Year 2009

VCE

Mathematical Methods and Mathematical Methods (CAS)

Trial Examination 1

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Victorian Certificate of Education
2009

STUDENT NUMBER

										Letter
Figures										
Words										

MATHEMATICAL METHODS
AND
MATHEMATICAL METHOD (CAS)

Trial Written Examination 1

Reading time: 15 minutes
Total writing time: 1 hour

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
11	11	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are NOT permitted to bring into the examination room: notes of any kind, blank sheets of paper, white out liquid/tape or a calculator of any type.

Materials supplied

- Question and answer book of 14 pages with a detachable sheet of miscellaneous formulas at the end of this booklet.
- Working space is provided throughout the booklet.

Instructions

- Detach the formula sheet from the end of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f(x)=\dfrac{\log_e(2x)}{2x^2}$. Find $f'(x)$.

2 marks

Question 2

Solve the equation $\log_6(x+2)+\log_6(2x-2)=2$.

2 marks

Question 3

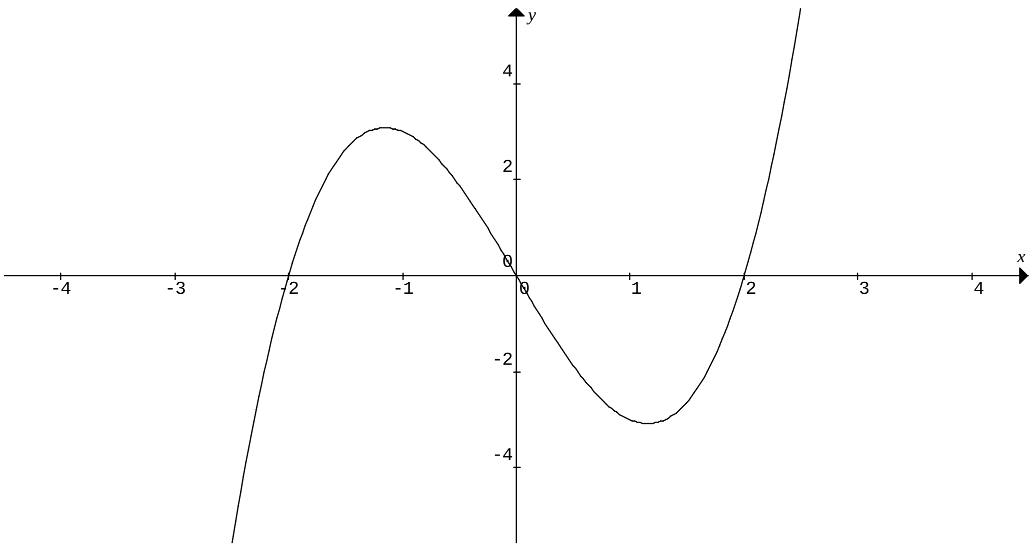
Given the function $f(x) = \begin{cases} a\sqrt{x+2} & \text{for} \quad x \ge 2 \\ mx+3 & \text{for} \quad x < 2 \end{cases}$

Find the values of *a* and *m*, if f is a differentiable function.

4 marks

Question 4

Part of the graph of $y = x^3 - 4x$ is shown below.



a. On the same set of axes sketch the graph of $y = \left|x^3 - 4x\right|$

b. Find the area of the region between the two curves $y = x^3 - 4x$ and $y = \left|x^3 - 4x\right|$ and $x = 0$ and $x = 2$.

1 + 2 = 3 marks

Question 5

For the function $f(x)=\frac{1}{3}\left(1-e^{-2x}\right)$. Find the inverse function f^{-1}.

3 marks

Question 6

A drinking cup contains water to depth of h cm, the volume V cm^3 of water in the cup is given by $V = \frac{\pi h}{45}\left(h^2 + 75h\right)$ for $0 \le h \le 8$.

a. Find, using calculus, the approximate change in the volume of the water in the cup, when the depth of the water in the cup increases from 1 to 1.01 cm.

2 marks

b. If the depth of the water is increasing at a rate of 10 cm/minute, find the rate (in cm^3 per minute) at which the volume of the water is increasing, when the depth is 1 cm.

