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MATHS METHODS (CAS) 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2010

Question 1

a.

$$y = e^{3x}(x^2 - 1)$$

$$\frac{dy}{dx} = 3e^{3x}(x^2 - 1) + e^{3x} \times 2x$$ **(1 mark)** –using product rule

$$= 3x^2 e^{3x} - 3e^{3x} + 2xe^{3x}$$ **(1 mark)** – correct answer

$$= e^{3x}(3x^2 + 2x - 3) \quad \text{(optional line)}$$

b. $f(x) = \log_e(\cos(x))$

Method 1 – short way

$$f(x) = \log_e(\cos(x))$$

$$f'(x) = \frac{-\sin(x)}{\cos(x)}$$

$$= -\tan(x)$$ **(1 mark)**

$$f'(\pi) = -\tan(\pi)$$

$$= 0$$

(1 mark)

Method 2 – long way

Let $y = \log_e(\cos(x))$

$y = \log_e(u)$ where $u = \cos(x)$

$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = -\sin(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{(chain rule)}$$

$$= \frac{1}{u} \times -\sin(x)$$

$$= \frac{1}{\cos(x)} \times -\sin(x)$$

$$= -\tan(x)$$ **(1 mark)**

So $f'(\pi) = -\tan(\pi)$

$$= 0$$

(1 mark)

Question 2

a. $\int\left(\sqrt{x}+e^{2x}\right)dx$

$=\int\left(x^{\frac{1}{2}}+e^{2x}\right)dx$

$=\frac{2x^{\frac{3}{2}}}{3}+\frac{1}{2}e^{2x}+c$

(1 mark)

b. $f'(x)=\sin(3x)$

$f(x)=\int\sin(3x)dx$

$=-\frac{1}{3}\cos(3x)+c$

(1 mark)

Given $f(\pi)=\frac{4}{3}$

$\frac{4}{3}=-\frac{1}{3}\cos(3\pi)+c$

$\frac{4}{3}=-\frac{1}{3}\times-1+c$

$\frac{4}{3}=\frac{1}{3}+c$

$c=1$

So $f(x)=-\frac{1}{3}\cos(3x)+1$

(1 mark)

Question 3

$f(x)=\log_e(x),\quad x>0$

Show $f(u)-2f\left(\frac{1}{v}\right)=f(uv^2)$

$LHS=f(u)-2f\left(\frac{1}{v}\right)$

$=\log_e(u)-2\log_e\left(\frac{1}{v}\right)$

$=\log_e(u)-\log_e\left(\frac{1}{v}\right)^2$ **(1 mark)**

$=\log_e(u)-\log_e\left(\frac{1}{v^2}\right)$

$=\log_e\left(u\div\frac{1}{v^2}\right)$

$=\log_e(uv^2)$

$=f(uv^2)$

$=RHS$ as required.

(1 mark)

Question 4

a. Draw a diagram.

		BLACK					
		1	2	3	4	5	6
RED	1	x	.	.	.	.	.
	2	.	x	.	.	.	.
	3	.	.	x	.	.	.
	4	.	.	.	x	.	.
	5	.	.	.	.	x	.
	6	.	.	.	.	.	x

$\Pr(1,1)+\Pr(2,2)+\Pr(3,3)+\Pr(4,4)+\Pr(5,5)+\Pr(6,6)$

$=\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}$

$=\frac{6}{36}$

$=\frac{1}{6}$ **(1 mark)**

b.

		BLACK					
		1	2	3	4	5	6
RED	1	.	x	x	x	x	x
	2	.	.	x	x	x	x
	3	.	.	.	x	x	x
	4	.	.	.	.	x	x
	5	.	.	.	.	.	x
	6	.	.	.	.	.	.

$\Pr(\text{no. on red} < \text{no. on black})$

$=\frac{5}{36}+\frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36}$

$=\frac{15}{36}$

$=\frac{5}{12}$ **(1 mark)**

c. This represents a binomial distribution with $n = 4$.
Since odd numbers occur on both die on 9 occasions (from the diagram),

		BLACK					
		1	2	3	4	5	6
	1	x	.	x	.	x	.
R	2	.	.	.	.	.	.
E	3	x	.	x	.	x	.
D	4	.	.	.	.	.	.
	5	x	.	x	.	x	.
	6	.	.	.	.	.	.

