THE SCHOOL FOR EXCELLENCE
UNIT 3 & 4 MATHEMATICAL METHODS 2009
COMPLIMENTARY WRITTEN EXAMINATION 2 - SOLUTIONS

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MARKING SCHEME (EXTENDED ANSWER QUESTIONS)

- ($A\,4\times\frac{1}{2}\downarrow$) means four answer half-marks rounded **down** to the next integer. Rounding occurs at the end of a part of a question.
- M1 = 1 **M**ethod mark.
- **A**1 = 1 **A**nswer mark (it **must** be this or its equivalent).
- **H**1 = 1 consequential mark (**H**is/**H**er mark…correct answer from incorrect statement or slip).

SECTION 1 – MULTIPLE CHOICE QUESTIONS

1	2	3	4	5	6	7	8	9	10	11
D	C	D	E	E	C	A	C	C	D	B

12	13	14	15	16	17	18	19	20	21	22
E	D	A	C	B	D	E	A	A	E	B

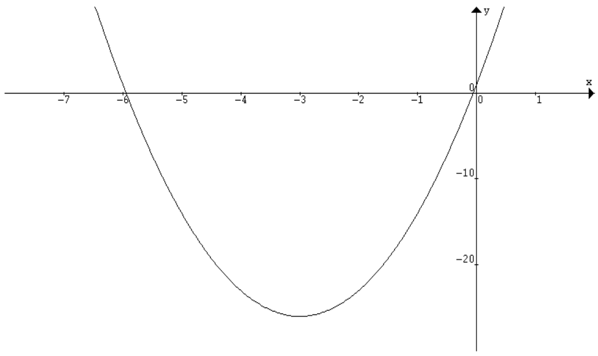
QUESTION 1

The vertical asymptote of a rational function occurs when the denominator (bottom line) of the fraction is zero. So $(x+b)$ must be the bottom line. The rest of the equation (ignoring the fraction) must read $y=a$ for the horizontal asymptote.

The answer is D.

QUESTION 2

The graph of $f(x)=3x^2+18x+1$ is shown below.



An inverse function exists if the function is one-to-one. The turning point of $f(x)$ occurs at $x=-3$ so any interval to the left of this value, or right of it, will ensure that the function is one-to-one.

The answer is C.

QUESTION 3

Because the graph touches the *x*-axis at $x=b$ there must be a factor of $(x-b)^2$ in the equation.

$\pm(x-c)$ is also a factor as is $\pm(x-a)$. The choices are between **D** or **E**.

Y Intercept:

Substituting $x=0$ into $y=(x-a)(x-b)^2(x-c)$ gives the value ab^2c which is negative as $a<0$. As the graph cuts the *y*-axis at a negative value, this is consistent with option D.

The answer is D.

QUESTION 4

$$g\left(h(x)\right) = \log_e\left(\frac{1}{|x+1|} + 1 - 1\right)$$

$$= \log_e\left(\frac{1}{|x+1|}\right)$$

$$= \log_e 1 - \log_e(|x+1|)$$

$$= -\log_e(|x+1|)$$

The domain of *g* (*h*(*x*)) is the same as the domain of *h*(*x*) which is ***R*** / {−1}.

The answer is E.

QUESTION 5

Let $m = a^x$ in the equation $a^{2x} - 5a^x + 4 = 0$.

Then $m^2 - 5m + 4 = 0$ and so $(m-4)(m-1) = 0$.

Therefore $m = 4$ and so $a^x = 4$ which means that $x = \log_a 4$

Also $m = 1$ and so $a^x = 1$ which means that $x = \log_a 1 = 0$.

The answer is E.

QUESTION 6

$\log_e(x^2)$ is defined for $R / \{0\}$ and $\log_e(1-x)$ is defined for $(-\infty, 1)$.
The expression is defined for the intersection of these two sets which is $(-\infty, 0) \cup (0, 1)$.

The answer is C.

