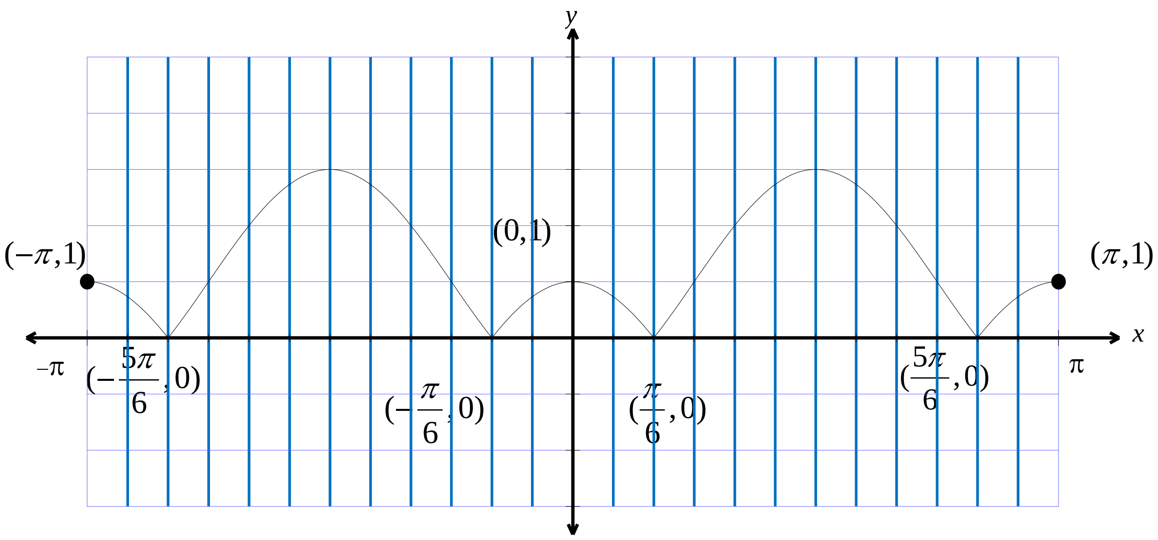
For the function $f:[-\pi,\pi]\to R,\ f(x)=|2\cos(2x)-1|$,

a. Sketch the graph of the function f on the set of axes below. Label axes intercepts and endpoints with their coordinates. y



3 marks

Solution



Mark allocation

- 1 mark for correct shape.
- 1 mark for intercepts labelled correctly.
- 1 mark for endpoints labelled correctly.

Question 4 – continued

b. State the equation of the tangent to the curve at $x = \frac{3\pi}{4}$.

3 marks

Solution

At $x = \frac{3\pi}{4}$, $f\left(\frac{3\pi}{4}\right) = 1$ and the graph behaves like $y = -(2\cos(2x) - 1) = -2\cos(2x) + 1$.

So $\frac{dy}{dx} = 4\sin(2x)$, and when $x = \frac{3\pi}{4}$, $\frac{dy}{dx} = 4\sin\left(2 \times \frac{3\pi}{4}\right) = 4\sin\left(\frac{3\pi}{2}\right) = -4$.

This gives $y - y_1 = m(x - x_1)$ as $y - 1 = -4\left(x - \frac{3\pi}{4}\right)$.

So $y = -4x + 3\pi + 1$ is the tangent.

Mark allocation

- 1 mark for finding $\frac{dy}{dx} = -4$.
- 1 mark for finding correct equation of the tangent.
- 1 mark for correct working.

[Total: 3 + 3 = 6 marks]

TURN OVER

Question 5

The weights of the adult males of a species of Alaskan huskies are normally distributed, with a mean of 72 kg and a standard deviation of 3 kg. **Use the result that** $\Pr(Z < 1) = 0.84$, **correct to two decimal places**, to find

a. the probability that a particular Alaskan husky weighs more than 75 kg.

1 mark

Solution

Let X = weight of adult male Alaskan husky.

$$\Pr(X > 75) = \Pr(Z > \frac{75-72}{3}) = \Pr(Z > 1)$$

$$\begin{aligned}\Pr(Z > 1) &= 1 - \Pr(Z < 1)\\ &= 1 - 0.84 = 0.16\end{aligned}$$

Mark allocation

- 1 mark for the correct answer.

b. the probability that an Alaskan husky weighs less than 69 kg if it is known that it weighs less than 72 kg.

2 marks

Solution

$$\begin{aligned}\Pr(X < 69 \mid X < 72) = \Pr(Z < -1 \mid Z < 0) &= \frac{\Pr(Z < -1 \cap Z < 0)}{\Pr(Z < 0)}\\ &= \frac{\Pr(Z < -1)}{\Pr(Z < 0)}\\ &= \frac{0.16}{0.5}\\ &= 0.32\end{aligned}$$

Mark allocation

- 1 mark for recognising conditional event and writing a correct conditional statement.
- 1 mark for the correct answer.

