MAV Trial Examination Papers 2009
Mathematical Methods (CAS)
Examination 2 – SOLUTIONS

Section 1

Answers

1. E	**2. D**	**3. D**	**4. B**	**5. C**	**6. B**
7. E	**8. C**	**9. D**	**10. A**	**11. C**	**12. E**
13. C	**14. A**	**15. A**	**16. B**	**17. C**	**18. B**
19. A	**20. D**	**21. A**	**22. E**		

Solutions

Question 1 **Answer E**

$f:[-4,2)\to R, f(x)=(x-1)^2$

The local minimum occurs at the turning point $(1,0)$ and the endpoint maximum occurs at $(-4, f(-4))=(-4,25)$. Hence the range is [0, 25].

Question 2 **Answer D**

The graph of g is a transformation of the graph of $y=|x|$, as follows.

Reflection in the x-axis: $y=-|x|$

Translation 2 units right: $y=-|x-2|$

Translation 3 units up: $y=-|x-2|+3$

The rule is $g(x)=-|x-2|+3$

Question 3 **Answer D**

The transformation of $y=f(x)$ to $y=-f\left(\frac{x}{2}\right)+3$ involves:

a reflection in the x-axis;
dilation by a factor of 2 from the y-axis and
a translation 3 units up.

By recognition, this corresponds to

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}0\\3\end{bmatrix}$$

This can be confirmed by finding (x', y'), the image of (x, y) under this transformation.

$x'=2x \Rightarrow x=\frac{x'}{2}$ (equation 1)

$y'=-y+3 \Rightarrow y=-(y'-3)$ (equation 2)

Substitute equations 1 and 2 in $y=f(x)$.

$$-(y'-3)=f\left(\frac{x'}{2}\right)$$

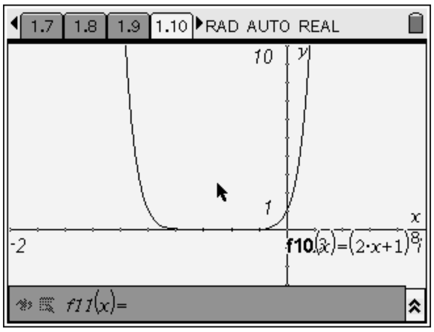
$y' = -f\left(\frac{x'}{2}\right) + 3$, as required.

Question 4 **Answer B**

From the graph of the quartic function the coefficient of x^4 is negative and the single roots are $x = a$ and $x = c$. The corresponding factors are $(x-a)$ and $(x-c)$. The double root is $(x-b)$ (notwithstanding the fact that *a* and *b* are negative numbers).
A possible rule is
$f(x) = -(x-a)(x-c)(x-b)^2$

Question 5 **Answer C**

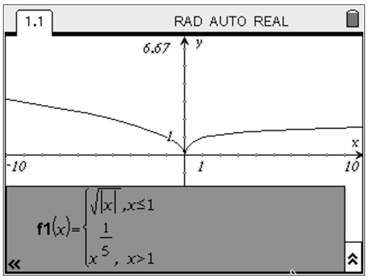
From the graph of $f(x) = (2x+1)^8$, note that *f* will be a one-to-one function for $x \in \left[-\frac{1}{2}, \infty\right)$.

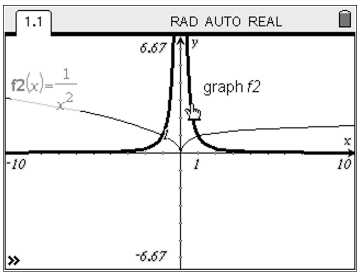


Therefore *f* will have an inverse function when $m \geq -\frac{1}{2}$.

Question 6 **Answer B**

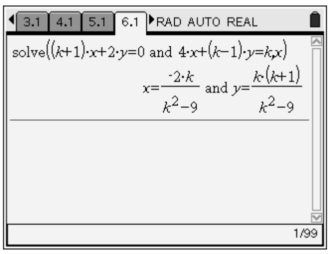
The domain of *f* is $R \setminus \{0\}$ and the domain of *g* is *R*. Hence, the domain of *f* − *g* is where the dom *f* ∩ dom *g* which is $R \setminus \{0\}$ .The graphs of *f* and *g* is shown below.