QUESTION 7

$$4\log_3(x-1) + 2 = \log_3(x-1)^4 + 2\log_3 3$$

$$= \log_3(x-1)^4 + \log_3 3^2$$

$$= \log_3(x-1)^4 + \log_3 9$$

$$= \log_3 9(x-1)^4$$

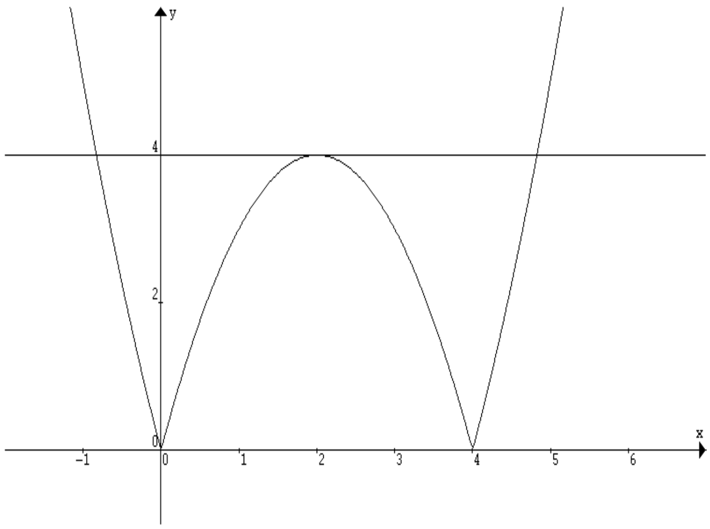
Therefore $3^{4\log_3(x-1)+2} = 3^{\log_3 9(x-1)^4} = 9(x-1)^4$

The answer is A.

QUESTION 8

$$\left|a^2-4a\right| = a^2-4a$$
$$= -(a^2-4a) = 4a-a^2$$

To solve $\left|a^2-4a\right|=4$, find the points of intersection of the graphs $y=\left|a^2-4a\right|$ and $y=4$.



Either $a^2-4a=4$ or $4a-a^2=4$

If $a^2-4a=4$ then $a^2-4a-4=0$ so $a=\dfrac{4\pm\sqrt{32}}{2} = 2\pm2\sqrt{2}$

If $4a-a^2=4$ then $a^2-4a+4=0$. Hence $a=2$.

From the graph, $\left|a^2-4a\right|\geq 4$ if $a\geq 2+2\sqrt{2}$ or $a=2$ or $a\leq 2-2\sqrt{2}$

The answer is C.

QUESTION 9

$$2\sin^2(\theta)=3-3\cos(\theta)$$
$$2(1-\cos^2(\theta))=3-3\cos(\theta)$$
$$2-2\cos^2(\theta)=3-3\cos(\theta)$$
$$2\cos^2(\theta)-3\cos(\theta)+1=0$$
$$(2\cos(\theta)-1)(\cos(\theta)-1)=0$$

Hence $\cos(\theta)=0.5$ or $\cos(\theta)=1$

For $-\pi\leq\theta\leq\pi$, $\cos(\theta)=0.5$ has solutions $-\dfrac{\pi}{3},\dfrac{\pi}{3}$ and $\cos(\theta)=1$ has solution 0.

The answer is C.

QUESTION 10

If $f(x) = a\sin(x) - b\sqrt{3}\cos(x)$ then $f'(x) = a\cos(x) + b\sqrt{3}\sin(x)$

$0 = a\cos(x) + b\sqrt{3}\sin(x)$ at any turning points.

$b\sqrt{3}\sin(x) = -a\cos(x)$

$$\frac{\sin(x)}{\cos(x)} = \tan(x) = -\frac{a}{b\sqrt{3}}$$

Now $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ and so $-\frac{a}{b\sqrt{3}} = \sqrt{3}$

Therefore $a = -3b$ and so the answer is either alternative C or D.

Only alternative D gives a minimum value at the required value of x.

The answer is D.