2 marks

Question 7

a. The probability density function of a continuous random variable X is given by

$$f(x)=\begin{cases} k\sin(3x) & 0\le x\le \dfrac{\pi}{3} \\ 0 & \text{otherwise} \end{cases}$$

a. Show that $k=\dfrac{3}{2}$.

1 mark

b. Differentiate $x\cos(3x)$ and hence find $E(X)$.

5 marks

Question 8

A discrete random variable X has a probability distribution given by

X	1	2
$\Pr(X = x)$	$\sin(k)$	$2\sin^2(k)$

a. Find the possible values of k, given $0 \le k \le 2\pi$.

3 marks

b. Find $E(X)$.

1 mark

Question 9

Let $f(x)=\tan\left(\frac{\sqrt{x}}{2}\right)$.

a. If $f(x)=g(h(x))$ write down, the rules for the functions $g(x)$ and $h(x)$.

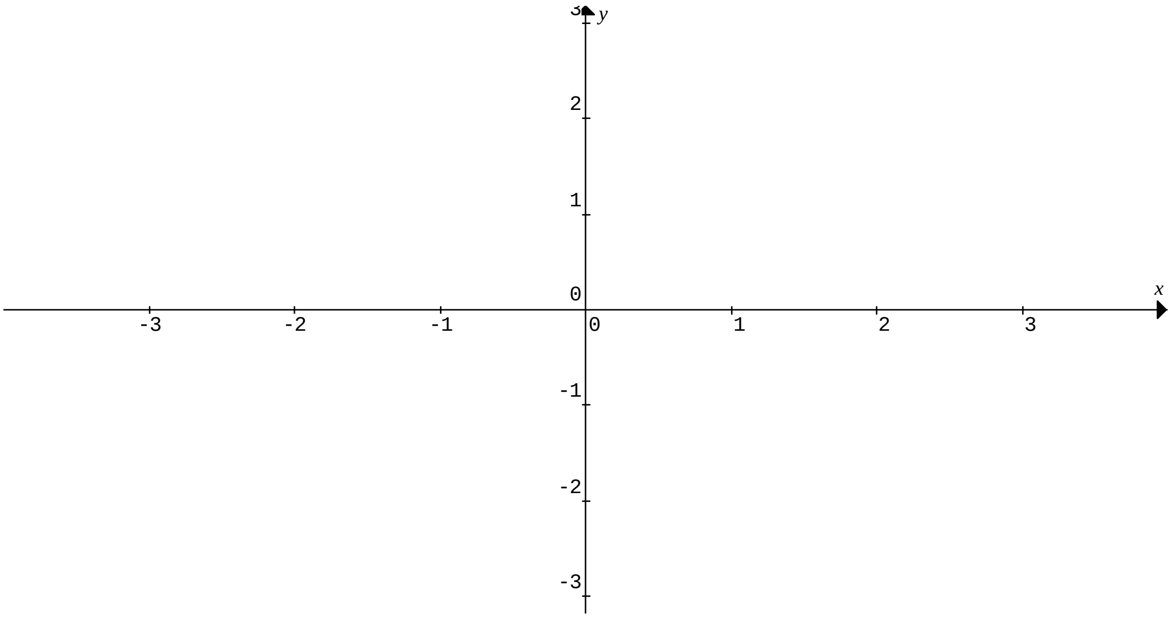
1 mark

b. Evaluate $f'\left(\frac{\pi^2}{9}\right)$.

3 marks

Question 10

a. Sketch the graph of function $y=1-\frac{4}{(2x+3)^2}$ on the axes below, clearly indicating all axial cuts and equation of any asymptotes.



2 marks

b. Find the area bounded by the curve $y=1-\frac{4}{(2x+3)^2}$, the co-ordinates axes and $x=1$.

3 marks

Question 11

Ashley has either vegemite or jam on his toast for breakfast every morning. If he has vegemite on his toast one morning, the probability he has jam on his toast the next morning is 0.20. If he has jam on his toast one morning, the probability he has vegemite on his toast the next morning is 0.75. Suppose he has vegemite on his toast on a Monday morning. What is the probability that from Monday to Wednesday inclusive he has vegemite on his toast for breakfast exactly twice?