Question 5 – continued

c. Five Alaskan huskies are used to pull a sled through the snow. Find the probability that exactly three of them weigh more than 72 kg.

2 marks

Solution

Let Y = Number of Alaskan huskies pulling sled through the snow that weigh > 72 kg.

$Y \sim Bi(n=5,\ p=0.5)$

So $\Pr(Y=3) = {}^5C_3\,(0.5)^3(0.5)^2$

$$= 10 \times \frac{1}{2}^5 = \frac{10}{32} = \frac{5}{16}$$

Mark allocation

- 1 mark for identifying binomial and stating parameters.
- 1 mark for the correct answer.

[Total: 1 + 2 + 2 = 5 marks]

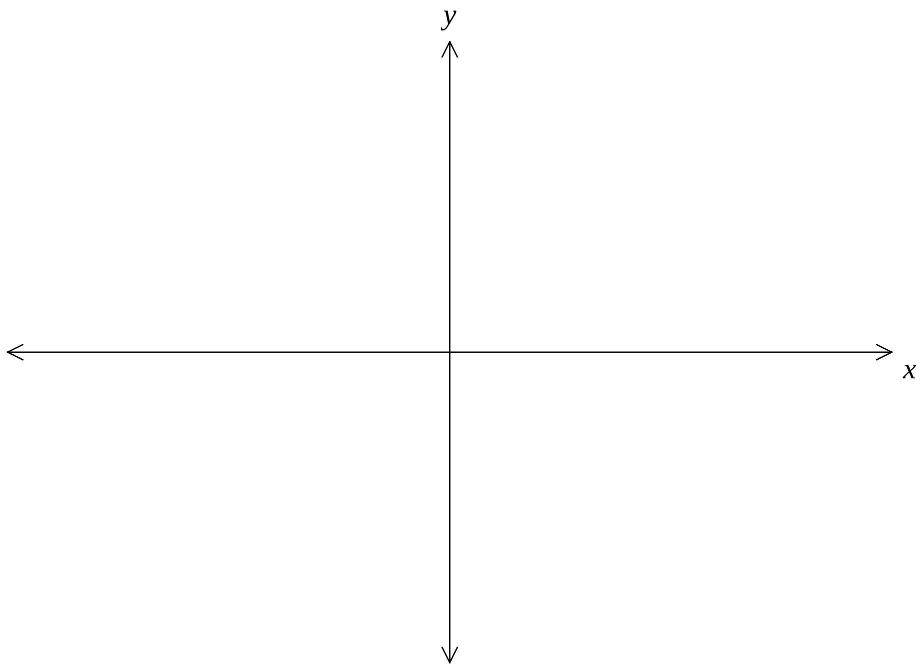
TURN OVER

Question 6

The probability density function of a continuous random variable X is given by

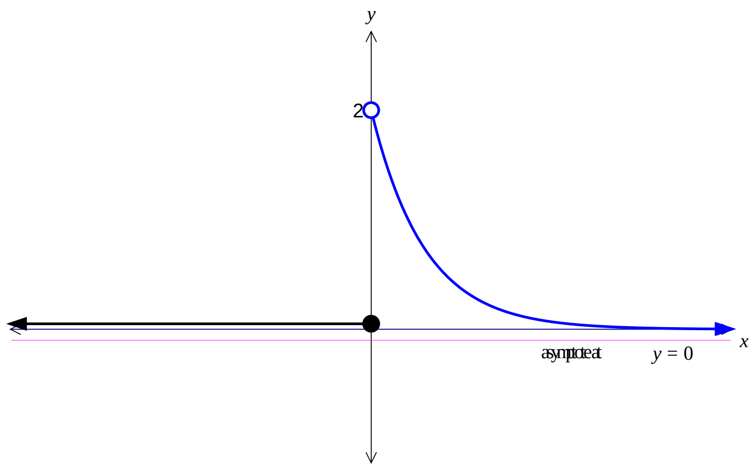
$$f(x)=\begin{cases}2e^{-2x}, & x>0\\ 0, & \text{otherwise}\end{cases}$$

a. Sketch the graph of f.



1 mark

Solution



Mark allocation

- 1 mark for the correct answer.

Question 6 – continued

b. Find $\Pr(X < 3)$.

2 marks

Solution

$$\Pr(X < 3) = \int_0^3 2e^{-2x} dx$$
$$= [-e^{-2x}]_0^3$$
$$= (-e^{-6}) - (-e^0)$$
$$= -e^{-6} + 1 = 1 - \frac{1}{e^6}$$

Mark allocation

- 1 mark for setting up integral.
- 1 mark for the correct answer.

c. If $\Pr(X \geq a) = \frac{1}{e^2}$, find the value of a.

2 marks

Solution

If $\Pr(X \geq a) = \frac{1}{e^2}$, then $\Pr(X < a) = 1 - \frac{1}{e^2}$.