$p = \frac{9}{36} = \frac{1}{4}$. **(1 mark)**

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \Pr(X = 0)\\ &= 1 - {}^4C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^4\\ &= 1 - \left(\frac{3}{4}\right)^4 \qquad \text{Note: } {}^4C_0 = 1 \text{ and } \left(\frac{1}{4}\right)^0 = 1\\ &= 1 - \frac{81}{256}\\ &= \frac{175}{256}\end{aligned}$$

(1 mark)

Question 5

$g: R \to R,\ g(x) = e^{x+1} - 2$

Let $y = e^{x+1} - 2$

Swap x and y for inverse

$x = e^{y+1} - 2$

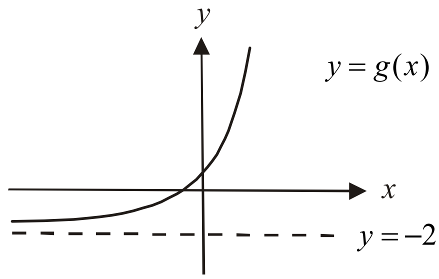
(1 mark)

Rearrange

$$\begin{aligned} x + 2 &= e^{y+1}\\ \log_e(x+2) &= y + 1\\ y &= \log_e(x+2) - 1\end{aligned}$$

So $g^{-1}(x) = \log_e(x+2) - 1$

Do a quick sketch of $y = g(x)$.



$d_g = R, \qquad r_g = (-2, \infty)$

So, $d_{g^{-1}} = (-2, \infty)$ and $r_{g^{-1}} = R$

So $g^{-1}: (-2, \infty) \to R,\ g^{-1}(x) = \log_e(x+2) - 1$

Note that to define a function you must give the rule (equation) **and** the domain.

(1 mark) – correct domain
(1 mark) – correct rule

Question 6

$$mx + y = 2 \qquad -(1)$$
$$2x + (m-1)y = m \qquad -(2)$$

For no solutions or infinite solutions the determinant of the matrix $\begin{bmatrix} m & 1 \\ 2 & m-1 \end{bmatrix}$ equals zero.

That is, $\begin{vmatrix} m & 1 \\ 2 & m-1 \end{vmatrix} = 0$

(1 mark)

$$m(m-1) - 2 = 0$$
$$m^2 - m - 2 = 0$$
$$(m-2)(m+1) = 0$$
$$m = 2 \text{ or } m = -1$$

(1 mark)

If $m = 2$,
in (1) $\quad 2x + y = 2$
in (2) $\quad 2x + y = 2$
They are the same equation hence there are an infinite number of solutions.

If $m = -1$,
in (1) $\quad -x + y = 2 \qquad -(3)$
in (2) $\quad 2x - 2y = -1 \qquad -(4)$
$(4) \div -2 \quad -x + y = \frac{1}{2} \qquad -(5)$
(3) and (5) describe parallel lines with different y-intercepts so there are no points of intersection and hence no solutions.

So for $m = -1$ there is no solution.

(1 mark)

Question 7

X	2	3	4	5	6
$\Pr(X = x)$	0.2	0.4	0.1	0.2	0.1

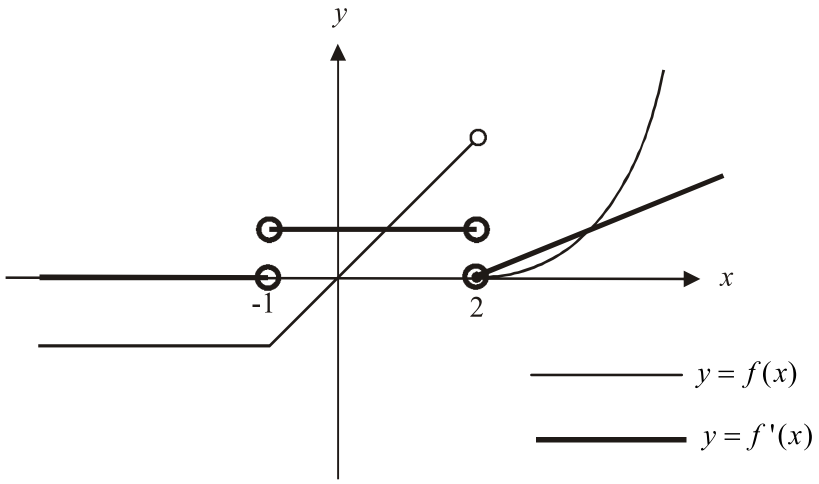
a. Median = 3

(1 mark)