QUESTION 11

Using the Quotient Rule:
$$\frac{dy}{dx} = \frac{x \times \frac{d}{dx}(\log_e(2x)) - \log_e(2x) \times \frac{d}{dx}(x)}{x^2}$$
$$= \frac{x \times \frac{2}{2x} - \log_e(2x) \times 1}{x^2}$$
$$= \frac{1 - \log_e(2x)}{x^2}$$

The answer is B.

QUESTION 12

If $y = x^3 - 4x^2 + 7x - 5$ then $\frac{dy}{dx} = 3x^2 - 8x + 7$

At $x = 2$, $\frac{dy}{dx} = 3 \times 4 - 8 \times 2 + 7 = 3$

Gradient of tangent is 3. Therefore, gradient of the normal is $-\frac{1}{3}$.

As $y = 1$ at $x = 2$:

Equation of the normal is:
$$y - 1 = -\frac{1}{3}(x - 2)$$
$$3y - 3 = -x + 2$$
$$3y + x - 5 = 0$$

The answer is E.

QUESTION 13

$$\int_3^1 (5-3f(x))dx = -\int_1^3 (5-3f(x))dx$$
$$= \int_1^3 (3f(x)-5)dx$$
$$= 3\int_1^3 f(x)dx - \int_1^3 5dx$$
$$= 3\times 10 - \left[5x\right]_1^3$$
$$= 30-(15-5)$$
$$= 20$$

The answer is D.

QUESTION 14

The function *g* (*x*) is obtained from *f* (*x*) through the following three transformations:

- A dilation from the *x* – axis (or parallel to the *y* – axis) by a factor of 5 which results in the minimum value being at $(2\sqrt{3}, -5)$.
- A reflection in the *y* – axis which now means that the minimum is at $(-2\sqrt{3}, -5)$.
- Finally there is a translation of 1 unit to the right which results in the minimum now being at $(-2\sqrt{3}+1, -5)$

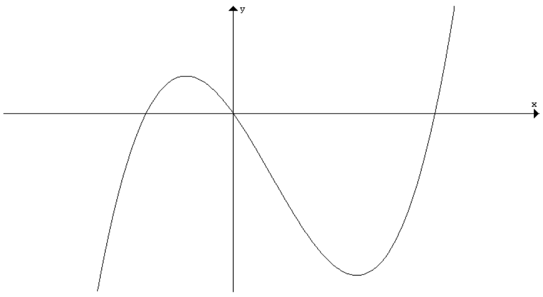
The answer is A.

QUESTION 15

$\int\left(\frac{f'(x)}{f(x)}\right)dx = \log_e|f(x)|$ and so $\int\left(\frac{2}{1-2x}+e^{3x+1}\right)dx = -\log_e|\ 2x-1\ |+\frac{1}{3}e^{3x+1}+c$

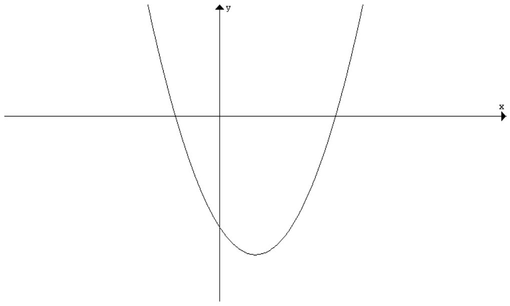
The answer is C.

QUESTION 16



The graph shown above resembles that of a cubic function and so its derivative function will resemble a parabola.

The gradient on the left and right of the function is positive and so the best alternative is shown alongside.



The answer is B.

QUESTION 17

$f(x) = (2x-1)e^{3x}$

Substitute $x=0$ into $(2x - 1)\, e^{3x}$: $-e^0 = -1$

Substitute $x=2$ into $(2x - 1)\, e^{3x}$: $3e^6$

The average rate of change is $\dfrac{3e^6 - (-1)}{2-0} = \dfrac{3e^6+1}{2}$

The answer is D.

QUESTION 18

If two events X and Y are independent then $\Pr(X \cap Y) = \Pr(X).\Pr(Y)$.