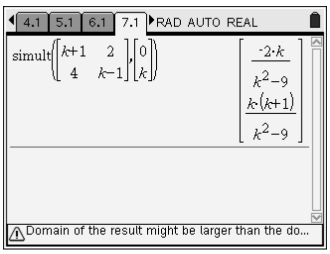




Question 7 **Answer E**

Solve the system of equations (using the Solve command or matrices)

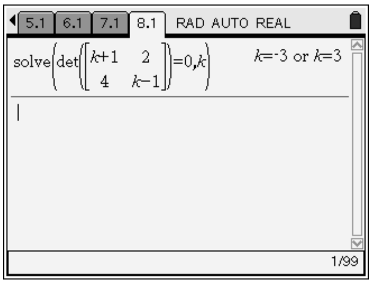




$x = \frac{-2k}{k^2-9}$, $y = \frac{k(k+1)}{k^2-9}$

Hence there will be a unique solution, unless $k^2 - 9 = 0$. That is $k = \pm 3$.
Hence a unique exists for $k \in R \setminus \{-3, 3\}$.

Alternatively, there is **no** unique solution when the determinant of the coefficient matrix is zero. This occurs when $k = \pm 3$. Unique solution $R \setminus \{-3, 3\}$.



Alternatively, there is no unique solution when both lines have the same gradient. That is,

$$\frac{k+1}{2} = \frac{4}{k-1}$$

$$(k+1)(k-1) = 8$$

$$k^2 - 1 = 8$$

$$k^2 - 9 = 0$$

$$k = \pm 3$$

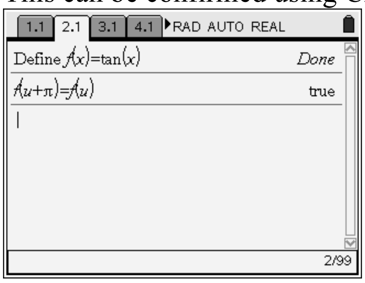
Hence a unique solution exists for $k \in R \setminus \{-3, 3\}$.

Question 8 **Answer C**

To satisfy $f(u+\pi) = f(u)$, f must be a function with a period of π. Some possibilities are $f(x) = \tan(x)$, $f(x) = |\cos(x)|$, $f(x) = \sin^2(x)$.

From the options provided, $f(x) = \tan(x)$ is the only function with a period of π.

This can be confirmed using CAS.



Question 9 **Answer D**

Use the CAS or solve by hand.

$$\cos(3x) = -\frac{1}{2}$$

$$3x = \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$$

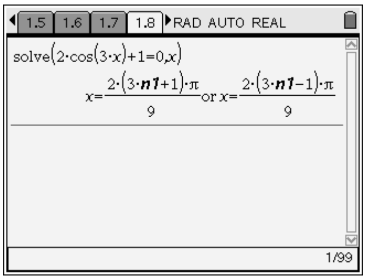
$$x = \ldots -\frac{2\pi}{3}, \frac{2\pi}{9}, \frac{4\pi}{9}, \ldots$$

The period is $\frac{2\pi}{3}$.

The general solutions are

$$x = \frac{2\pi}{9} + \frac{2\pi}{3}k = \frac{2\pi + 6\pi k}{9} = \frac{2\pi(3k+1)}{9}, \; k \,\varepsilon\, Z$$

$$x = -\frac{2\pi}{9} + \frac{2\pi}{3}k = \frac{-2\pi + 6\pi k}{9} = \frac{2\pi(3k-1)}{9}, \; k \,\varepsilon\, Z$$



Question 10 **Answer A**

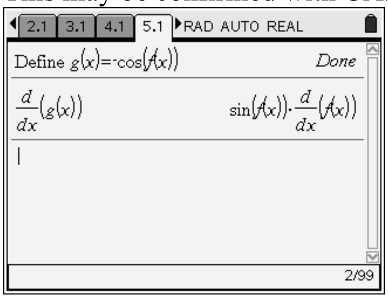
Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}, \text{ where } y=g(x) \text{ and } u=f(x)$$

$$\frac{d\left(g(x)\right)}{dx}=\frac{d\left(-\cos\left(f(x)\right)\right)}{d\left(f(x)\right)}\times\frac{d\left(f(x)\right)}{dx}$$

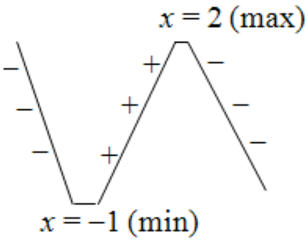
$$=\sin\left(f(x)\right)\times f'(x)$$

This may be confirmed with CAS.



Question 11 **Answer C**

Consider the sign of the gradient.



Local min. at $x=-1$ and max. at $x=2$.

Question 12 **Answer E**

The derivative of f is not defined at $x=1$ as there is a cusp at $x=1$.

f' exists for $x \in R\setminus\{1\}$.

