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Mathematical Methods(CAS)

2010

Trial Examination 2

SECTION 1 Multiple-choice questions

Instructions for Section 1

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The domain and range of $\log\frac{a}{\sqrt{a-x^2}}$, $a>0$, are respectively

A. $\left(-\infty,-\sqrt{a}\right)\cup\left(\sqrt{a},\infty\right)$ and $\left(\frac{1}{2}\log a,\infty\right)$
B. $\left[-\sqrt{a},\sqrt{a}\right]$ and $\left(\frac{1}{2}\log a,\infty\right)$
C. $\left(-\infty,-a\right]\cup\left[a,\infty\right)$ and $\left(-\infty,\log a\right]$
D. $\left[-a,a\right]$ and $\left(\log a,\infty\right)$
E. $\left(-\sqrt{a},\sqrt{a}\right)$ and $\left[\log\sqrt{a},\infty\right)$

Question 2

The graphs of $y=b-|x-a|$ and $y=|x+a|$ for $b>2a>0$ intersect at (x_1,y_1) and (x_2,y_2). Which one of the following statements is true?

A. $x_1+x_2=a$ and $y_1+y_2=b$
B. $x_1+x_2=-a$ and $y_1+y_2=-b$
C. $x_1+x_2=0$ and $y_1+y_2=b$
D. $x_1+x_2=b$ and $y_1+y_2=a$
E. $x_1+x_2=0$ and $y_1+y_2=\frac{b}{2}$

Question 3

If $4x-4y-3z=0$ and $3x+3y+2z=0$, the x and y values that can ***never*** satisfy both equations simultaneously are

A. $x=0.2$ and $y=-3.4$
B. $x=-1$ and $y=17$
C. $x=\frac{4}{5}$ and $y=-\frac{68}{5}$
D. $x=-2.1$ and $y=35.5$
E. $x=0.05$ and $y=-0.85$

Question 4

A transformation $T: R^2 \to R^2$ that maps the curve $y = f(x)$ onto the curve $y = 2 - f\left(-\frac{x+1}{3}\right)$ is given by

A. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

B. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

C. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

D. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

E. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Question 5

All parabolas of the form $y = ax^2$ can be changed to $y = x^2$ under

A. a dilation from the y-axis by a factor of a

B. a dilation from the y-axis by a factor of $\frac{1}{a}$

C. a dilation from the x-axis by a factor of $\frac{1}{a}$

D. a dilation from the x-axis and a dilation from the y-axis by the same factor a

E. a dilation from the x-axis and a dilation from the y-axis by the same factor $\frac{1}{a}$

Question 6

$\sin(2x)$ is equivalent to

A. $(1-\sin x+\cos x)(1+\sin x+\cos x)$

B. $(1-\sin x+\cos x)(1+\sin x-\cos x)$

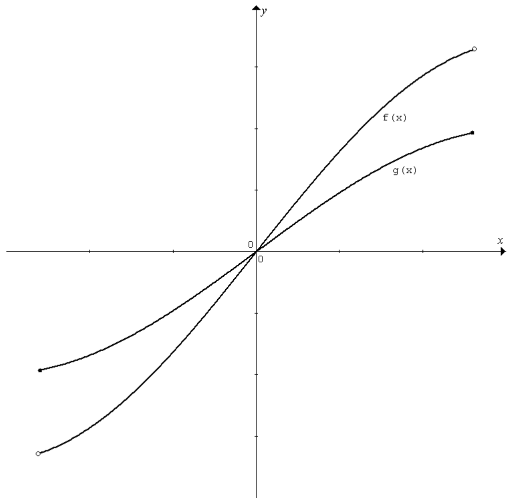
C. $(\sin x+\cos x)(\sin x-\cos x)$

D. $(\cos x-\sin x)(\sin x+\cos x)$

E. $(\sin x+\cos x)^2$

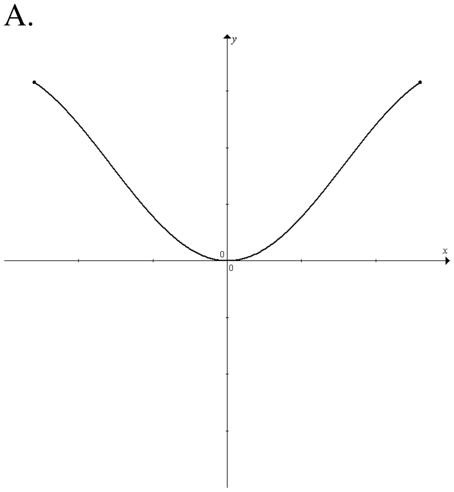
Question 7

The graphs of $f(x)$ and $g(x)$ are shown below.



