INSIGHT

Trial Exam Paper

2009

MATHEMATICAL METHODS (CAS)

Written examination 2

STUDENT NAME:

QUESTION AND ANSWER BOOK

Reading time: 15 minutes
Writing time: 2 hours

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
		Total	80

- Students are permitted to bring the following items into the examination: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved **graphics** calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.
- Students are NOT permitted to bring the following items into the examination: blank sheets of paper and/or white out liquid/tape.

Materials provided

- The question and answer book of 21 pages, with a separate sheet of miscellaneous formulas.
- An answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the box provided and on the answer sheet for multiple-choice questions.
- Remove the formula sheet during reading time.
- You must answer the questions in English.

At the end of the exam

- Place the answer sheet for multiple-choice questions inside the front cover of this question book.

Students are NOT permitted to bring mobile phones or any other electronic devices into the examination.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the multiple-choice answer sheet.
Select the response that is **correct** for the question.
A correct answer scores 1 mark, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
If more than one answer is selected no marks will be awarded.

Question 1

The function $f:[0,2\pi]\to R,\ f(x)=1-3\sin(2x-\frac{\pi}{2})$ has a range and period respectively of

A. $[0,3]$ and 2

B. $[-1,3]$ and 2π

C. $[-2,4]$ and π

D. $[-3,3]$ and π

E. $[-2,4]$ and 2π

Question 2

The range of the function $f(x)=e^{|x|}-3$ is

A. $[-2,\infty)$

B. R^+

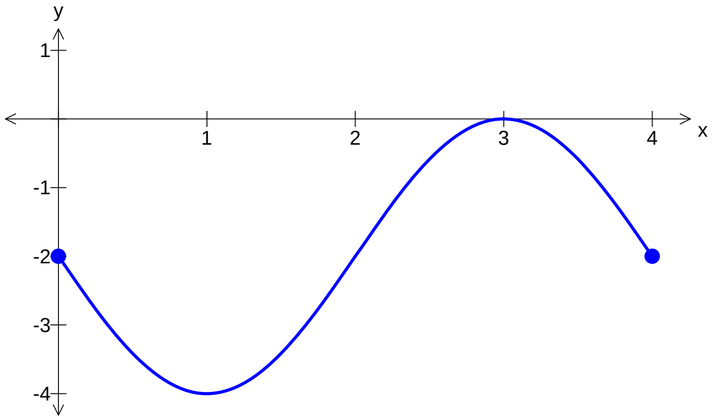
C. $(-2,0]$

D. $(-2,0)$

E. $(-3,\infty)$

Question 3

The diagram below shows one cycle of the graph of a circular function.



A possible equation for the function whose graph is shown is

A. $y = 2 - 2\sin(4x)$

B. $y = -2 - 2\sin(4x)$

C. $y = 4 - 2\sin(4\pi x)$

D. $y = -2 - 2\sin(\frac{\pi}{2}x)$

E. $y = -2 - 2\sin(\frac{1}{2}x)$

Question 4

One cycle of the graph of the function with the equation $y = \tan(ax)$ has successive vertical asymptotes at $x = \frac{3}{8}$ and $x = \frac{5}{8}$.

A possible value for a is

A. 2

B. 2π

C. 4

D. 4π

E. 8

Question 5

For the system of simultaneous linear equations

$$z = -3$$
$$x - y = 5$$
$$x + y = 1$$

an equivalent matrix equation is

A. $\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} z \\ y \\ x \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$

E. $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$

Question 6

The transformation $T: R^2 \to R^2$ which maps the curve with the equation $y = e^x$ to the curve with the equation $y = e^{2x-4} + 3$ could have the rule

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

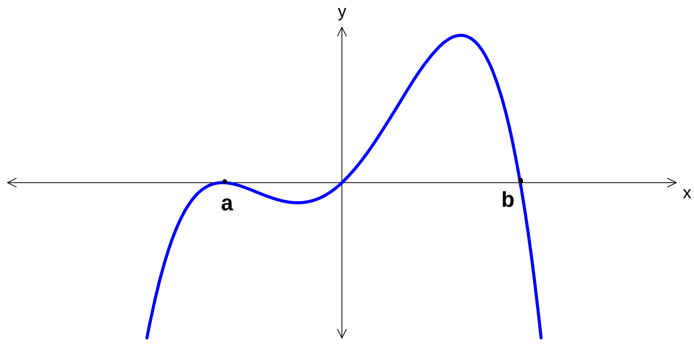
B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Question 7