Question 13 **Answer C**

The area is given by

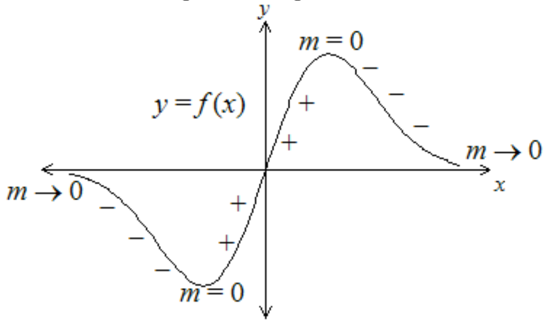
$$\int_{-2}^{0} f(x)dx-\int_{0}^{1} f(x)dx+\int_{1}^{2} f(x)dx$$

Note that $\int_{1}^{0} f(x)dx=-\int_{0}^{1} f(x)dx$

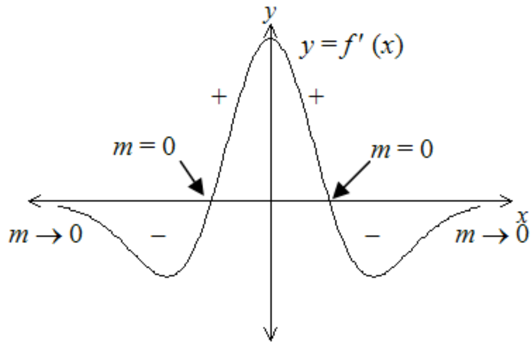
The equivalent option is $\int_{-2}^{0} f(x)dx+\int_{1}^{0} f(x)dx+\int_{1}^{2} f(x)dx$

Question 14 **Answer A**

Consider the sign of the gradient function.



Option **A** shows the graph of the gradient function with this sign profile.



Question 15 **Answer A**

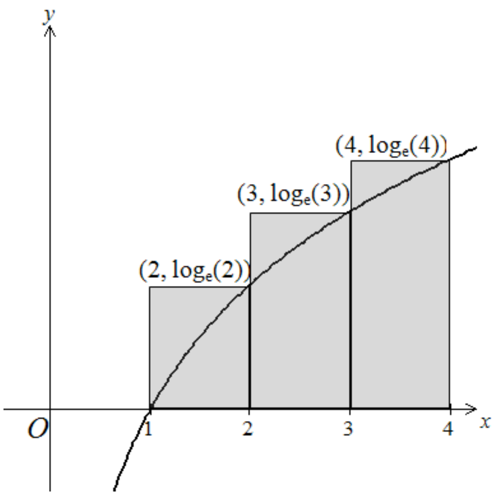
$$\int_{-2}^{1}\left(2x-\frac{u(x)}{2}\right)dx=\int_{-2}^{1}(2x)\,dx-\frac{1}{2}\int_{-2}^{1}u(x)\,dx$$

$$=\left[x^2\right]_{-2}^{1}-\frac{1}{2}\times 8$$

$$=[1-4]-4$$

$$=-7$$

Question 16 **Answer B**



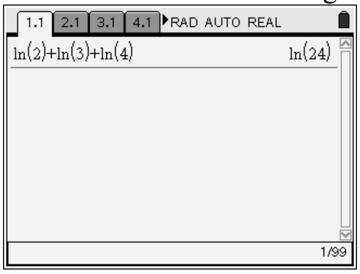
The area of each rectangle = length × width

$$\text{Area } =\log_e(2)\times 1+\log_e(3)\times 1+\log_e(4)\times 1$$

$$=\log_e(2\times 3\times 4)$$

$$=\log_e(24)$$

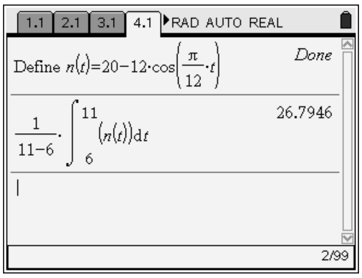
This can be confirmed using CAS.



Question 17 **Answer C**

The average value of $n(t)=20-12\cos\left(\frac{\pi}{12}t\right)$ in the interval [6, 11] is given by

$$\frac{1}{11-6}\int_6^{11} n(t)dt \approx 27$$



Question 18 **Answer B**

Let R represent the probability it will rain.