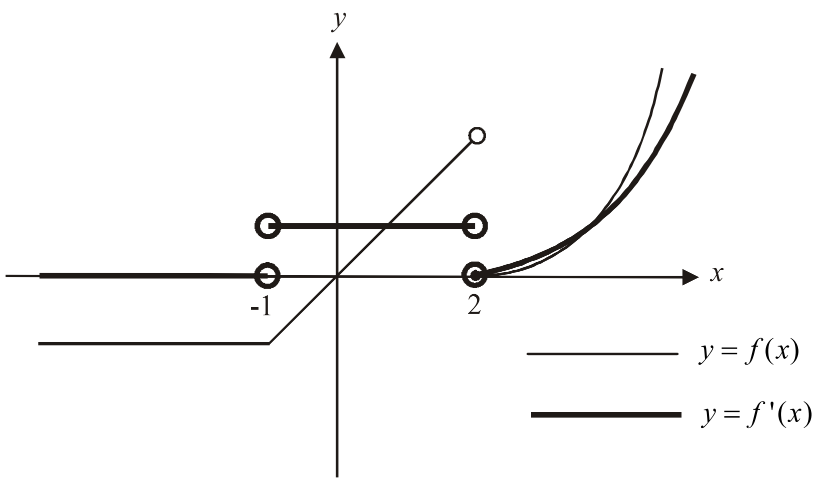
b.
$$\Pr(X \geq 3 | X < 6)$$
$$= \frac{\Pr(X \geq 3) \cap \Pr(X < 6)}{\Pr(X < 6)}$$ **(1 mark)**
$$= \frac{\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)}{1 - \Pr(X = 6)}$$
$$= \frac{0.4 + 0.1 + 0.2}{0.9}$$
$$= \frac{7}{9}$$ **(1 mark)**

Question 8

a.



OR



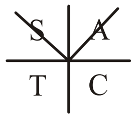
(1 mark) – correct left branch
(1 mark) – correct middle branch
(1 mark) – correct right branch
(curved or straight)

b. $d_{f'} = R \setminus \{-1, 2\}$

(1 mark)

Question 9

<u>Method 1</u> – intuitively



$$\sin(2x) = \frac{1}{\sqrt{2}}$$

If we were given a restricted domain like $x \in [0, 2\pi]$ then $2x \in [0, 4\pi]$ so

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$$

(1 mark)

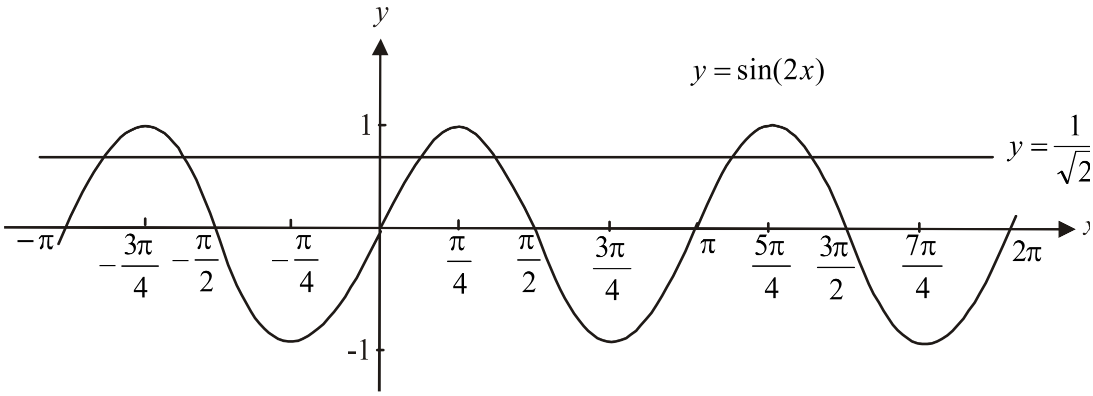
However, the domain is not restricted; we are looking for the general solution where each of the solutions found above is repeated for every clockwise and anticlockwise rotation by 2π.
We can express these values as

$$x = \frac{\pi}{8} + n\pi \quad \text{or} \quad x = \frac{3\pi}{8} + n\pi \qquad n \in Z \text{ (the set of integers)}$$

(1 mark) **(1 mark)**

<u>Method 2</u> – graphically

Sketch the graphs of $y = \sin(2x)$ and $y = \frac{1}{\sqrt{2}}$.