The graph of $f(x) \times g(x)$ is closest to

A.



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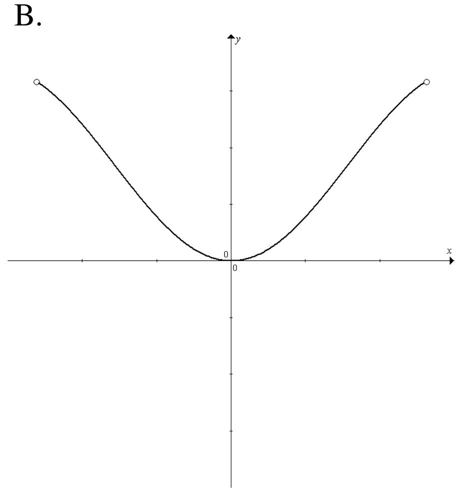
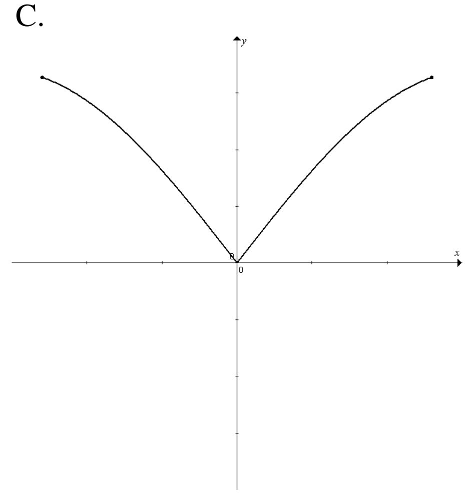
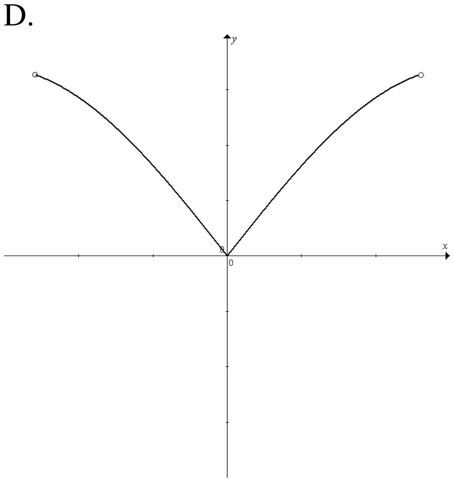


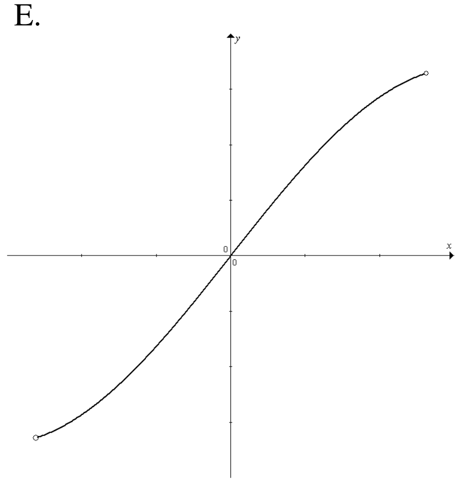
C.



D.



E.



Question 8

Which one of the following functions does **NOT** have the property $f(f(x)) = x$?