The probability it will rain on Wednesday $= RRR + RR'R$

$= 0.35\times0.35 + 0.65\times0.22$

$= 0.2655$

Question 19 **Answer A**

$a+b+0.5=1$

$a+b=0.5$ **…(1)**

$0.1+2a+4b+1.2+1.4=3.7$

$2a+4b=1$ **…(2)**

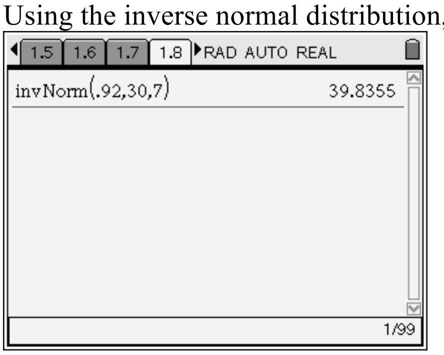
$$\begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0.5\\ 1.0\end{bmatrix}$$

Question 20 **Answer D**

Let X represent the Exam Result

$X \sim N(30, 49)$

Using the inverse normal distribution, 40 is the cut off score.

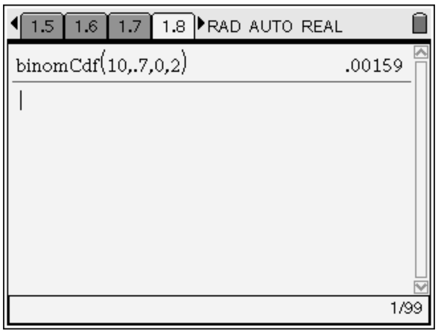


Question 21 **Answer A**

Let X represent being well

$X \sim \text{Bi}(10, 0.7)$

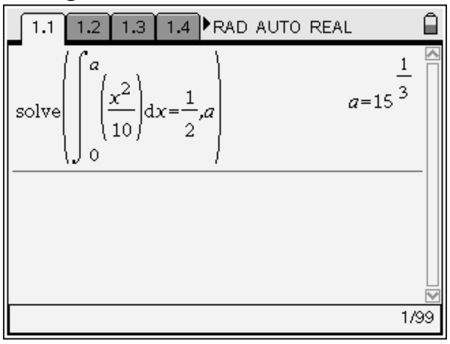
$\Pr(X \le 2) = 0.0016$ correct to 4 decimal places.



Question 22 **Answer E**

Solve $\int_{-\infty}^{a} f(x) = \frac{1}{2}$ for a where a is the median

Using CAS, the median is $\sqrt[3]{15}$.



Section 2

Question 1

a. A is the amplitude which is 10. B is the vertical translation which is 10.

Both correct **[1A]**

$$P=\frac{2\pi}{n}=\frac{4}{3}\times 20=\frac{80}{3}$$

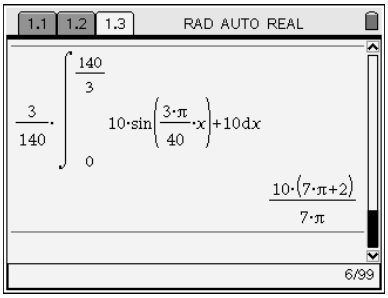
$$n=\frac{2\pi\times 3}{80}=\frac{3\pi}{40}$$ **[1A]**

b. Average value = $\dfrac{1}{b-a}\displaystyle\int_a^b (d(x))dx$

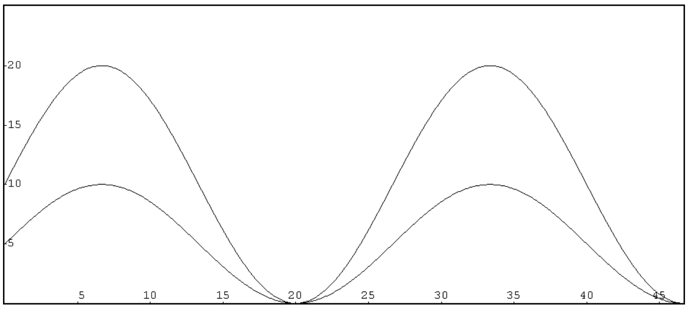
$$=\frac{1}{\frac{140}{3}-0}\int_0^{\frac{140}{3}}\left(10\sin\left(\frac{3\pi}{40}x\right)+10\right)dx$$ **[1A]**

$$=10+\frac{20}{7\pi}\ \text{cm}$$ **[1A]**

Note $b=\dfrac{7}{4}\times\dfrac{80}{3}=\dfrac{140}{3}$



c. **i.**



Correct shape **[1A]**

Coordinates labelled **[1A]**

$$(0,5),\left(\frac{20}{3},10\right),(20,0),\left(\frac{100}{3},10\right),\left(\frac{140}{3},0\right)$$