We know that in the first quadrant $\sin(2x) = \frac{1}{\sqrt{2}}$

$$2x = \frac{\pi}{4}$$

$$x = \frac{\pi}{8}$$

By symmetry and using the graph, the points of intersection occur at

$$x = \ldots -\frac{3\pi}{4} - \frac{\pi}{8}, -\frac{3\pi}{4} + \frac{\pi}{8}, \frac{\pi}{4} - \frac{\pi}{8}, \frac{\pi}{4} + \frac{\pi}{8}, \frac{5\pi}{4} - \frac{\pi}{8}, \frac{5\pi}{4} + \frac{\pi}{8}, \ldots$$

$$= \ldots -\frac{7\pi}{8}, -\frac{5\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \ldots$$

We can express these values as

$$x = \frac{\pi}{8} + n\pi \quad \text{or} \quad x = \frac{3\pi}{8} + n\pi \qquad n \in Z \text{ (the set of integers)}$$

(1 mark) **(1 mark)** **(1 mark)** for graph

<u>Method 3</u> – using a formula

For $\sin(x)=a$

$x=2n\pi+\sin^{-1}(a)$ or $x=(2n+1)\pi-\sin^{-1}(a)$, $n\in Z$

For $\sin(2x)=\frac{1}{\sqrt{2}}$

$2x=2n\pi+\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ or $2x=(2n+1)\pi-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$2x=2n\pi+\frac{\pi}{4}$

$x=n\pi+\frac{\pi}{8}, n\in Z$

$$
\begin{aligned}
2x&=(2n+1)\pi-\frac{\pi}{4}\\
x&=\frac{(2n+1)\pi}{2}-\frac{\pi}{8}\\
&=\frac{4(2n+1)\pi-\pi}{8}\\
&=\frac{8n\pi+4\pi-\pi}{8}\\
&=\frac{8n\pi+3\pi}{8}\\
&=n\pi+\frac{3\pi}{8}, \quad n\in Z
\end{aligned}
$$

(1 mark) **(1 mark)** **(1 mark)** for showing $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$

<u>Method 4</u> – using a formula

For $\sin(x)=a$, $x=n\pi+(-1)^n\sin^{-1}(a)$, $n\in Z$

For $\sin(2x)=\frac{1}{\sqrt{2}}$

$$
\begin{aligned}
2x&=n\pi+(-1)^n\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\\
2x&=n\pi+(-1)^n\frac{\pi}{4}\\
x&=\frac{n\pi}{2}+(-1)^n\frac{\pi}{8}, \quad n\in Z
\end{aligned}
$$

(1 mark) – correct first term
(1 mark) – correct second term
(1 mark) for showing $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$

(Note that for this formula

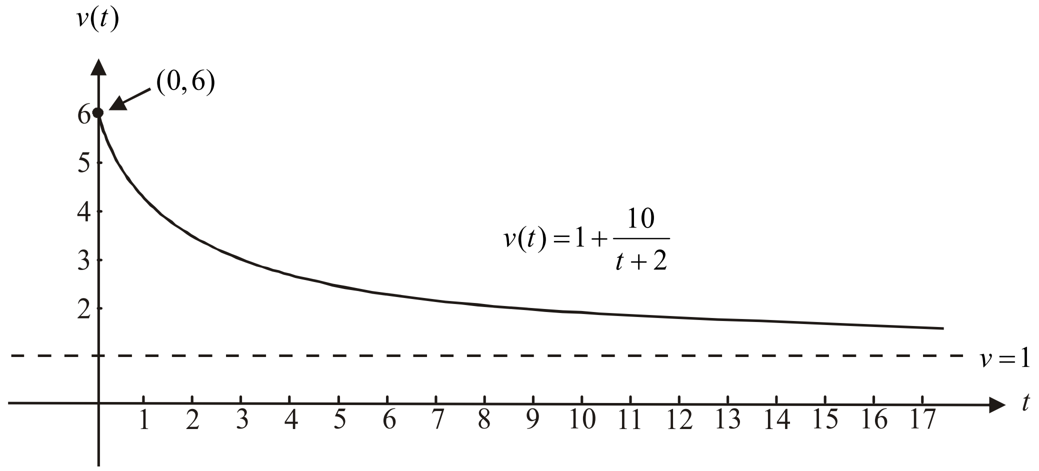
when $n=-2$, $x=-\pi+\frac{\pi}{8}=-\frac{7\pi}{8}$, when $n=0$, $x=\frac{\pi}{8}$

when $n=-1$, $x=-\frac{\pi}{2}-\frac{\pi}{8}=-\frac{5\pi}{8}$, when $n=1$, $x=\frac{\pi}{2}-\frac{\pi}{8}=\frac{3\pi}{8}$

and so on.)