3 marks

END OF QUESTION AND ANSWER BOOKLET

END OF EXAMINATION

MATHEMATICAL METHODS

Written examination 1

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

Mathematical Methods and CAS Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a+b)h$

curved surface area of a cylinder: $2\pi rh$

volume of a cylinder: $\pi r^2 h$

volume of a cone: $\frac{1}{3}\pi r^2 h$

volume of a pyramid: $\frac{1}{3}Ah$

volume of a sphere: $\frac{4}{3}\pi r^3$

area of triangle: $\frac{1}{2}bc\sin(A)$

Calculus

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$$

$$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$$

$$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$$

$$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$$

$$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$$

$$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$$

$$\int x^n dx=\frac{1}{n+1}x^{n+1}+c\ ,\ n\neq -1$$

$$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$$

$$\int \frac{1}{x}dx=\log_e|x|+c$$

$$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$$

$$\int \cos(ax)dx=\frac{1}{a}\sin(ax)+c$$

product rule: $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

Chain rule: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h)\approx f(x)+h\,f'(x)$

Probability

$\Pr(A)=1-\Pr(A')$

$\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$

$\Pr(A/B)=\frac{\Pr(A\cap B)}{\Pr(B)}$

mean: $\mu=E(X)$

variance: $\operatorname{var}(X)=\sigma^2=E\left((X-\mu)^2\right)=E(X^2)-\mu^2$

probability distribution		mean	variance
discrete	$\Pr(X=x)=p(x)$	$\mu=\sum x\,p(x)$	$\sigma^2=\sum (x-\mu)^2\,p(x)$
continuous	$\Pr(a<X<b)=\int_a^b f(x)dx$	$\mu=\int_{-\infty}^{\infty} x\,f(x)dx$	$\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$

Year 2009

VCE

Mathematical Methods
and
Mathematical Methods
(CAS)
Solutions
Trial Examination 1

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Question 1

$y=\dfrac{\log_e(2x)}{2x^2}$ differentiating using the quotient rule

let $u=\log_e(2x)$ $\quad v=2x^2$

$\dfrac{du}{dx}=\dfrac{1}{x}$ $\quad \dfrac{dv}{dx}=4x$ M1

$\dfrac{dy}{dx}=\dfrac{2x^2 \times \dfrac{1}{x}-4x\log_e(2x)}{4x^4}=\dfrac{2x-4x\log_e(2x)}{4x^4}$

$\dfrac{dy}{dx}=\dfrac{1}{2x^3}\left(1-2\log_e(2x)\right)$ A1

Question 2

$\log_6(x+2)+\log_6(2x-2)=2$

$\log_6((x+2)(2x-2))=2$

$2(x+2)(x-1)=6^2=36$

$x^2+x-2=18$

$x^2+x-20=0$

$(x+5)(x-4)=0$ M1

$x=-5 \quad x=4$

but $x>-2$ so there is only one answer $x=4$ A1

Question 3

For the function to be differentiable it must be continuous and the gradients must match.

$f(2)=a\sqrt{4}=2a=2m+3$ M1

$f'(x)=\begin{cases}\dfrac{a}{2\sqrt{x+2}} & x>2\\ m & x<2\end{cases} \Rightarrow \quad f'(2)=\dfrac{a}{4}=m$ A1