The graph shown could be that of a function f with the equation

A. $y = -x(x+a)^2(x-b)$

B. $y = -x(x+a)(x-b)$

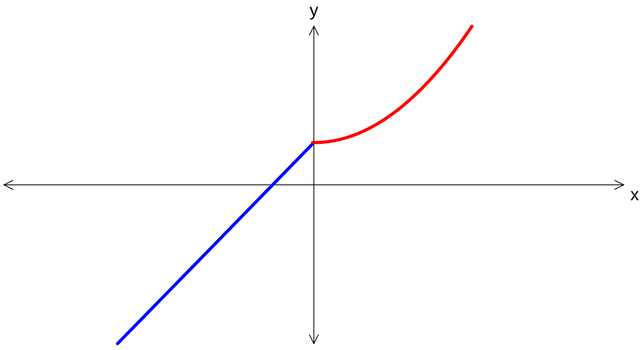
C. $y = x(x-a)^2(x-b)$

D. $y = -x(x+a)^2(b-x)$

E. $y = x(x-a)^2(b-x)$

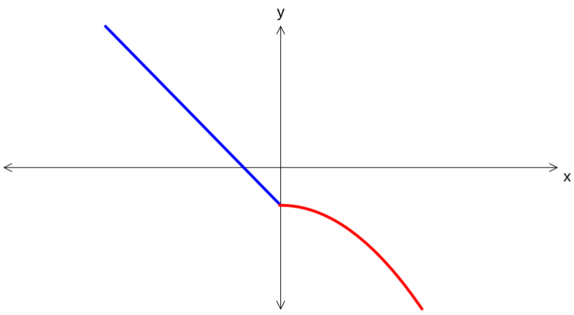
Question 8

The graph of the function with the equation $y = f(x)$ is shown below.

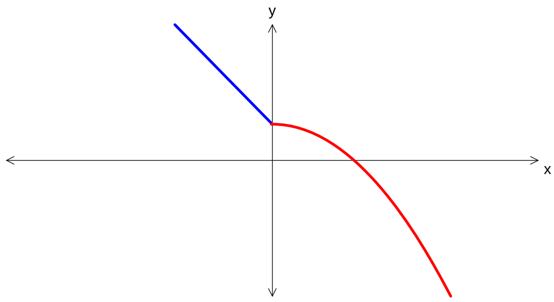


Which of the following is most likely to be the graph of the inverse function?

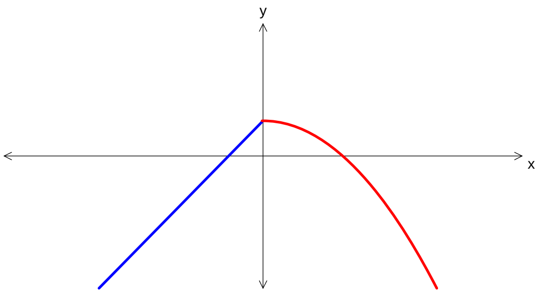
A.



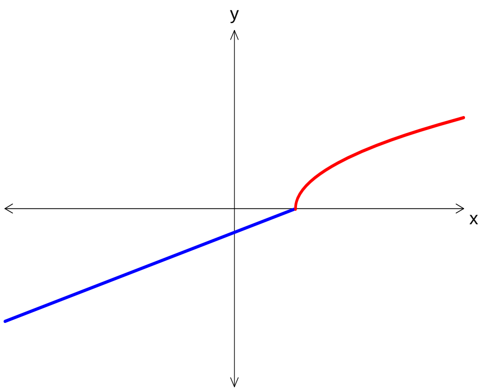
B.