ii. $d_b=5\sin\left(\dfrac{3\pi}{40}x\right)+5$ **[1A]**

iii. The areas are the same. **[1A]**

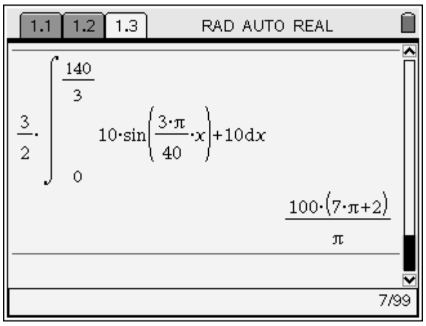
iv. $\dfrac{3}{2}\displaystyle\int_0^{\frac{140}{3}} d\,dx$

correct terminals with dx **[1A]**

$$=\frac{3}{2}\int_0^{\frac{140}{3}}\left(10\sin\left(\frac{3\pi}{40}x\right)+10\right)dx$$

(Could use $\frac{3}{2}d$ or $3(d-d_b)$ or $3d_b$).

$= 700+\frac{200}{\pi}\text{ cm}^3$ **[1A]**



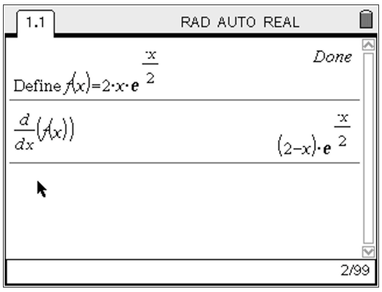
d. $d_b = 5\sin\left(\frac{3\pi}{40}\left(\frac{20}{3}-2\right)\right)+5$ **[1H]**

The height is 9.46 cm correct to two decimal places. **[1A]**

Question 2

Consider the function $f:R\rightarrow R, f(x)=2xe^{-\frac{x}{2}}$.

a. $f'(x)=(2-x)e^{-\frac{x}{2}}$ **[1A]**

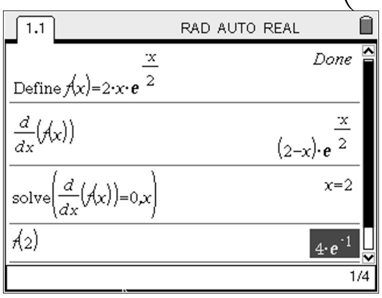


$f'(x)=2e^{-\frac{x}{2}}-xe^{-\frac{x}{2}}$

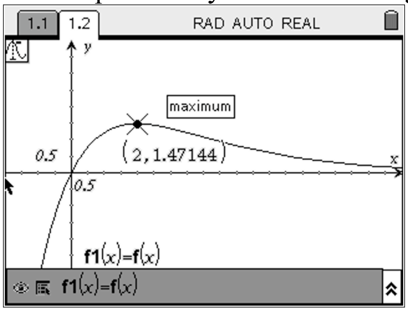
b. For a stationary point, $f'(x)=0$. This occurs at $x=2$.

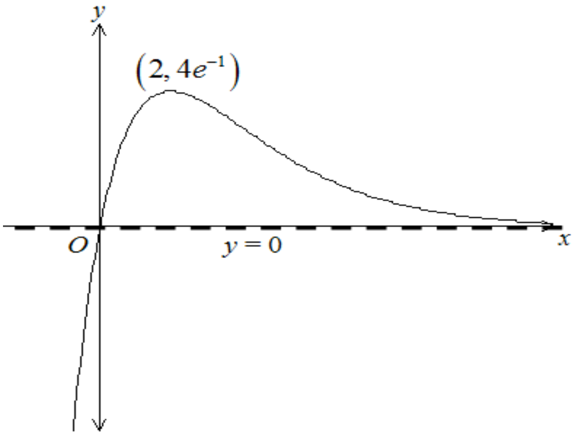
$f(2)=4e^{-1}=\frac{4}{e}$ **[1M]**

Turning point at $\left(2,4e^{-1}\right)=\left(2,\frac{4}{e}\right)$ **[1A]**



c. The shape and key features of the graph may be obtained by graphing the function.