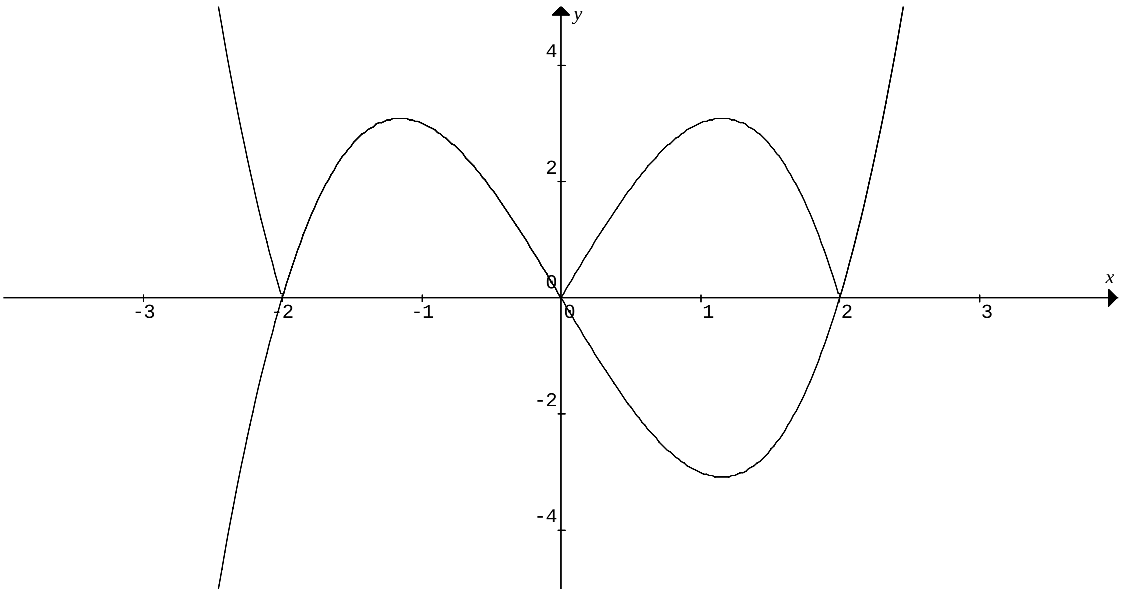
$a=4m \quad \Rightarrow 8m=2m+3 \quad 6m=3$

$m=\dfrac{1}{2}$ A1

$a=2$ A1

Question 4

a. G1



b. $A = 2\int_0^2 \left(4x - x^3\right) dx$ by symmetry

$A = 2\left[2x^2 - \frac{x^4}{4}\right]_0^2$ M1

$A = 2\left[(8-4)-(0)\right]$

$= 8 \text{ units}^2$ A1

Question 5

$f \quad y = \frac{1}{3}\left(1 - e^{-2x}\right)$ interchanging y and x

$f^{-1} \quad x = \frac{1}{3}\left(1 - e^{-2y}\right)$ transposing to make y the subject M1

$f^{-1} \quad 3x = 1 - e^{-2y} \quad \Rightarrow \quad e^{-2y} = 1 - 3x \quad \Rightarrow \quad -2y = \log_e\left(1-3x\right)$

$y = f^{-1}(x) = -\frac{1}{2}\log_e\left(1-3x\right) = \frac{1}{2}\log_e\left(\frac{1}{1-3x}\right)$ A1

the domain of f^{-1}, needs to be stated as we are asked for a function

since $1 - 3x > 0 \quad \Rightarrow x < \frac{1}{3}$ dom $f^{-1} = \left(-\infty, \frac{1}{3}\right)$

$f^{-1} : \{x : x < \frac{1}{3}\} \to R$, $f^{-1}(x) = -\frac{1}{2}\log_e\left(1-3x\right)$ A1

Question 6

a. $V=\frac{\pi h}{45}\left(h^2+75h\right)=\frac{\pi}{45}\left(h^3+75h^2\right)$

when $h=1$ $\Delta h=0.01$ find ΔV

$\frac{dV}{dh}=\frac{\pi}{45}\left(3h^2+150h\right)=\frac{\pi}{15}\left(h^2+50h\right)\approx\frac{\Delta V}{\Delta h}$ M1

$\Delta V\approx\frac{\pi}{15}(1+50)\text{ x }0.01=\frac{51\pi}{1500}\text{ cm}^3$

the volume increases by $\frac{17\pi}{500}\text{ cm}^3$ A1

b. Now given $\frac{dh}{dt}=10$ cm/min

By the chain rule $\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}=\frac{10\pi}{15}\left(h^2+50h\right)$ M1

when $h=1$ $\left.\frac{dV}{dt}\right|_{h=1}=\frac{2\pi(1+50)}{3}=34\pi\text{ cm}^3/\text{min}$ A1