A. $f(x) = \frac{1}{x}$

B. $f(x) = -x$

C. $f(x) = x$

D. $f(x) = 1$

E. $f(x) = -\frac{1}{x}$

Question 9

If $e^{x-1} - e^{2x} = \frac{1}{(2e)^2}$, then

A. $x = -(\log_e 2 + 1)$

B. $x = \log_e 2 - 1$

C. $x = \log_e 2 + 1$

D. $x = \log_e 2 + e$

E. $x = -(\log_e 2 + e)$

Question 10

The general solution to the equation $\sqrt{3}\sin(5x) + \cos(5x) = 0$ is

A. $x = \frac{2\pi}{15} + \frac{n\pi}{5}, \ n \in Z$

B. $x = \frac{n\pi}{5} - \frac{2\pi}{15}, \ n \in Z$

C. $x = \frac{n\pi}{5} \pm \frac{2\pi}{15}, \ n \in Z$

D. $x = \frac{\pi}{6} + \frac{n\pi}{5}, \ n \in Z$

E. $x = \frac{n\pi}{5} - \frac{\pi}{6}, \ n \in Z$

Question 11

The tangent at the point $(-2,4)$ on the curve $y=f(x)$ has equation $y=\frac{1}{2}x+5$.

The tangent at the point $(-2,1)$ on the curve $y=-f(x)+5$ has equation

A. $y=-\frac{1}{2}x$

B. $y=-2x-3$

C. $y=2x+5$

D. $y=-\frac{1}{2}x+3$

E. $y=-2x+1$

Question 12

The inverse of the function $f:[-3,0)\to R$, $f(x)=-3\sqrt{1-\frac{x^2}{9}}$ is

A. $f^{-1}:(-3,0]\to R$, $f^{-1}(x)=-\sqrt{9-x^2}$

B. $f^{-1}:[-3,0)\to R$, $f^{-1}(x)=-3\sqrt{1-\frac{x^2}{9}}$

C. $f^{-1}:[-3,0)\to R$, $f^{-1}(x)=3\sqrt{1-\frac{x^2}{9}}$

D. $f^{-1}:(-3,0]\to R$, $f^{-1}(x)=\sqrt{9-x^2}$

E. $f^{-1}:(0,3]\to R$, $f^{-1}(x)=\sqrt{9-x^2}$

Question 13

Given $g(x)=(1+f(x))\log_e(1+f(x))$ and $f(1)=e-1$, $g'(1)$ is equal to

A. $f'(1)$

B. $2f'(1)$

C. $1+f'(1)$

D. $e+f'(1)$

E. $e-1+f'(1)$

Question 14

The graphs of $y=\frac{1}{x}$ and $y=ax^2-1$ have exactly one intersection when

A. $a>\frac{3}{25}$

B. $a>\frac{3}{26}$

C. $a>\frac{6}{47}$

D. $a>\frac{7}{48}$

E. $a>\frac{4}{27}$

Question 15

The function $f:(-\infty,0]\rightarrow R$, $f(x)=\frac{1}{2}(x+1)^5-1$ is increasing on

A. $[-\infty,0]$

B. $[-\infty,0)$

C. $(-\infty,1]$

D. $(-\infty,-1]$ and $[-1,0]$

E. R

Question 16

$g(x)$ is continuous and differentiable over the interval $(-1,1)$.

If $g(0)=g'(0)=\frac{1}{2}$, the value of $g(0.01)$ by Euler's linear approximation method is closest to

A. 1.01

B. 0.505

C. 0.483

D. 0.482

E. 0.481

Question 17

The speed of a particle moving in a straight line is given by $v(t)=10\pi\cos\left(\frac{\pi t}{2a}\right)$, $t\in[0,a]$.

The average speed of the particle over the interval $[0,a]$ is

A. 20
B. $20a$
C. 20π
D. $\frac{10\pi}{a}$
E. $\frac{10a}{\pi}$

Question 18

At time t minutes water is drained from a large tank at a rate given by $20e^{-0.25t}$ litres per minute. The volume of water (in litres) drained from the tank in the first 4 minutes is

A. $20(e-1)$
B. $20(1-e^{-1})$
C. $80(1-e^{-1})$
D. $80e^{-1}$
E. $80e$

Question 19

18 unbiased dice are rolled. The expected number of 1 or 6 appearing uppermost is closest to