Now $A \cap B = \{2, 4\}, A \cap C = \{3\}, B \cap C = \{6\}, A \cap D = \{1, 2, 5\}, B \cap D = \{2, 10\}$

Test whether:

$\Pr(A).\Pr(B) = \Pr(A \cap B)$? Left side = $\dfrac{5}{10} \times \dfrac{5}{10} = \dfrac{1}{4}$ Right side = $\dfrac{2}{10}$ No!

$\Pr(A).\Pr(C) = \Pr(A \cap C)$? Left side = $\dfrac{5}{10} \times \dfrac{3}{10} = \dfrac{3}{20}$ Right side = $\dfrac{1}{10}$ No!

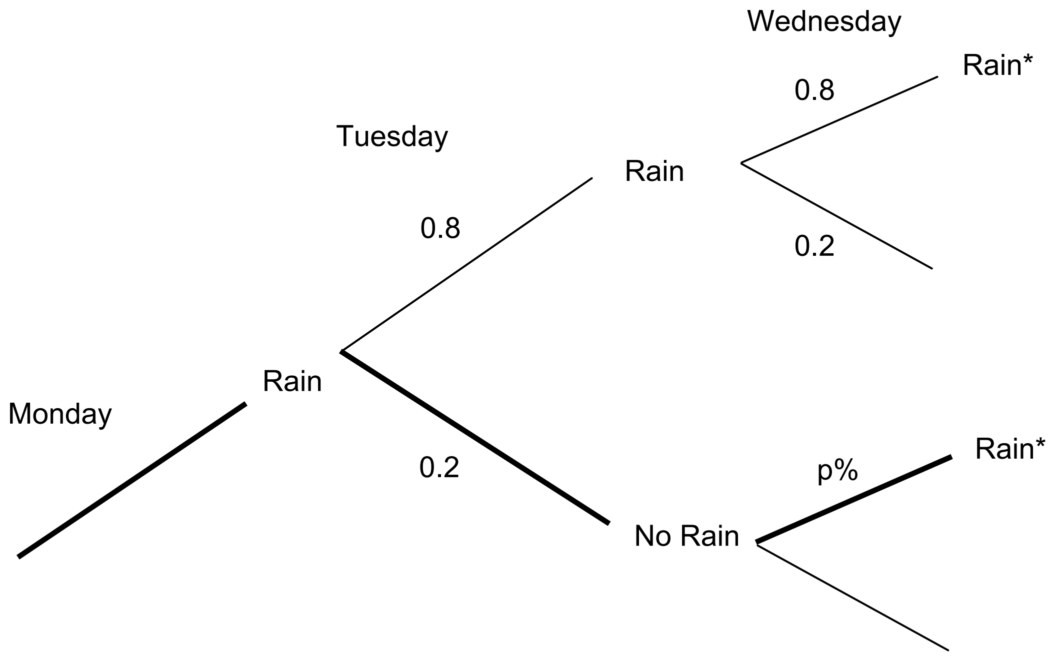
$\Pr(B).\Pr(C) = \Pr(B \cap C)$? Left side = $\frac{5}{10} \times \frac{3}{10} = \frac{3}{20}$ Right side = $\frac{1}{10}$ No!

$\Pr(A).\Pr(D) = \Pr(A \cap D)$? Left side = $\frac{5}{10} \times \frac{4}{10} = \frac{1}{5}$ Right side = $\frac{3}{10}$ No!

$\Pr(B).\Pr(D) = \Pr(B \cap D)$? Left side = $\frac{5}{10} \times \frac{4}{10} = \frac{1}{5}$ Right side = $\frac{2}{10} = \frac{1}{5}$ Yes!

The answer is E.

QUESTION 19



Probability of rain on Wednesday $= 0.8 \times 0.8 + 0.2 \times \frac{p}{100}$

$= \frac{64}{100} + \frac{2p}{1000}$

$\frac{65}{100} = \frac{64}{100} + \frac{2p}{1000}$ and so $p = 5$.