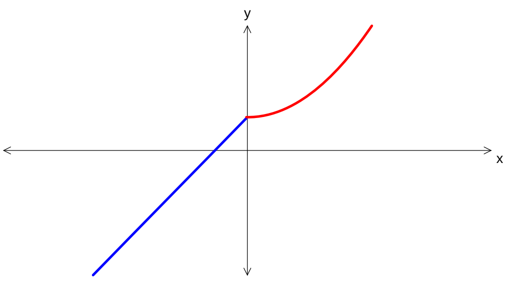
C.



D.

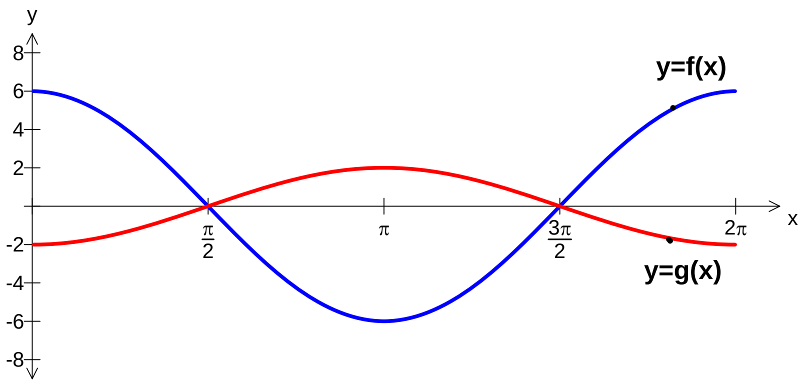


E.



Question 9

The diagram below shows the graphs of two circular functions, *f* and *g*.



The graph of the function with the equation $y = f(x)$ is transformed into the graph of the function with the equation $y = g(x)$ by

A. a dilation by a scale factor of $\frac{1}{3}$ from the *y*- axis followed by a reflection in the *x*-axis.

B. a dilation by a scale factor of 3 from the *y*- axis followed by a reflection in the *x*-axis.

C. a dilation by a scale factor of 3 from the *y*- axis followed by a reflection in the *y*-axis.

D. a dilation by a scale factor of $\frac{1}{3}$ from the *x*- axis followed by a reflection in the *x*-axis.

E. a dilation by a scale factor of 3 from the *x*- axis followed by a reflection in the *y*-axis.

Question 10

The function defined by $f : A \to R,\ f(x) = e^{(x-b)^2}$, $b \in R$, will have an inverse function for all values of *b*, if its domain *A* is

A. R

B. $R \setminus \{b\}$

C. $[b, \infty)$

D. R^+

E. $[-b, \infty)$

Question 11

If $f(x) = e^{2x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ then $g(f(1))$ is

A. $\dfrac{1}{e}$

B. $-e$

C. $\sqrt{e}$

D. 1

E. -1

Question 12

If $y = \log_e(\sin(3x))$ then $\dfrac{dy}{dx}$ is equal to

A. $\dfrac{3\cos(3x)}{\sin(3x)}$

B. $\dfrac{1}{\sin(3x)}$

C. $\dfrac{1}{3\cos(3x)}$

D. $3\tan(3x)$

E. $-3\tan(3x)$

Question 13

The equation of the normal to the curve with equation $y = x\cos(x)$, at the point on the curve with x-coordinate 2π is

A. $y = \dfrac{1}{x\sin(x)}$

B. $y = -x$

C. $y = x$

D. $y = -x + 4\pi$

E. $y = x - 4\pi$

Question 14

If $y = 2e(e^x - 1)$ then the rate of change of y with respect to x when $x = 0$ is

A. $2e^2$

B. 0

C. $2e - 1$

D. $2e^2 - 2e$

E. $2e$

Question 15

The interval $[0, 5]$ is divided into n equal subintervals by the points $x_0, x_1, ...x_{n-1}, x_n$ where $0 = x_0 < x_1 < .... < x_{n-1} < x_n = 5$. Let $\partial x = x_i - x_{i-1}$ for $i = 1, 2, ..., n$.

