2010 Mathematical Methods (CAS) Trial Exam 2 Solutions

SECTION 1

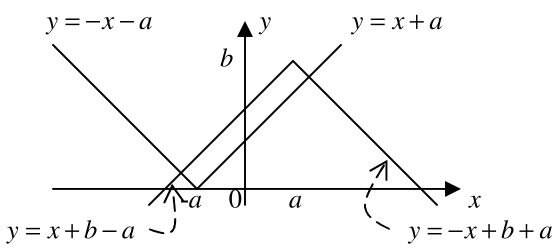
1	2	3	4	5	6	7	8	9	10	11
E	C	D	C	D	B	B	D	A	D	A

12	13	14	15	16	17	18	19	20	21	22
A	B	E	D	B	A	C	D	D	A	C

Q1 $\log\frac{a}{\sqrt{a-x^2}}$, $a>0$ is defined when $a-x^2>0$,

$\therefore -\sqrt{a}<x<\sqrt{a}$. The value of $\log\frac{a}{\sqrt{a-x^2}}$ is minimum when $x=0$, i.e. $\log\sqrt{a}$. E

Q2



Solve $y=-x-a$ and $y=x+b-a$ simultaneously,

$x_1=-\frac{b}{2}$ and $y_1=\frac{b}{2}-a$.

Solve $y=x+a$ and $y=-x+b+a$ simultaneously,

$x_2=\frac{b}{2}$ and $y_2=\frac{b}{2}+a$.

$\therefore x_1+x_2=0$ and $y_1+y_2=b$ C

Q3

$4x-4y-3z=0$ ……(1)

$3x+3y+2z=0$ ……(2)

$2\times eq(1)+3\times eq(2)$: $17x+y=0$, $\therefore \frac{y}{x}=-17$ ……(3)

$x=-2.1$ and $y=35.5$ do not satisfy $eq(3)$. D

Q4 $y=f(x)\rightarrow y=2-f\left(-\frac{x+1}{3}\right)$.

Reflection in the x-axis, reflection in the y-axis, dilation from the y-axis by a factor of 3, left translation by a unit, and upward translation by 2 units

$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-3 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}-1\\2\end{bmatrix}$ C

Q5 A dilation of $y=ax^2$ by a factor of a from the x-axis gives $y=a^2x^2$, and then a dilation by a factor of a from the y-axis gives $y=a^2\left(\frac{x}{a}\right)^2=x^2$. D

Q6 $\sin(2x)=2\sin x\cos x=1-(\sin x-\cos x)^2$

$=(1-(\sin x-\cos x))(1+(\sin x-\cos x))$

$=(1-\sin x+\cos x)(1+\sin x-\cos x)$ B

Q7 For example, sketch the graph of $y=(2\sin x)\times(\sin x)$. B

Q8 $f(x)=1$ is a constant function, $\therefore f(f(x))\neq x$. D

Q9 $e^{x-1}-e^{2x}=\frac{1}{(2e)^2}$, $4e^2e^{x-1}-4e^2e^{2x}=1$

$4e^{2x+2}-4e^{x+1}+1=0$, $4\left(e^{x+1}\right)^2-4\left(e^{x+1}\right)+1=0$

$\left(2\left(e^{x+1}\right)-1\right)^2=0$, $\therefore 2\left(e^{x+1}\right)-1=0$

$e^{x+1}=\frac{1}{2}$, $x+1=-\log_e 2$

$\therefore x=-\log_e 2-1=-(\log_e 2+1)$ A

Q10 $\sqrt{3}\sin(5x)+\cos(5x)=0$

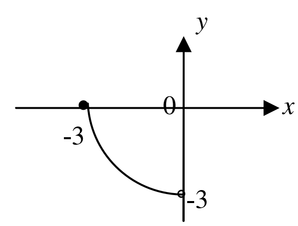
$\tan(5x)=-\frac{1}{\sqrt{3}}$, $\therefore 5x=\frac{5\pi}{6}+n\pi$

$x=\frac{\pi}{6}+\frac{n\pi}{5}$ D

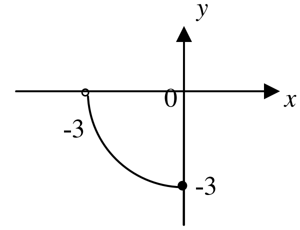
Q11 Equation of the tangent is $y=-\left(\frac{1}{2}x+5\right)+5$,

i.e. $y=-\frac{1}{2}x$. A

Q12 The graph of $f:[-3,0)\rightarrow R$, $f(x)=-3\sqrt{1-\frac{x^2}{9}}$ is:



The graph of its inverse is:



The inverse is $f^{-1}:(-3,0]\rightarrow R$, $f^{-1}(x)=-\sqrt{9-x^2}$. A

Q13 $g(x)=(1+f(x))\log_e(1+f(x))$ and $f(1)=e-1$

Apply the product rule:

$g'(x)=(1+f(x))\dfrac{f'(x)}{1+f(x)}+f'(x)\times\log_e(1+f(x))$

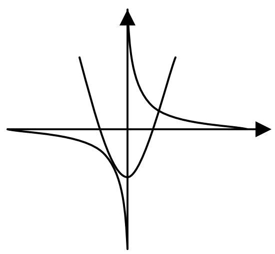
$=f'(x)+f'(x)\times\log_e(1+f(x))$

$=f'(x)(1+\log_e(1+f(x)))$

$\therefore g'(1)=f'(1)(1+\log_e(1+f(1)))$

$=f'(1)(1+\log_e e)=2f'(1)$ B

Q14 The two graphs have exactly two intersections when they touch each other at one point and cross each other at another point.



At the point where they touch each other $ax^2-1=\dfrac{1}{x}$ (same coordinates) and $2ax=-\dfrac{1}{x^2}$ (same gradient).

$\therefore ax^2=-\dfrac{1}{2x}=1+\dfrac{1}{x}$

$\therefore -\dfrac{1}{2}=x+1$, $x=-\dfrac{3}{2}$. $\therefore a=\dfrac{4}{27}$.

Hence there is only one intersection when $a>\dfrac{4}{27}$. E

Q15 By definition D

Q16 $g(a+h)\approx g(a)+hg'(a)$

$g(0.01)=g(0+0.01)\approx g(0)+0.01g'(0)=0.5+0.01\times0.5=0.505$

B

Q17 $v(t)=10\pi\cos\dfrac{\pi t}{2a}$ is ≥ 0 over the interval $[0,a]$.

Average speed $=\dfrac{\int_0^a\left(10\pi\cos\dfrac{\pi t}{2a}\right)dt}{a}=\dfrac{\left[\dfrac{10\pi\sin\dfrac{\pi t}{2a}}{\dfrac{\pi}{2a}}\right]_0^a}{a}=20$ A

Q18 $\dfrac{dV}{dt}=20e^{-0.25t}$

$V=\int_0^4 20e^{-0.25t}dt=\left[\dfrac{20e^{-0.25t}}{-0.25}\right]_0^4=80(1-e^{-1})$ C

Q19 Binomial: $n=18$, $p=\dfrac{2}{6}$, $\mu=np=6$ D

Q20 $\int_0^\pi |a(x\cos(2x)+1)|dx=1$

$|a|\int_0^\pi |x\cos(2x)+1|dx=1$, $|a|\times3.8637112=1$

$|a|\approx0.26$, $a\approx\pm0.26$ D

Q21 $\Pr(A\cap B)=\Pr(B\cap A)=\Pr(B\mid A)\Pr(A)=\dfrac{1}{2}\times\dfrac{2}{3}=\dfrac{1}{3}$

$\Pr(B)=\dfrac{\Pr(A\cap B)}{\Pr(A\mid B)}=\dfrac{\frac{1}{3}}{\frac{1}{2}}=\dfrac{2}{3}$

	A	A'	
B	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
B'	$\frac{1}{3}$	0	$\frac{1}{3}$
	$\frac{2}{3}$	$\frac{1}{3}$	1