A. 3
B. 4
C. 5
D. 6
E. 7

Question 20

$f(x)=|a(x\cos(2x)+1)|$, where $x\in[0,\pi]$, is a probability density function. The value of $|a|$ is closest to

A. 0.15
B. 0.16
C. 0.25
D. 0.26
E. 0.35

Question 21

If $\Pr(A)=\frac{2}{3}$ and $\Pr(A \mid B)=\Pr(B \mid A)=\frac{1}{2}$, then $\Pr(A'\cap B')=$

A. 0
B. $\frac{1}{6}$
C. $\frac{1}{5}$
D. $\frac{1}{4}$
E. $\frac{2}{5}$

Question 22

$X \sim \mathrm{N}(3,3)$ can be transformed to $\mathrm{Z} \sim \mathrm{N}(0,1)$.

If a probability is given by $\Pr(Z<\sqrt{3})$ after the transformation and the corresponding probability before the transformation is given by $\Pr(X<x)$, then the value of x is

A. 9
B. $3(\sqrt{3}+1)$
C. 6
D. $3(\sqrt{3}-1)$
E. 2

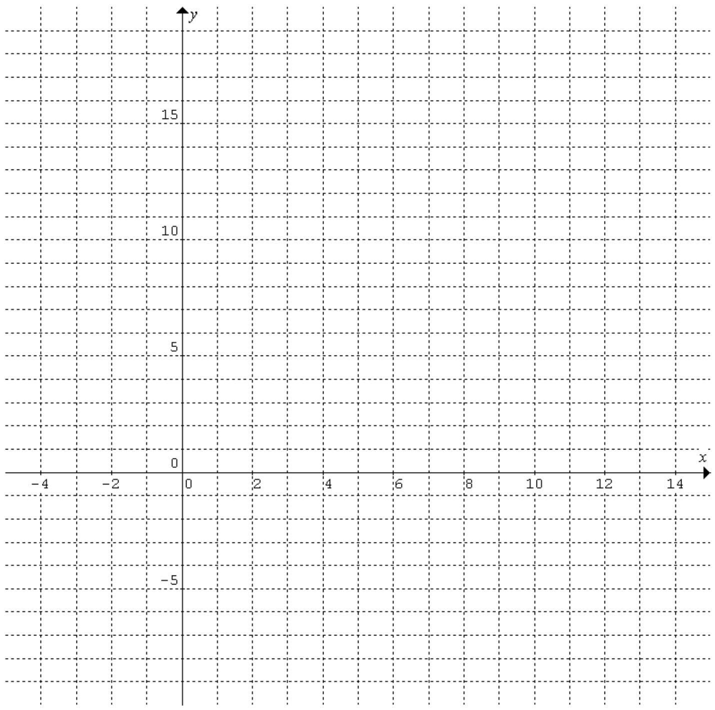
Instructions for Section 2

Answer **all** questions.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.

Question 1

Let $f:[0,6]\rightarrow R,\ f(x)=6x-x^2$.

a. Sketch the graph of function *f*. Label the maximum with its coordinates. 2 marks



b. A trapezium is to be fitted in the region bounded by $y=f(x)$ and the *x*-axis.
Let $x=a$, where $3<a<6$, be the *x*-coordinate of one of the vertex of the trapezium on the curve.

i. Find the area of the trapezium in terms of *a*. 2 marks

ii. State the area of the largest trapezium and the value of *a* when it occurs. 1 mark

c. Let $P(b,(6b-b^2))$ be a point on the curve $y=f(x)$, where $3<b<6$.

i. Show that the equation of the tangent to the curve $y=f(x)$ at P is $y=2(3-b)x+b^2$. 2 marks

ii. Find the x and y-intercepts of the tangent in terms of b. 1 mark

iii. Hence show that the area enclosed by the tangent and the axes is a minimum when $b=a$, the value found in Question 1bii.

3 marks

d. $A(6,0)$ and $B(2,8)$ are two points on the curve $y=f(x)$.

i. Show that the chord AB is parallel to the tangent at P. 2 marks

ii. Hence explain that ΔPAB has the greatest area among the triangles in the region bounded by the chord AB and the curve $y=f(x)$.