Question 7

a. $\int_0^{\frac{\pi}{3}}k\sin(3x)\,dx=1$

$-\frac{k}{3}\left[\cos(3x)\right]_0^{\frac{\pi}{3}}=1$

$-\frac{k}{3}\left(\cos(\pi)-\cos(0)\right)=\frac{2k}{3}=1$ M1

$k=\frac{3}{2}$

b. $\frac{d}{dx}\left(x\cos(3x)\right)=\cos(3x)-3x\sin(3x)$ A1

$3\int x\sin(3x)\,dx=\int\cos(3x)\,dx-x\cos(3x)=\frac{1}{3}\sin(3x)-x\cos(3x)$ M1

Now $E(X)=\frac{3}{2}\int_0^{\frac{\pi}{3}}x\sin(3x)\,dx$ A1

$E(X)=\frac{3}{2}\text{ x }\frac{1}{3}\left[\frac{1}{3}\sin(3x)-x\cos(3x)\right]_0^{\frac{\pi}{3}}$ M1

$E(X)=\frac{1}{2}\left[\left(\frac{1}{3}\sin(\pi)-\frac{\pi}{3}\cos(\pi)\right)-\left(\frac{1}{3}\sin(0)-0\right)\right]$

$E(X)=\frac{\pi}{6}$ A1

Question 8

X	1	2
$\Pr(X=x)$	$\sin(k)$	$2\sin^2(k)$

a. $\sum \Pr(X=x)=\sin(k)+2\sin^2(k)=1$

$2\sin^2(k)+\sin(k)-1=0$

$(2\sin(k)-1)(\sin(k)+1)=0$ M1

$\sin(k)=-1$ not possible, since, probabilities must be positive A1

and $\sin(k)=\frac{1}{2}$

$k=\frac{\pi}{6}\ ,\ \frac{5\pi}{6}$ A1

b. $E(X)=\sum x\Pr(X=x)=\sin(k)+4\sin^2(k)$

$E(X)=\frac{1}{2}+4\times\frac{1}{4}$

$E(X)=1.5$ A1

Question 9

a. $g(x)=\tan(x)$ and $h(x)=\frac{\sqrt{x}}{2}$ A1

b. let $f(x)=y=\tan(u)$ $u=\frac{\sqrt{x}}{2}=\frac{1}{2}x^{\frac{1}{2}}$ chain rule M1

$\frac{dy}{du}=\frac{1}{\cos^2(u)}$ $\frac{du}{dx}=\frac{1}{4}x^{-\frac{1}{2}}=\frac{1}{4\sqrt{x}}$

$f'(x)=\frac{dy}{dx}=\frac{1}{4\sqrt{x}\cos^2\left(\frac{\sqrt{x}}{2}\right)}$ A1

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}}=f'\left(\frac{\pi^2}{9}\right)=\frac{1}{4\times\frac{\pi}{3}\cos^2\left(\frac{\pi}{6}\right)}=\frac{3}{4\pi\times\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{3}{4\pi}\times\frac{4}{3}$