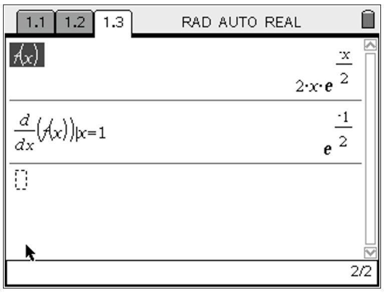


Correct shape and skew **[1A]**

Asymptote clearly shown and labelled **[1A]**

Axes intercept at the origin and turning point labelled: **[1A]**

d. **i**. $f'(1) = e^{-\frac{1}{2}} = \dfrac{1}{\sqrt{e}}$ **[1A]**



ii. $m = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$\dfrac{f(a) - 0}{a - 0} = e^{-\frac{1}{2}}$ **[1M]**

Solve for a, $\dfrac{2ae^{-\frac{a}{2}}}{a} = e^{-\frac{1}{2}}$ **[1M]**

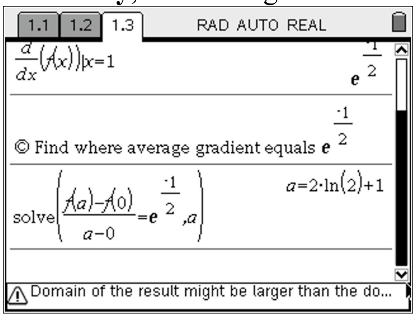
$2e^{-\frac{a}{2}} = e^{-\frac{1}{2}}$

$\log_e\left(2e^{-\frac{a}{2}}\right) = \log_e\left(e^{-\frac{1}{2}}\right)$

$\log_e(2) - \dfrac{a}{2} = -\dfrac{1}{2}$

$a = 1 + 2\log_e(2)$, as required **[1M]**

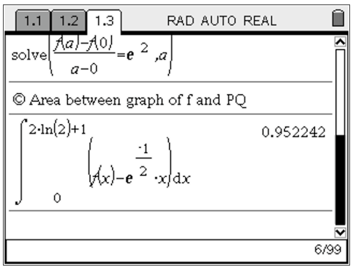
Alternatively, solve using CAS



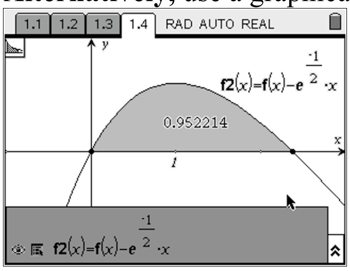
iii. The equation of the line PQ is $y=\left(e^{-\frac{1}{2}}\right)x$

Area $=\int_{0}^{1+2\log_e(2)}\left(\left(2xe^{-\frac{x}{2}}\right)-\left(\left(e^{-\frac{1}{2}}\right)x\right)\right)$ **[1M]**

$=0.9522$ **[1A]**



Alternatively, use a graphical approach.



e. The gradient of the tangent: $m_T=e^{-\frac{1}{2}}$.

Gradient of normal: $m_N=-\frac{1}{m_T}=-e^{\frac{1}{2}}$

Normal at $(1,f(1))=\left(1,2e^{-\frac{1}{2}}\right)$ **[1M]**

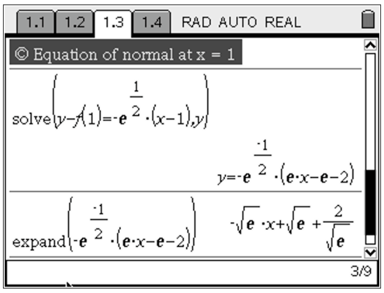
The equation of the normal is

$y-2e^{-\frac{1}{2}}=-e^{\frac{1}{2}}(x-1)$

$y=-e^{\frac{1}{2}}x+e^{\frac{1}{2}}+2e^{-\frac{1}{2}}$ **[1A]**

Alternatively

$$y = -\sqrt{e}\,x + \sqrt{e} + \frac{2}{\sqrt{e}}$$



f. LHS $= 2xyf(x+y)$

$= 2xy \times 2(x+y)e^{-\left(\frac{x+y}{2}\right)}$ **[1M]**

$= 2xy\left(2xe^{-\frac{x}{2}}e^{-\frac{y}{2}} + 2ye^{-\frac{y}{2}}e^{-\frac{x}{2}}\right)$ **[1M]**

$= xf(x)f(y) + yf(y)f(x)$ **[1M]**

$= (x+y)f(x)f(y) =$ RHS as required

Question 3

a. Using Pythagoras' theorem,

$BD = \sqrt{x^2+6^2} = \sqrt{x^2+36}$ **[1A]**

b. **i.** $\text{time} = \dfrac{\text{distance}}{\text{speed}}$ **[1M]**

Time from A to B = $\dfrac{14-x}{20}$

Time from B to D = $\dfrac{\sqrt{x^2+36}}{20}$ **[1M]**

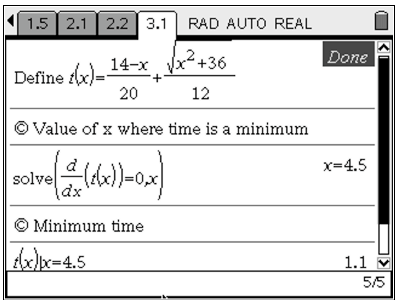
$T(x) = \dfrac{14-x}{20} + \dfrac{\sqrt{x^2+36}}{12},\ 0 \le x \le 14$