1 mark

iii. Find the area of the region bounded by the chord AB and the curve $y=f(x)$. 2 marks

Question 2

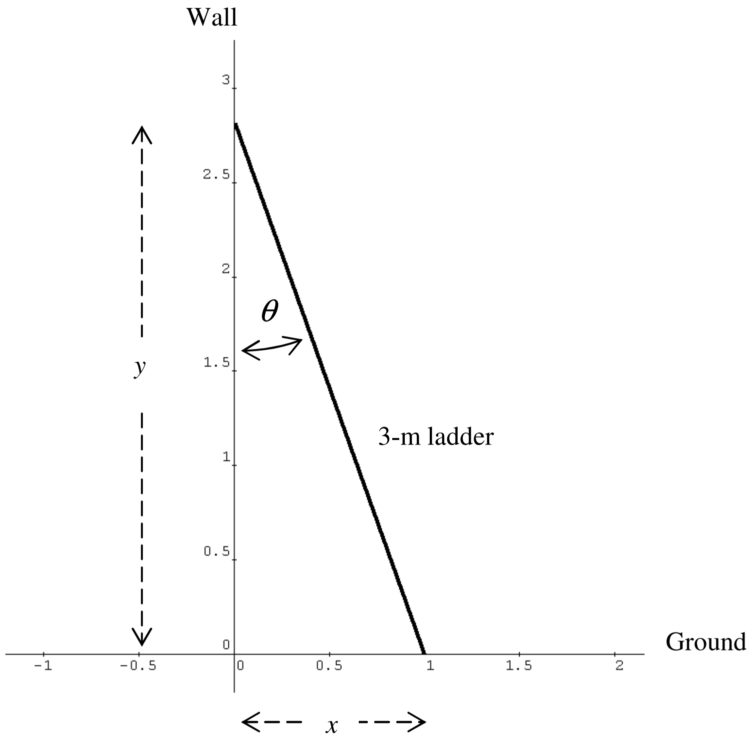
Let $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $a \in R \setminus \{0\}$.

a. Find $\cos\theta$ in terms of parameter *a* and variable *x*. 2 marks

b. Given that $\frac{dx}{d\theta} = k\sqrt{b - x^2}$, find *k* and *b* in terms of *a*. 3 marks

c. Given that $\frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}}$, find the derivative of $\sin^{-1}\left(\frac{x}{a}\right)$ for (i) $a > 0$ and (ii) $a < 0$. 2 marks

A 3-m ladder leans against a vertical wall. The bottom of the ladder slides away from the base of the wall at a speed of 0.6 m/s.



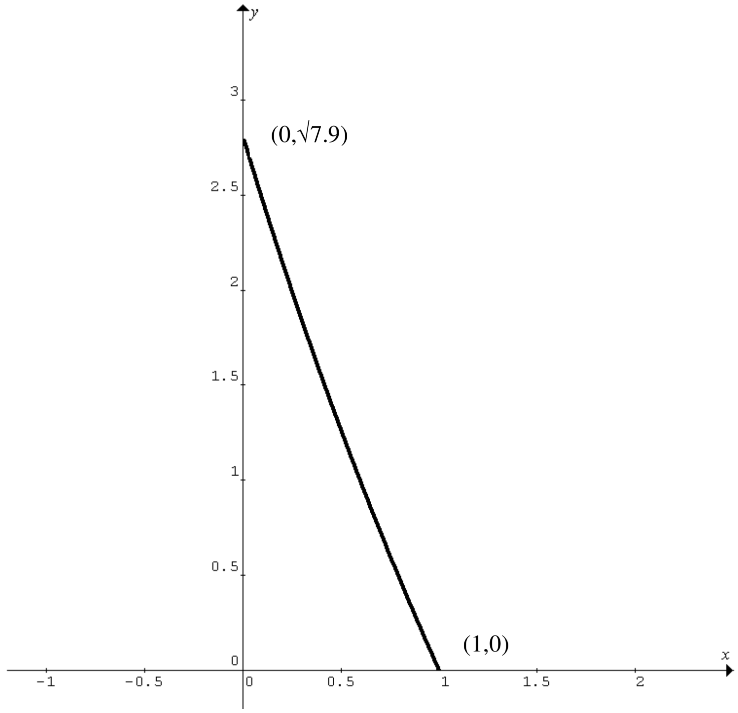
d. How fast is θ (the angle between the ladder and the wall) changing when the bottom of the ladder is 1 m from the base of the wall? Express your answer in degrees per second, correct to 1 decimal place.