Using the probability function gives:

$$\Pr(X < a) = \int_0^a 2e^{-2x}\, dx$$
$$= [-e^{-2x}]_0^a$$
$$= (-e^{-2a}) - (-e^0)$$
$$= 1 - \frac{1}{e^{2a}}$$

So $a = 1$.

Mark allocation

- 1 mark for setting up integral involving *a*.
- 1 mark for the correct answer.

[Total: 1 + 2 + 2 = 5 marks]

TURN OVER

Question 7

a. Find the general solution to the equation $\sin(x) = \sqrt{3}\cos(x)$.

2 marks

Solution

$$\sin(x) = \sqrt{3}\cos(x)$$
$$\frac{\sin(x)}{\cos(x)} = \sqrt{3}$$
$$\tan(x) = \sqrt{3}$$
$$x = \frac{\pi}{3} + n\pi, \text{ for } n \in Z.$$

Mark allocation

- 1 mark for identifying basic angle of $\frac{\pi}{6}$.
- 1 mark for the correct answer.

b. Find the average value of the function $y = \sin(2x)$ over the interval $\left[0, \frac{\pi}{8}\right]$.

3 marks

Solution

$$\text{Average value of a function} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\frac{\pi}{8}-0}\int_0^{\frac{\pi}{8}} \sin(2x)\,dx$$
$$= -\frac{8}{\pi}\left[\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{8}}$$
$$= -\frac{4}{\pi}\left(\cos\frac{\pi}{4} - \cos 0\right)$$
$$= -\frac{4}{\pi}\left(\frac{1}{\sqrt{2}} - 1\right)$$

Mark allocation

- 1 mark for setting up correct equation for average value.
- 1 mark for antidifferentiating $\sin(2x)$ correctly.
- 1 mark for the correct answer.

[Total: 2 + 3 = 5 marks]

Question 8

Suppose that the probability of snow at a particular resort is dependent on whether or not it has snowed on the previous day. If it has snowed the previous day, then the probability of snow is 0.7. If it has not snowed the previous day, then the probability of snow is 0.1.

If it has snowed on a Thursday

a. What is the probability that it doesn't snow again until Sunday?

2 marks

Solution

We want NNS, so $S \xrightarrow{0.3} N \xrightarrow{0.9} N \xrightarrow{0.1} S$; i.e. $0.3 \times 0.9 \times 0.1 = 0.027$

Mark allocation

- 1 mark for multiplication of three values or valid tree diagram.
- 1 mark for the correct answer.

b. What is the probability that it will snow in the long term?

2 marks

Solution

Consider the transition matrix $\begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}$ and look at the diagonal $\begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}$.

$\Pr(snow) = \dfrac{0.1}{0.1 + 0.3} = \dfrac{1}{4}$

Mark allocation

- 1 mark for transition matrix.
- 1 mark for the correct answer.

[Total: 2 + 2 = 4 marks]

TURN OVER

Question 9

A **normal** to the curve $y = e^{x+1} - 1$ has the equation $y = -\frac{x}{e} + a$, where a is a real constant.

Find the value of a.

4 marks

Solution

Gradient of normal is $\frac{-1}{e}$, therefore gradient of tangent is e.

$\frac{dy}{dx} = e^{x+1}$

Let $\frac{dy}{dx} = m_{\text{tangent}}$, so $e = e^{x+1} \Rightarrow x = 0$.

When $x = 0$, $y = e - 1$ and $y = a$.

So $e - 1 = a$

$a = e - 1$

Mark allocation

- 1 mark for stating gradient of normal.
- 1 mark for finding gradient of tangent.
- 1 mark for equating $\frac{dy}{dx} = m_{\text{tangent}}$.
- 1 mark for the correct answer.

Question 10

For the function $f(x) = \frac{x+1}{x-1}$, show that $f(f(x)) = x$ for $x \in R \backslash \{1\}$.

2 marks

Solution

$$f(f(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{\frac{x+1}{x-1} + \frac{x-1}{x-1}}{\frac{x+1}{x-1} - \frac{x-1}{x-1}} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = \frac{2x}{x-1} \times \frac{x-1}{2} = x$$

Mark allocation

- 1 mark for forming $f(f(x))$.
- 1 mark for the correct answer.

Question 11

Find the values of m such that the system of linear simultaneous equations

$$mx + 12y = 24$$
$$3x + my = m$$

has a unique solution.

2 marks

Solution

Unique solution means that the two lines intersect, so must have different gradients.

Line 1: $12y = -mx + 24$, so the gradient is $\frac{-m}{12}$.

Line 2: $my = -3x + m$, so the gradient is $\frac{-3}{m}$.

To have a unique solution:

$$\frac{-m}{12} \neq \frac{-3}{m}$$
$$m^2 \neq 36$$
$$m \neq \pm 6$$

Mark allocation

- 1 mark for determining gradients of both lines.
- 1 mark for the correct answer.

END OF SOLUTIONS