Then $\lim\limits_{\partial x \to 0} \sum\limits_{i=1}^{n} (2x_i \partial x)$ is equal to

A. $\int_5^0 2x\, dx$

B. $\int_0^5 x^2\, dx$

C. 25

D. 12.5

E. 10

Question 16

If $f'(x) = 5e^{2x}$ and c is a real constant, then $f(x)$ is equal to

A. $\frac{5}{2}e^{2x} + c$

B. $10e^{2x} + c$

C. $\frac{5}{2}e^{3x} + c$

D. $\frac{5}{2}e^{x^2} + c$

E. $5e^{x^2} + c$

Question 17

Let p be a function defined on the interval $[3,5]$ and q a function such that $q'(x) = p(x)$, for all $x \in [3,5]$

$\int_3^5 p(x)\,dx$ is equal to

A. $q(x)+c$

B. $p(5)-p(3)$

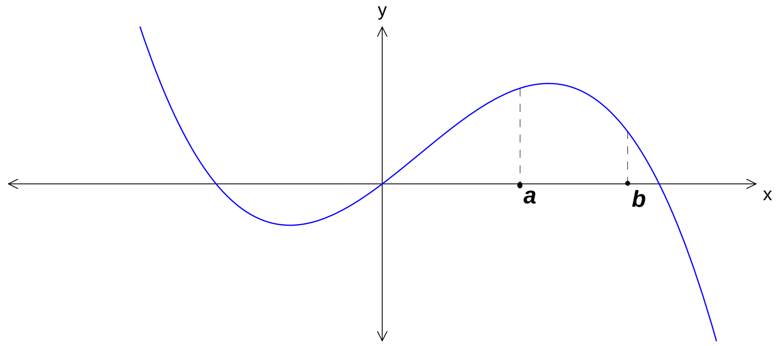
C. $q(5)-q(3)$

D. $q'(5)-q'(3)$

E. $p(x)+c$

Question 18

The graph of the function with equation $y = f(x)$ is shown below.



Let g be a function such that $g'(x) = f(x)$.

On the interval (a,b), the graph of g will have a

A. maximum turning point.

B. minimum turning point.

C. negative gradient.

D. positive gradient.

E. stationary point of inflection.

Question 19

The number of defective batteries in a box of batteries ready for sale is a random variable with a binomial distribution with mean 8.1 and standard deviation 0.9.

If a battery is drawn at random from the box, the probability that it is not defective is

A. 0.1

B. 0.9

C. 0.09

D. 0.3

E. 0.7

Question 20

Let X be a normally distributed random variable with mean μ and standard deviation σ.
Which one of the following is not always true?

A. $\Pr(X > \mu) = 0.5$

B. $\Pr(X > a) = 1 - \Pr(X < a)$

C. $\Pr(\mu - \sigma < X < \mu + \sigma) \approx 0.68$

D. $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$

E. $\Pr(a < X < b) = \Pr(X > b) - \Pr(X > a)$

Question 21

The random variable X has the following probability distribution.

X	1	2	3
$\Pr(X=x)$	a	b	0.1

If the mean of X is 1.3, then the values of a and b respectively are—

A. 0.5, 0.4

B. 0.8, 0.1

C. 0.1, 0.8

D. 0.3, 0.7

E. 0.4, 0.5

Question 22

Juicy Giant orange juice is packed in small glass bottles labelled as containing 250 *ml*. The packing process produces bottles that are normally distributed with a standard deviation of 3 *ml*. In order to guarantee that only 1% of bottles are under-volume, the actual mean volume, in *ml*, would be required to be closest to

A. 256

B. 257

C. 258

D. 250.01

E. 247

END OF SECTION 1
TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal answer will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise stated diagrams are not drawn to scale.

Question 1

Grace shops each day at one of two supermarkets, Costless and Spendway. Her choice of supermarket each day depends only on which supermarket she has shopped at on the previous day. If she shops at Costless one day then the chance of her shopping at Costless the next day is $\frac{3}{5}$.

The transition matrix for the probabilities of Grace shopping at either supermarket given the supermarket she shopped at on the previous day is

$$\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}$$

a. If she shops at Spendway one day, what is the chance she shops at Costless the next day?

1 mark

b. Suppose she shops at Costless one Friday.

What is the exact probability she shops at Costless the next Monday?

3 marks

c. In the long term, what percentage of days, to the nearest percent, will Grace shop at Spendway?