[1A]

Using technology to find the coordinates of the local minimum of the graph of T,

ii. Minimum time occurs when $x = 4.5$ km. **[1A]**

iii. The minimum time is 1.1 hours **[1A]**



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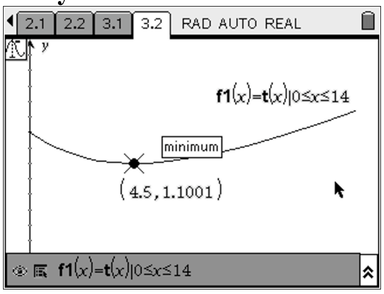
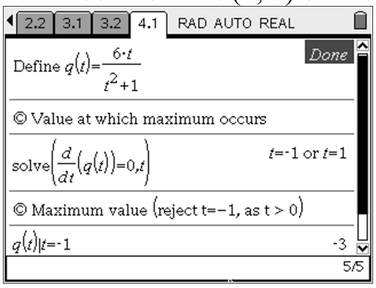


c. **i.** The maximum concentration is 3 units/cm^3 and it occurs 1 hour after the dose is administered. **[2A]**

ii. Label the max. (1, 3) on the graph **[1A]**



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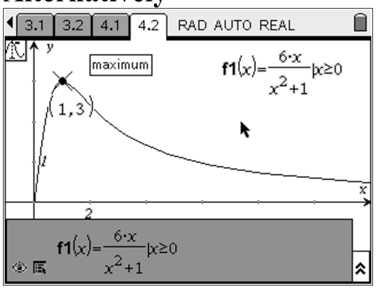
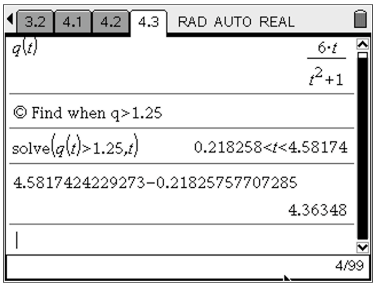
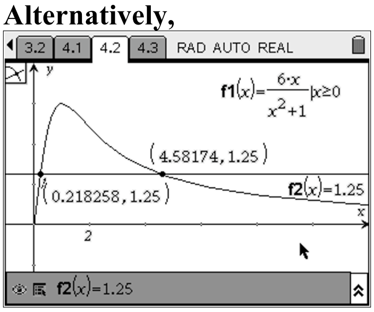


d. $Q(t) > 1.25$ when $0.22 < t < 4.58$. **[1M]**

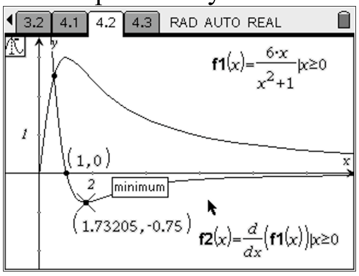
Pain relief: 4.58 – 0.22 = 4.36 hours **[1A]**

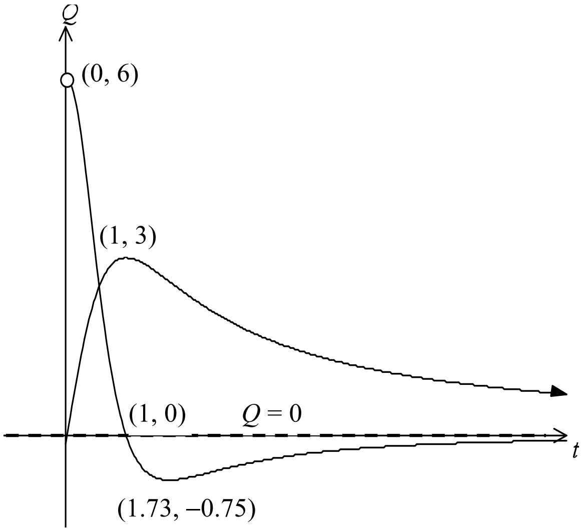


Alternatively,



e. The shape and key features of the graph of S can be obtained by graphing it on the CAS device.