The answer is A.

QUESTION 20

Binomial Distribution with np = 5 and npq = 4

So 5q = 4 (substituting np = 5). Therefore q = $\frac{4}{5}$ which gives p = $\frac{1}{5}$

If p = $\frac{1}{5}$ then $\frac{n}{5} = 5$ and so n = 25

$\mu - \sigma = 5 - 2 = 3$ and $\mu + \sigma = 5 + 2 = 7$ and so find the Binomial cdf for $3 \le X \le 7$.

This is binomcdf(25, 0.2, 7) – binomcdf(25, 0.2, 2) = 0.8909 – 0.0982 = 0.7927

The answer is A.

QUESTION 21

$$Normalcdf\,(4, 5, 5, 0.5) = 0.4772499$$

$$\Pr(X > 4 \mid X < 5) = \frac{0.4772499}{0.5} = 0.954499$$

The answer is E.

QUESTION 22

The sum of the probabilities is 1.

Therefore $\int_0^b ax\,\mathrm{dx} = \left[\frac{a}{2}x^2\right]_0^b = 1$ and so $\frac{ab^2}{2} = 1$ (Equation 1)

Now $\int_0^{\frac{4}{3}} ax\ dx = 0.5$

$\left[\frac{a}{2}x^2\right]_0^{\frac{4}{3}} = 0.5$ and so $\frac{a}{2} \times \frac{16}{9} = \frac{1}{2}$. (Equation 2)

Hence $a = \frac{9}{16}$

Substituting for a in equation 1 gives $b^2 = \frac{32}{9}$ and so $b = \frac{4\sqrt{2}}{3}$.

The answer is B.

SECTION 2 – EXTENDED ANSWER QUESTIONS

QUESTION 1

a. $f'(x) = (x+a)\times 2(x-b) + 1\times (x-b)^2$ (using the Product Rule) **M1**

$= (x-b)[2(x+a)+(x-b)]$

$= (x-b)(3x+2a-b)$ **A1**

$= 0$ if $x = b$ or $x = \dfrac{b-2a}{3}$ **A1**

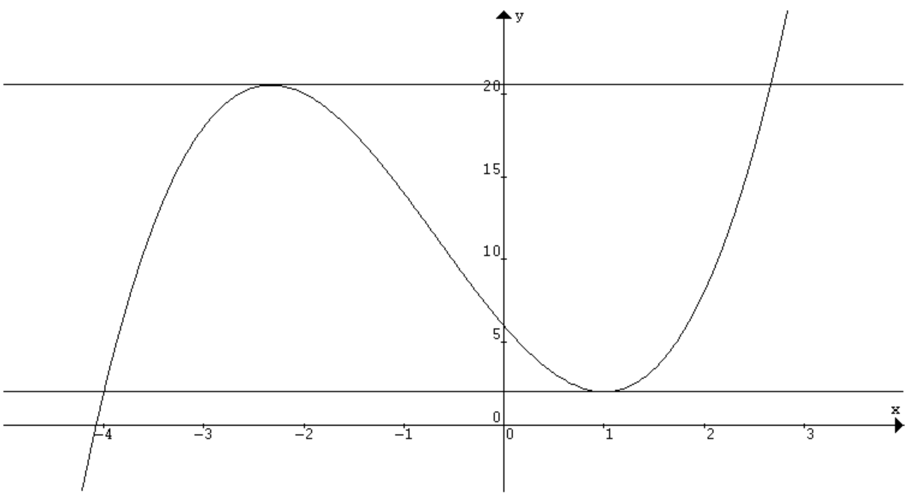
Since $f'(1) = 0$ then b could be 1. If this is the case, see if $a = 4$ satisfies the other stationary value. $-\dfrac{7}{3} = \dfrac{1-2a}{3}$ so $-7 = 1-2a$ which means that $a = 4$, as req. **M1**

b. $f(x) = (x+4)(x-1)^2 + 2$

When $x = 1$, $y = (1+4)(1-1)^2 + 2 = 2$ and so the turning point is at $(1, 2)$. **A1**

c. When $x = -\dfrac{7}{3}$, $y = (-\dfrac{7}{3}+4)(-\dfrac{7}{3}-1)^2 + 2 = 20.52$ and so $c = 20.52$ **A1**

d. The lines $y = 2$ and $y = 20.52$ have been drawn showing that each of them meets the graph at two points.