2 marks

The time, t in minutes, she spends in the supermarket is independent of the type of supermarket she shops at and is a random variable with probability density function

$$f(t) = \begin{cases} at(60-t) & if\ \ 0 \le t \le 60 \\ 0 & otherwise \end{cases}$$

d. Show that $a = \dfrac{1}{36000}$.

2 marks

e. What is the probability, correct to 3 decimal places, that Grace spends longer than 45 minutes shopping in the supermarket?

2 marks

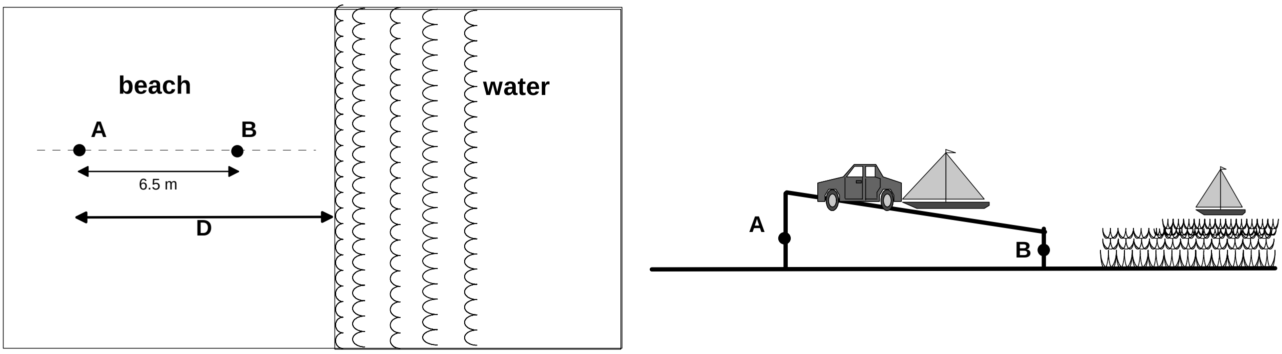
f. On 5% of occasions Grace spends between n and $2n$ minutes shopping. Find the value of n, correct to 3 decimal places.

2 marks
Total 12 marks

Question 2

Keith likes to holiday at Point Roadknight beach where the waves roll on to the beach at regular intervals.

The diagrams below show the beach with two markers, *A* and *B* at the start and end of the boat ramp, where *A* is 6.5 metres further up the beach than *B*. The line *AB* is perpendicular to the water's edge.

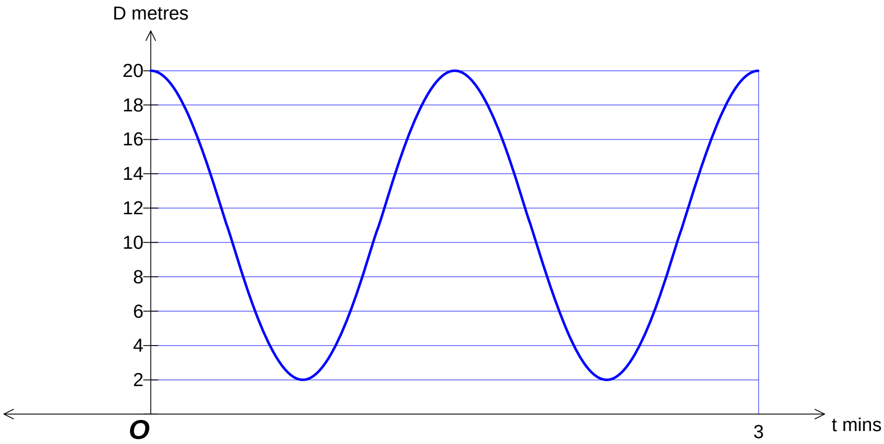


View from above **Side view**

Keith records the distance of the water's edge from the top marker *A*. He calculates that on one particular day the distance *D* metres of the water's edge from the top marker *A* is a function of time *t* (in minutes from when he starts to observe the waves). It can be modelled exactly by the equation

$$D = a\cos(bt) + c$$

where a, b and c are positive constants. The graph of *D* as a function of time *t* is shown below:



a. State the maximum and minimum distances of the water's edge from marker *A* on this day.

2 marks

b. Find the number of waves that hit the beach in one hour on this day.