Correct shape **[1A]**
Local min. and x-intercept labelled **[1A]**
Asymptote labelled and (0, 6) shown as an "open circle" **[1A]**

Question 4

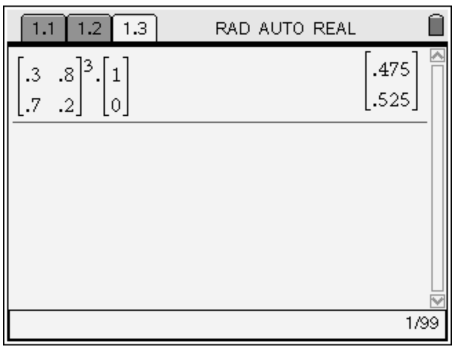
a. $0.3^3 = 0.027$ **[1A]**

b.
$$\begin{array}{c} \\ t_{i+1} \\ t'_{i+1} \end{array}\begin{array}{c} \begin{matrix} t_i & t'_i \end{matrix} \\ \begin{bmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{bmatrix} \end{array}$$

1A for each correct column **[2A]**

c. $\begin{bmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.475 \\ 0.525 \end{bmatrix}$ **[1M]**

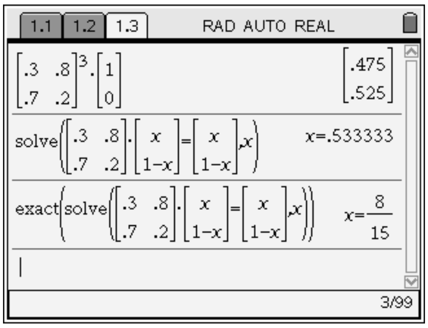
The probability Grandma bottles tomatoes in 2011 is 0.475. **[1A]**



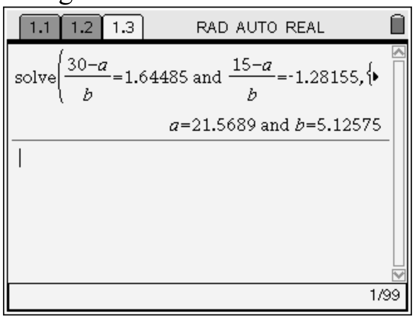
d. Solve $\begin{bmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix}$ for x. **[1M]**

$x = \frac{8}{15}$

The probability she will bottle tomatoes in the long term, assuming she does not die is $\frac{8}{15}$. **[1A]**



e. Using the inverse normal distribution, with $\mu = 0$ and σ = 1.



$$\frac{30-\mu}{\sigma} = 1.64485 \ldots(1)$$

$$\frac{15-\mu}{\sigma} = -1.28155 \ldots(2)$$

Both equations correct **[1M]**

$\sigma = 5.1$ g

$\mu = 21.6$ g

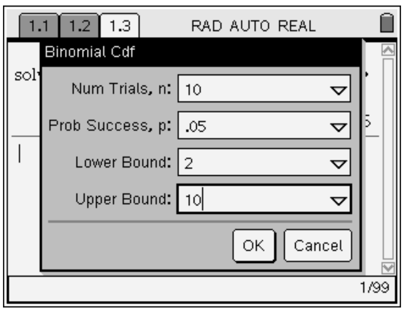
Both answers correct **[1A]**

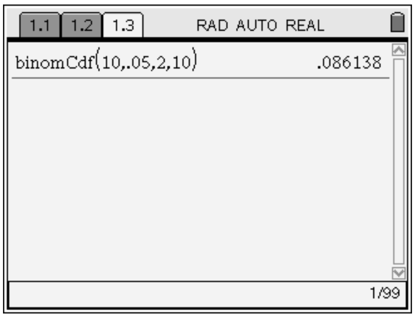
f. 0.85^4 **[1M]**

≈ 0.5220 **[1A]**

g. Let $Y \sim Bi(10, 0.05)$ **[1M]**

$\Pr(Y \geq 2) \approx 0.0861$ **[1A]**





h. Let $W \sim Bi(n, 0.05)$

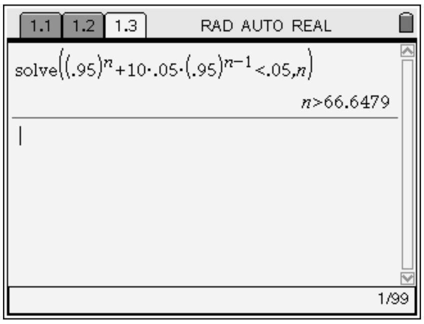
$\Pr(W \geq 2) > 0.95$

$1 - (\Pr(W = 0) + \Pr(W = 1)) > 0.95$ **[1M]**

$\Pr(W = 0) + \Pr(W = 1) < 0.05$

$0.95^n + \binom{10}{1}(0.05)(0.95)^{n-1} < 0.05$ **[1M]**

$n = 67$ **[1A]**



END OF SOLUTIONS