If $2 < m < 20.52$ then the equation $f(x) = m$ will have three distinct solutions.

A $4\times\dfrac{1}{2}\downarrow$ **(2 , < , <, 20.52)**

e. If the turning points of $f(x)$ are at $(1, 2)$ and $\left(-\frac{7}{3}, 20.52\right)$ then the horizontal distance between them is $1 - -\frac{7}{3} = \frac{10}{3}$ units. This would need to be multiplied by 3 to give the required result of being 10 units apart. Hence $k = 3$.

M1 (horizontal distance idea)
A1 (k = 3)

f. (-7, 20.52) and (3, 2) **A2** (1 for each pair)

Total = 12 marks

QUESTION 2

a. Total area = Two end semi-circles + flat surface + curved surface **M1**

$$A = 2\times(\frac{1}{2}\pi r^2) + 2r\times h + \frac{1}{2}\times 2\pi r h$$

$$A = \pi r^2 + 2rh + \pi r h$$ **A1**

b. Volume = 500 = $\frac{1}{2}\pi r^2 h$ and so $h = \frac{1000}{\pi r^2}$ **A1**

$$A = \pi r^2 + 2rh + \pi r h$$

$$= \pi r^2 + (2r + \pi r)h$$

$$= \pi r^2 + (2r + \pi r)\times\frac{1000}{\pi r^2}$$ **M1**

$= \pi r^2 + (2 + \pi)r \times \frac{1000}{\pi r^2}$ which when cancelling the r gives

$A = \pi r^2 + \frac{1000(2+\pi)}{\pi r}$, as required.

c. $$A = \pi r^2 + \frac{1000(2+\pi)}{\pi}\times r^{-1}$$

$$\frac{dA}{dr} = 2\pi r - \frac{1000(2+\pi)}{\pi}\times r^{-2}$$ **H1**

$= 0$ for a minimum value

Therefore $2\pi^2 r^3 = 1000(2+\pi)$ and so $r = \left(\frac{1000(2+\pi)}{2\pi^2}\right)^{\frac{1}{3}}$ = 6.39 cm **A1**

d. 384.40 cm^2 (do not accept 384.4 cm^2) **A1**

e. $C = p(\pi r^2 + 2rh) + q \times \pi r \times \frac{1000}{\pi r^2} = p(\pi r^2 + 2r \times \frac{1000}{\pi r^2}) + q \times \pi r \times \frac{1000}{\pi r^2}$ **M2**

(Give a method mark for each part, curved and flat)

$\therefore C = p\left(\pi r^2 + \frac{2000}{\pi r}\right) + \frac{1000}{r} q$

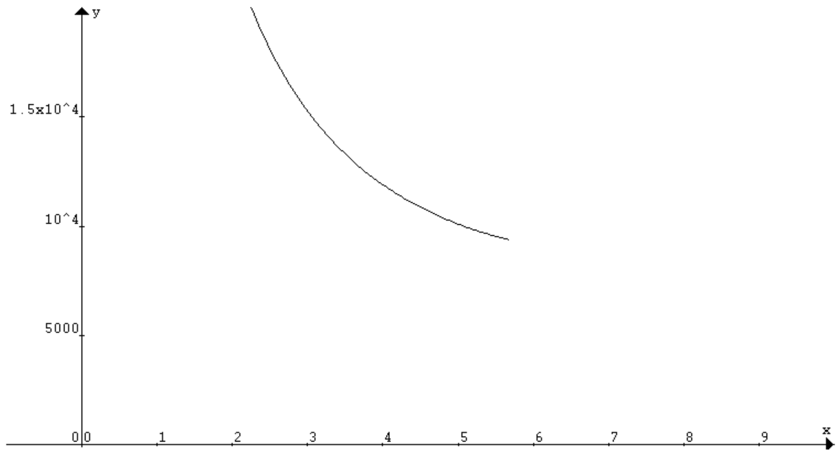
f. $\frac{dC}{dr} = p\left(2\pi r - \frac{2000}{\pi r^2}\right) - \frac{1000}{r^2} q$ **H1**