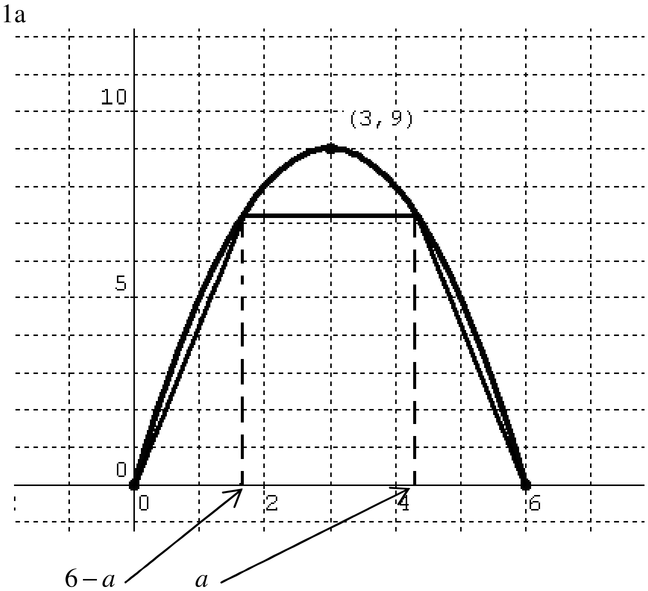
$\Pr(A'\cap B')=0$ A

Q22 $\mu_X=3$ and $\sigma_X=\sqrt{3}$

$z=\dfrac{x-\mu}{\sigma}$, $\sqrt{3}=\dfrac{x-3}{\sqrt{3}}$, $x=6$ C

SECTION 2

Q1a



Q1bi At $x=a$, $y=6a-a^2$

Area of the trapezium $A=\dfrac{1}{2}((a-(6-a)+6))(6a-a^2)=6a^2-a^3$

Q1bii $\dfrac{dA}{da}=12a-3a^2=3a(4-a)=0$, $a=4$.

When $a=4$, $A_{max}=6(4^2)-4^3=32$.

Q1ci $y = 6x - x^2$, $\frac{dy}{dx} = 6 - 2x$

At point P $x = b$ and $\frac{dy}{dx} = 6 - 2b$.

$y - y_1 = m(x - x_1)$, $y - (6b - b^2) = (6 - 2b)(x - b)$

Expand and simplify: $y = 2(3 - b)x + b^2$

Q1cii y-intercept: Let $x = 0$, $y = b^2$, $(0, b^2)$.

x-intercept: Let $y = 0$, $x = -\frac{b^2}{2(3-b)} = \frac{b^2}{2(b-3)}$, $\left(\frac{b^2}{2(b-3)}, 0\right)$.

Q1ciii Area $A = \frac{1}{2}(b^2)\left(\frac{b^2}{2(b-3)}\right) = \frac{b^4}{4(b-3)}$

$\frac{dA}{db} = \frac{1}{4} \times \frac{(b-3)(4b^3) - b^4}{(b-3)^2} = \frac{3b^2(b-4)}{4(b-3)}$, where $3 < b < 6$

Let $\frac{dA}{db} = 0$ to get $b = 4$.

$b < 4$	$b = 4$	$b > 4$
$A' < 0$	A'	$A' > 0$

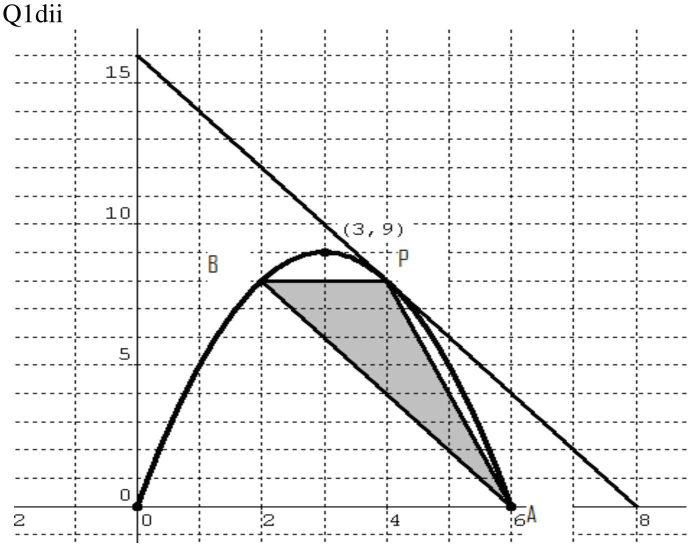
$\therefore$ A is a minimum when $b = 4$.

Q1di Gradient of chord AB $= \frac{8 - 0}{2 - 6} = -2$.

Gradient of the tangent at P $= 6 - 2b = -2$.

$\therefore$ chord AB and the tangent at P are parallel.

Q1dii



ΔPAB has the greatest area because it has the greatest altitude (distance between the chord and the tangent at P) among the triangles in the region bounded by AB and $y = f(x)$.