Question 10

a.



(1 mark) – correct endpoint
(1 mark) – correctly labelled asymptote
(1 mark) – correct shape

b.

$$v(t)=1+\frac{10}{t+2}<3$$

$$\frac{10}{t+2}<2$$

$$10<2(t+2) \quad \text{(note that } t\geq 0, \text{ so } t+2\geq 0 \text{ so we don't have to reverse the inequality sign)}$$

$$10<2t+4$$

$$6<2t$$

$$3<t$$

So $t>3$ or $t\in(3,\infty)$

(1 mark)

c.

$$\text{distance travelled}=\int_0^1\left(1+\frac{10}{t+2}\right)dt$$ **(1 mark)**

$$=\left[t+10\log_e(t+2)\right]_0^1$$

$$=\{(1+10\log_e(3))-(0+10\log_e(2))\}$$

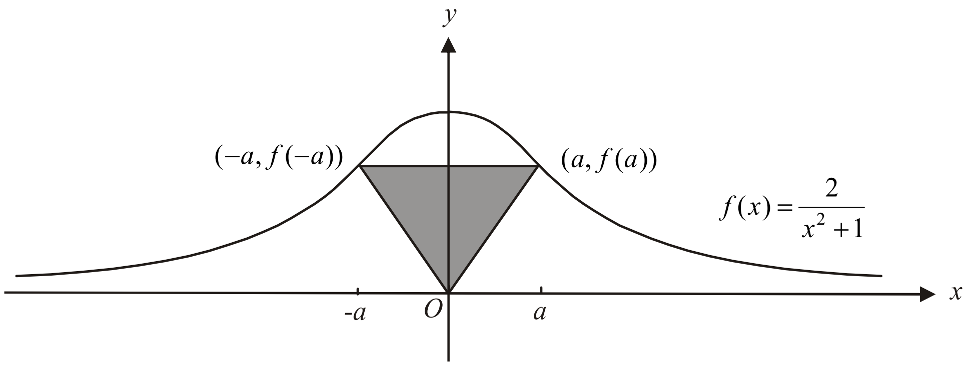
$$=1+10\log_e(3)-10\log_e(2)$$

$$=1+10\log_e\left(\frac{3}{2}\right)\text{ metres}$$

(1 mark)

Question 11

a.



$$A = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times 2a \times f(a)$$
$$= a \times f(a)$$
$$= a \times \frac{2}{a^2 + 1}$$
$$= \frac{2a}{a^2 + 1}$$

(1 mark)

b. $\frac{dA}{da} = \frac{(a^2+1) \times 2 - 2a \times 2a}{(a^2+1)^2}$ (quotient rule)

$$= \frac{2a^2 + 2 - 4a^2}{(a^2+1)^2}$$
$$= \frac{2 - 2a^2}{(a^2+1)^2}$$

(1 mark) correct use of quotient rule

For min/max $\frac{dA}{da} = 0$

$$\frac{2-2a^2}{(a^2+1)^2} = 0$$

So $2 - 2a^2 = 0$

$$2(1 - a^2) = 0$$
$$2(1-a)(1+a) = 0$$
$$a = 1 \text{ or } a = -1$$

but $a > 0$ so $a = 1$

(1 mark)

At $a = 0$, $\frac{dA}{da} = \frac{2-2a^2}{(a^2+1)^2}$

$= 2$

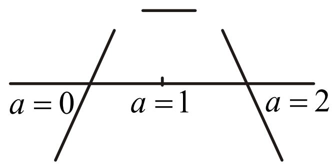
> 0

At $a = 2$, $\frac{dA}{da} = \frac{2-2a^2}{(a^2+1)^2}$

$= \frac{2-8}{25}$

$= \frac{-6}{25}$

< 0



So at $a = 1$ there is a local maximum. **(1 mark)**

From part **a.**, area $= \frac{2a}{a^2+1} = \frac{2}{2} = 1$ square unit.

So the maximum area is 1 square unit. **(1 mark)**