For minimum cost $\frac{dC}{dr} = 0$ and so $2\pi r\ p = \frac{2000p + 1000\pi q}{\pi r^2}$ **M1**

Hence $r = \left(\frac{1000(2p + q\pi)}{2\pi^2 p}\right)^{\frac{1}{3}}$ **A1**

g. **i.** \$8578
ii. 7.79 cm **A2**

h. Since $h = \frac{1000}{\pi r^2}$ then if $h = 10$, $r = \sqrt{\frac{100}{\pi}}$ (= 5.64 cm to 2 decimal places). **A1**



The minimum cost occurs when $r = 5.64\,cm$, and is approximately \$9396. **A1**

Total = 16 marks

QUESTION 3

a. Invnorm(0.8) = $\frac{94-p}{q}$ and Invnorm(0.99) = $\frac{122-p}{q}$ **M1**

$0.842q = 94 - p$ and $2.326q = 122 - p$ **A1**

b. (i) $0.842q = 94 - p$

(ii) $2.326q = 122 - p$

Taking (i) from (ii) gives $1.484q = 28$ and so $q = 18.8679...$ which rounds to 18.9 as required.

Substituting for q in (i) gives $p = 78.113$ which rounds to 78.1, as required. **M1**

Normalcdf(-10^{10}, 80, 78.1, 18.9) = 0.5400 so the answer is 54% **A1**

c. Normalcdf (100, 10^{10}, 78.1, 18.9) = 0.12328 **A1**

Required probability = $\frac{0.12328}{0.46}$ **M1**

$= 0.268$ **A1**

d.

Recorded speed of car	Amount of penalty	Probability
Below 80 km/h	zero	0.54
From 80 km/h to under 100 km/h	$220	0.34
From 100 km/h to under 110 km/h	$440	0.08
Over 110 km/h	$500	0.04 or 0.05

(**A** $4 \times \frac{1}{2} \downarrow$)

e. Mean = $\frac{0 \times 0.54 + \$220 \times 0.34 + \$440 \times 0.08 + \$500 \times 0.04}{0.46}$ **M1**

$= \frac{(0 + 68 + 35.2 + 20)}{0.46}$

$= \frac{123.2}{0.46}$

$= \$270$ to the nearest \$10 (or \$280 if 0.05 was used) **A1**

f. The proportion of the population exceeding 100 km/h is 0.12 (from the table).

Hence $0.12x = 48$ and so $x = \frac{48}{0.12} = 400$. **M1**

400 cars pass in the hour. **A1**

If students use 0.13 then the mark scheme is:

The proportion of the population exceeding 100 km/h is 0.13 (from the table).

Hence $0.13x = 48$ and so $x = \frac{48}{0.13} = 369.23$. **M1**

369 (or 370) cars pass in the hour. **A1**

g. Pr(Speeding) = 0.46

Binomial Distribution: $(0.46 + 0.54)^6$ **M1**

Pr(at least two) = Pr(2) + Pr(3) + Pr(4) + Pr(5) + Pr(6) or 1 – [Pr(0) + Pr(1)] **A1**

$$= 1 - \left[0.54^6 + \binom{6}{1} \times 0.54^5 \times 0.46 \right]$$

$$= 0.8485$$ **A1**

Total = 16 marks

QUESTION 4

a. **i.** $y = e^{\cos x}$

Let $u = \cos(x)$ and so $\frac{du}{dx} = -\sin(x)$ **M1**

$y = e^u$ and so $\frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= e^{\cos(x)} \times -\sin(x)$$