Q1diii Area $= \int_2^6 (6x - x^2)dx - \frac{1}{2}(6-2)(8-0)$

$= \left[3x^2 - \frac{x^3}{3}\right]_2^6 - 16 = \frac{32}{3}$

Q2a $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\therefore \cos\theta \ge 0$

$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{x}{a}\right)^2}$ or $\frac{1}{|a|}\sqrt{a^2 - x^2}$

Q2b $\frac{dx}{d\theta} = a\cos\theta$, $\therefore k\sqrt{b - x^2} = \frac{a}{|a|}\sqrt{a^2 - x^2}$

$\therefore b = a^2$ and $k = \frac{a}{|a|} = \begin{cases} -1 & a < 0 \\ 1 & a > 0 \end{cases}$

Q2c $x = a\sin\theta$, $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}} = \frac{|a|}{a} \times \frac{1}{\sqrt{a^2 - x^2}}$

(i) $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$ for $a > 0$

(ii) $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}}$ for $a < 0$

Q2d Given $\frac{dx}{dt} = 0.6$

$x = 3\sin\theta$, $\therefore \theta = \sin^{-1}\left(\frac{x}{3}\right)$, $\frac{d\theta}{dx} = \frac{1}{\sqrt{9 - x^2}}$

$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} = \frac{1}{\sqrt{9 - x^2}} \times 0.6$

At $x = 1$, $\frac{d\theta}{dt} = \frac{1}{\sqrt{9 - 1}} \times 0.6 \approx 0.212132$ radians per second or 12.2 degrees per second

Q2e $y = 3\cos\theta$, $\frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt} \approx -3\sin\theta \times 0.212132$

When $x = 1$, $\sin\theta = \frac{1}{3}$, $\frac{dy}{dt} \approx -3 \times \frac{1}{3} \times 0.212132 \approx -0.2$

The speed is 0.2 ms^{-1}.

Q2f $y = A[\log_e(x+2) + c]$

$(1,0)$, $0 = A[\log_e(1+2) + c]$, $\therefore c = -\log_e 3$

$(0, \sqrt{7.9})$, $\sqrt{7.9} = A[\log_e 2 - \log_e 3] = -A\log_e 1.5$

$\therefore A = -\frac{\sqrt{7.9}}{\log_e 1.5}$

Q2g $y = A[\log_e(x+2) + c] = -\frac{\sqrt{7.9}}{\log_e 1.5}\log_e\left(\frac{x+2}{3}\right)$

Average distance $= \frac{\int_0^1 -\frac{\sqrt{7.9}}{\log_e 1.5}\log_e\left(\frac{x+2}{3}\right)dx}{1} \approx 1.31\,\text{m}$

Q3a $\cos^4\left(\frac{\pi x}{2}\right)=\left(\cos^2\left(\frac{\pi x}{2}\right)\right)^2=\left(\frac{1}{2}(\cos(\pi x)+1)\right)^2$

$=\frac{1}{4}\left(\cos^2(\pi x)+2\cos(\pi x)+1\right)=\frac{1}{4}\left(\frac{1}{2}(\cos(2\pi x)+1)+2\cos(\pi x)+1\right)$

$=\frac{1}{8}(\cos(2\pi x)+4\cos(\pi x)+3)$

Q3b $\int_{-1}^{1} k\cos^4\left(\frac{\pi x}{2}\right)dx=\int_{-1}^{1}\frac{k}{8}(\cos(2\pi x)+4\cos(\pi x)+3)dx=1$

$\left[\frac{k}{8}\left(\frac{\sin(2\pi x)}{2\pi}+\frac{4\sin(\pi x)}{\pi}+3x\right)\right]_{-1}^{1}=1$

$\frac{6k}{8}=1$, $k=\frac{4}{3}$

Q3c $f(x)=k\cos^4\left(\frac{\pi x}{2}\right)=\frac{4}{3}\times\frac{1}{8}(\cos(2\pi x)+4\cos(\pi x)+3)$

$=\frac{1}{6}(\cos(2\pi x)+4\cos(\pi x)+3)$

$\int_{-c}^{c}\frac{1}{6}(\cos(2\pi x)+4\cos(\pi x)+3)dx=0.95$

$2\times\int_{0}^{c}\frac{1}{6}(\cos(2\pi x)+4\cos(\pi x)+3)dx=0.95$

$\frac{1}{3}\left[\frac{\sin(2\pi x)}{2\pi}+\frac{4\sin(\pi x)}{\pi}+3x\right]_{0}^{c}=0.95$

$\frac{\sin(2\pi c)}{2\pi}+\frac{4\sin(\pi c)}{\pi}+3c=2.85$

$c\approx 0.544$ by CAS/graphics calc.