$f'\left(\frac{\pi^2}{9}\right)=\frac{1}{\pi}$ A1

Question 10

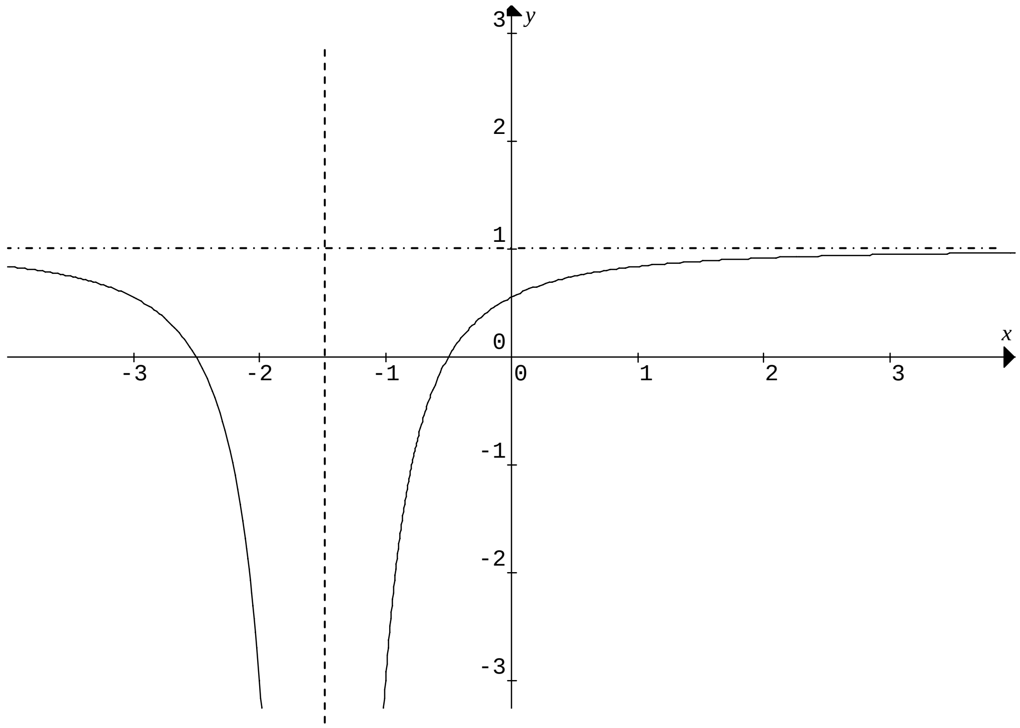
a. $x=-\frac{3}{2}$ is a vertical asymptote, $y=1$ is a horizontal asymptote

the y-intercept is $y=1-\frac{4}{9}=\frac{5}{9}$ $\left(0\,,\,\frac{5}{9}\right)$ A1

crosses the x-axis, when $y=0$, $(2x+3)^2=4$ $2x+3=\pm 2$

$2x=-1,-5$ $x=-\frac{1}{2}\,,\,-\frac{5}{2}$ $\left(-\frac{1}{2}\,,\,0\right)$ $\left(-\frac{5}{2}\,,\,0\right)$

correct graph, correct axial intercepts and correct asymptotes G1



b. The area is $A=\int_0^1\left(1-\frac{4}{(2x+3)^2}\right)dx$ A1

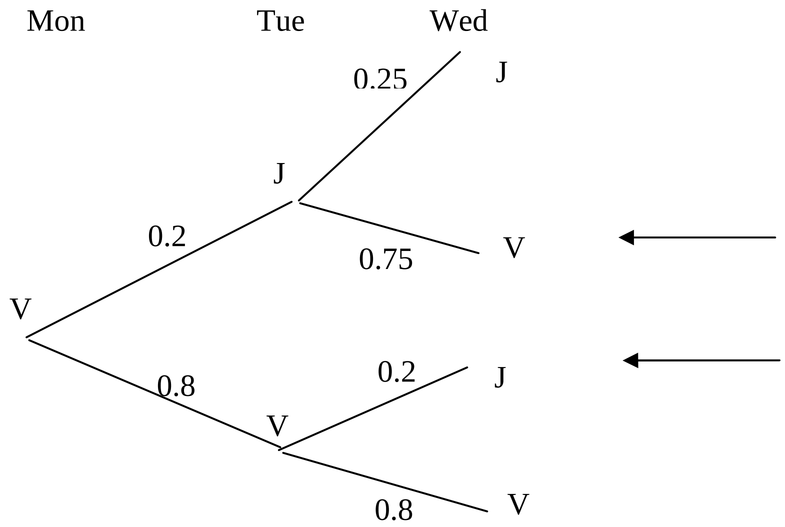
$A=\left[x+\frac{2}{2x+3}\right]_0^1=\left(\left(1+\frac{2}{5}\right)-\left(0+\frac{2}{3}\right)\right)$ M1

$A=\frac{11}{15}$ units^2 A1

Question 11

$\Pr(\text{Vegimite on two days})$ M1

$= V\ V\ J \text{ or } V\ J\ V$



$= 0.2 \times 0.75 + 0.8 \times 0.2$

$= \frac{1}{5} \times \frac{3}{4} + \frac{4}{5} \times \frac{1}{5} = \frac{3}{20} + \frac{4}{25} = \frac{15+16}{100}$ A1

$= 0.31$ A1

END OF SUGGESTED SOLUTIONS