3 marks

e. Hence or otherwise, find the speed of the top of the ladder when the bottom of the ladder is 1 m from the base of the wall. Express your answer in m/s, correct to 1 decimal place.

2 marks

Now the ladder is prevented from sliding. A heavy load is placed on the ladder causing it to bend. The bottom of the ladder is 1 m from the base of the wall, and the top of the ladder is $\sqrt{7.9}$ m from the ground. The shape of the ladder is given by the equation $y = A[\log_e(x+2)+c]$, where A and c are constants.
All length measures are in metres.



f. Show that $A = -\dfrac{\sqrt{7.9}}{\log_e 1.5}$ and $c = -\log_e 3$. 2 marks

g. Determine the average distance of the ladder from the ground. Express your answer in metres, correct to 2 decimal places.
2 marks

Question 3
Certain bolts manufactured by Company A have length $L = 180 + X$ mm, where X is a random variable with probability density function $f(x) = \begin{cases} k\cos^4\left(\dfrac{\pi x}{2}\right) & \textit{for } -1 \le x \le 1 \\ 0 & \textit{otherwise.} \end{cases}$

a. Use a double angle formula to show that $\cos^4\left(\dfrac{\pi x}{2}\right) = \dfrac{1}{8}\left(\cos(2\pi x) + 4\cos(\pi x) + 3\right)$. 2 marks

b. Hence or otherwise, show that $k = \dfrac{4}{3}$. 2 marks

c. Determine c, correct to 3 decimal places, so that with a probability of 95% a bolt will have any length between $180 - c$ and $180 + c$.
2 marks

d. Given that a bolt has a length between $180 - c$ and $180 + c$, find the probability, correct to 3 decimal places, that it is longer than 180.5 mm.
2 marks

e. Bolts from a very large batch made by Company A are inspected one by one.
Find the probability, correct to 3 decimal places, which a bolt outside the range from $180-c$ to $180+c$ first appears in the tenth inspection.

2 marks

f. The bolts are sold in boxes of 20. Find the probability, correct to 3 decimal places, that 95% or more of the bolts in a box are in the range from $180-c$ to $180+c$.

2 marks

Company B manufactures the same type of bolts. The length of the bolts has a normal distribution. The mean length is 180 mm and 95% of the bolts are in the range from $180-c$ to $180+c$, same as those manufactured by Company A.

g. Find the standard deviation, correct to 3 decimal places, of the length of the bolts made by Company B.

1 mark

h. From which company would you purchase the bolts if you need them as close to 180 mm as possible? Justify your answer with calculations.

2 marks

Question 4

Consider the equation $\cos x + e^{2y} + 1.5z^3 = 0$.

a. i. Express y in terms of x and z. 1 mark

ii. Find the exact value(s) of y when $x = \pi$ and $z = e^{\frac{1}{3}y}$. 2 marks

Let $\left\{\begin{array}{l} -\cos x + e^{2y} + 0.5z^3 = -2, \\ 2\cos x + 9e^{2y} + z^3 = -1, \\ 3\cos x + 8e^{2y} + 0.5z^3 = 1 \end{array}\right\}$ be a set of simultaneous equations and $0 \le x \le \pi$.

b. Show that there exists an infinite number of solutions to the set of simultaneous equations. 3 marks

c. If $x=\pi$, $y=m$ and $z=n$ are solutions to the equations, find m and n. 2 marks

d. Given that $x=\pi$, $y=0$ and $z=-2$ are the ***only*** solutions to the set of simultaneous equations

$$\begin{Bmatrix} -\cos x + e^{2y} + pz^3 = q, \\ 2\cos x + 9e^{2y} + z^3 = -1, \\ 3\cos x + 8e^{2y} + 0.5z^3 = 1 \end{Bmatrix}$$

, find the possible values of coefficients p and q. 3 marks

End of exam 2