Q3d $\Pr(X>0.5 \mid -0.544<X<0.544)$

$=\frac{\Pr(0.5<X<0.544)}{\Pr(-0.544<X<0.544)}=\frac{\int_{0.5}^{0.544}\frac{4}{3}\cos^4\left(\frac{\pi x}{2}\right)dx}{0.95}\approx 0.013$

Q3e $0.95^9(1-0.95)\approx 0.032$

Q3f 20 bolts picked in random from a very large batch are packed in a box. Binomial distribution: $n=20$, $p=0.95$, $X\ge 19$ (95% of 20 $=19$)

$\Pr(X\ge 19)=1-\Pr(X\le 18)\approx 1-0.264=0.736$

Q3g $2\sigma=0.544$, $\sigma=0.272$

Q3h Company B: 68% within 180 ± 0.272

Company A: $\Pr(-0.272<X<0.272)=0.645=64.5\%$

$\therefore$ Company B.

Q4ai $\cos x+e^{2y}+1.5z^3=0$, $e^{2y}=-\cos x-1.5z^3$,

$2y=\log_e\left(-\cos x-1.5z^3\right)$, $y=\frac{1}{2}\log_e\left(-\cos x-1.5z^3\right)$

Q4aii When $x=\pi$ and $z=e^{\frac{1}{3}y}$,

$\cos\pi+e^{2y}+1.5e^{y}=0$, $\left(e^{y}\right)^2+1.5e^{y}-1=0$

$\therefore 2\left(e^{y}\right)^2+3e^{y}-2=0$, $\left(2e^{y}-1\right)\left(e^{y}+2\right)=0$

Since $e^{y}+2>0$, $\therefore 2e^{y}-1=0$, $y=\log_e\frac{1}{2}=-\log_e 2$.

Q4b $\begin{Bmatrix} -\cos x+e^{2y}+0.5z^3=-2, \\ 2\cos x+9e^{2y}+z^3=-1, \\ 3\cos x+8e^{2y}+0.5z^3=1 \end{Bmatrix}$

Let $\cos x=p$, $e^{2y}=q$ and $z^3=r$.

$\therefore p\in[-1,1]$, $q\in R^+$ and $r\in R$.

$-p+q+0.5r=-2$ ……(1)

$2p+9q+r=-1$ ………(2)

$3p+8q+0.5r=1$ ……..(3)

2×eq(1) + eq(2): $11q+2r=-5$ …….(4)

3×eq(1) + eq(3): $11q+2r=-5$ …….(5)

Eq(4) − eq(5): $0=0$

$\therefore$there exists an infinite number of solutions to the simultaneous equations (1), (2) and (3). Hence there exists an infinite number of solutions to the given set of simultaneous equations.

For example, let $q=0.5$, $\therefore r=-5.25$ (from eq(4)) and $p=-0.125$ (from eq(1)).

$\therefore y=\frac{1}{2}\log_e 0.5$, $z=\sqrt[3]{-5.25}$ and $x=\cos^{-1}(-0.125)$.

Q4c Given that $x=\pi$, $y=m$ and $z=n$.

$e^{2m}+0.5n^3=-3$ ……(1)

$9e^{2m}+n^3=1$ ………..(2)

$8e^{2m}+0.5n^3=4$ …….(3)

Eq(3) − eq(1): $7e^{2m}=7$, $e^{2m}=1$, $m=0$

Eq(2) − 9×eq(1): $-3.5n^3=28$, $n^3=-8$, $n=-2$

Q4d $x=\pi$, $y=0$ and $z=-2$ are solutions to the set of simultaneous equations

$-\cos x+e^{2y}+pz^3=q$ ……….(1)

$2\cos x+9e^{2y}+z^3=-1$ ………(2)

$3\cos x+8e^{2y}+0.5z^3=1$ ……..(3)

From eq(1) $8p+q=2$

Eq(3) + eq(1): $2\cos x+9e^{2y}+(p+0.5)z^3=q+1$ ……(4), compare with eq(2).

If $x=\pi$, $y=0$ and $z=-2$ are the only solutions to the set of simultaneous equations, then $p+0.5\ne 1$ and $q+1\ne -1$.

$\therefore p\ne 0.5$ and $q\ne -2$

$\therefore$ any $p\ne 0.5$ and $q\ne -2$ satisfying $8p+q=2$ are the possible values of *p* and *q*.

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