$$= -\sin(x)e^{\cos(x)}$$ **A1**

ii. $2\int_o^{\pi} \sin(x)e^{\cos(x)}dx = 2\left[-e^{\cos(x)}\right]_0^{\pi}$ **A1**

$$= 2(-e^{-1} + e^1)$$ **A1**

b. **i.** $f(x)=\sin(x)e^{\cos(x)}=uv$

Let $u=\sin x$ and $v=e^{\cos x}$ then $\dfrac{du}{dx}=\cos(x)$ and $\dfrac{dv}{dx}=-\sin(x)e^{\cos(x)}$ **M1**

$f'(x)=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$ = $\cos(x)e^{\cos(x)}$ − $\sin^2(x)e^{\cos(x)}$ **A1**

ii. If $f'(x)=0$ then $\cos(x)e^{\cos(x)}$ − $\sin^2(x)e^{\cos(x)}=0$

Therefore $e^{\cos(x)}(\cos(x)$ − $\sin^2(x))$ = 0

Now $e^{\cos(x)}$ can never be zero so $\cos(x)-\sin^2(x)=0$

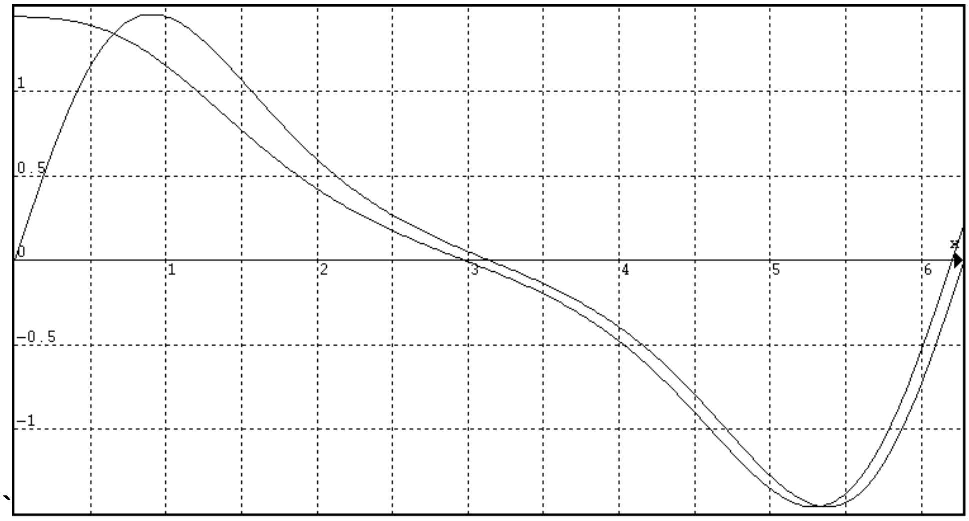
Hence $\cos(x)-(1-\cos^2(x))=0$ and so $\cos^2(x)+\cos(x)-1=0$ **M1**

Using the quadratic formula, $\cos(x)=\dfrac{-1\pm\sqrt{1+4}}{2}$ **A1**

One of these values corresponds with what needed to be found.

c. $f(g(x))$ = $\sin\sqrt{(x^2+1)}\,.\,e^{\cos\sqrt{(x^2+1)}}$ **A1**

d. **i.**



Intercepts (0, 1.44), (2.98, 0), (6.20, 0). Coordinate format not necessary here. **A1**

Shape with two points of intersection at approximately (0.7, 1.3) and (5.3, -1.4) **H1**

ii. (0.65, 1.34) and (5.33, 1.46) **A1**

e. $$\int_{0.65}^{2.98}[f(x)-g(f(x))]dx+\int_{2.98}^{\pi}f(x)dx+\int_{5.33}^{6.20}[f(x)-g(f(x))]dx+\left|\int_{6.20}^{2\pi}f(x)dx\right|$$

The two "difference integrals" with correct lower terminals. **M1**
All four integrals correct. **A1**

Total = 14 marks