2 marks

c. Find the values of *a*, *b* and *c*.

3 marks

d. **i.** Write down an equation, the solution of which gives the times when the marker at *B* is in the water on this day.

1 mark

ii. Find the exact percentage of time that the marker at B is in the water.

3 marks

Due to winds, tides and currents, on some days the waves come further up the beach and are closer together. Keith observes that on such a day the distance of the water's edge from marker A can be described by the equation

$D = (8 + S)\cos(\frac{8\pi t}{3}) + 11$ where S metres is the seasonal tidal factor which varies with the factors described above. S is normally distributed with a mean of 2 and a standard deviation of 0.3.

e. On a particular day the waves just reach the top of the boat ramp at marker A. Find the value of S on this day.

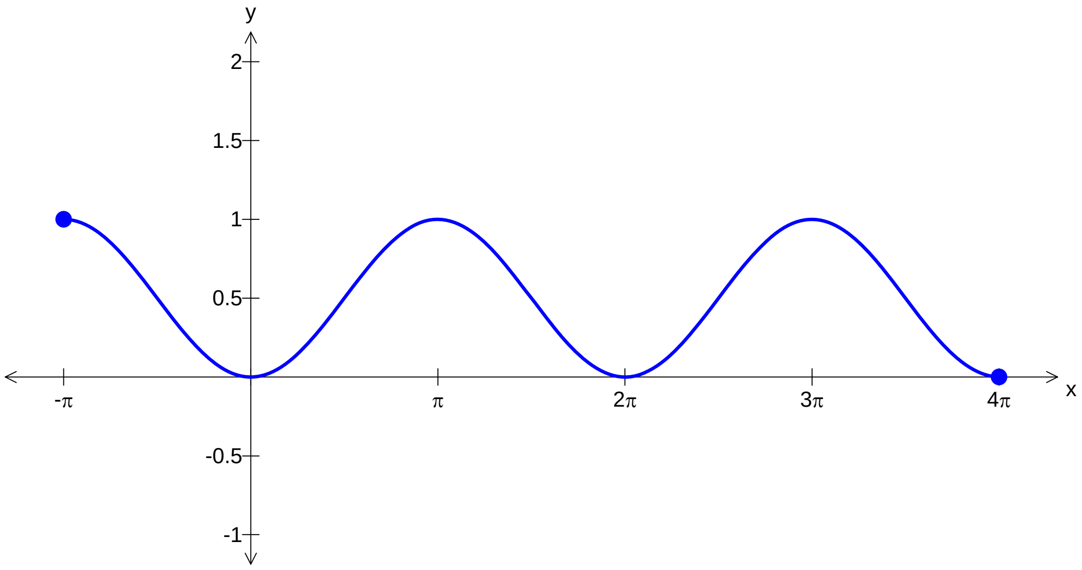
2 marks

f. Marcus decides to go fishing from the boat ramp at Point Roadknight beach, however he does not know the value of S. How many metres up the boat ramp from marker B should he stand so that he has an 80% chance of not getting wet from the waves? Answer correct to two decimal places.

3 marks
Total 16 marks

Question 3

The graph of $f:[-\pi,4\pi]\to R,\ f(x)=\sin^2\left(\frac{x}{2}\right)$ is shown below.



a. Find $f'(\frac{3\pi}{2})$ and $f'(\frac{5\pi}{2})$.

2 marks

b. **i.** Find the gradient of the normal to the curve at any point *x*.

2 marks

ii. Find exact coordinates of the points on the curve $y=f(x)$ where the gradient of the normal to the curve is equal to 2.

2 marks

iii. Find the exact equations of the **normals** to the curve at the points where $x = \frac{3\pi}{2}$ and $x = \frac{5\pi}{2}$.

2 marks

c. The graph of $y = f(x)$ is transformed to give the graph of $y = f(x) + b$. Find the exact value of *b*, such that the graph of the normal at $x = \frac{3\pi}{2}$ and the graph of the normal at $x = \frac{5\pi}{2}$ intersect on the *x*-axis.

2 marks

Let $g : [-\pi, 4\pi] \rightarrow R,\ g(x) = \sin^2\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) + 1$.

d. i. Find $g'(x)$.

1 mark

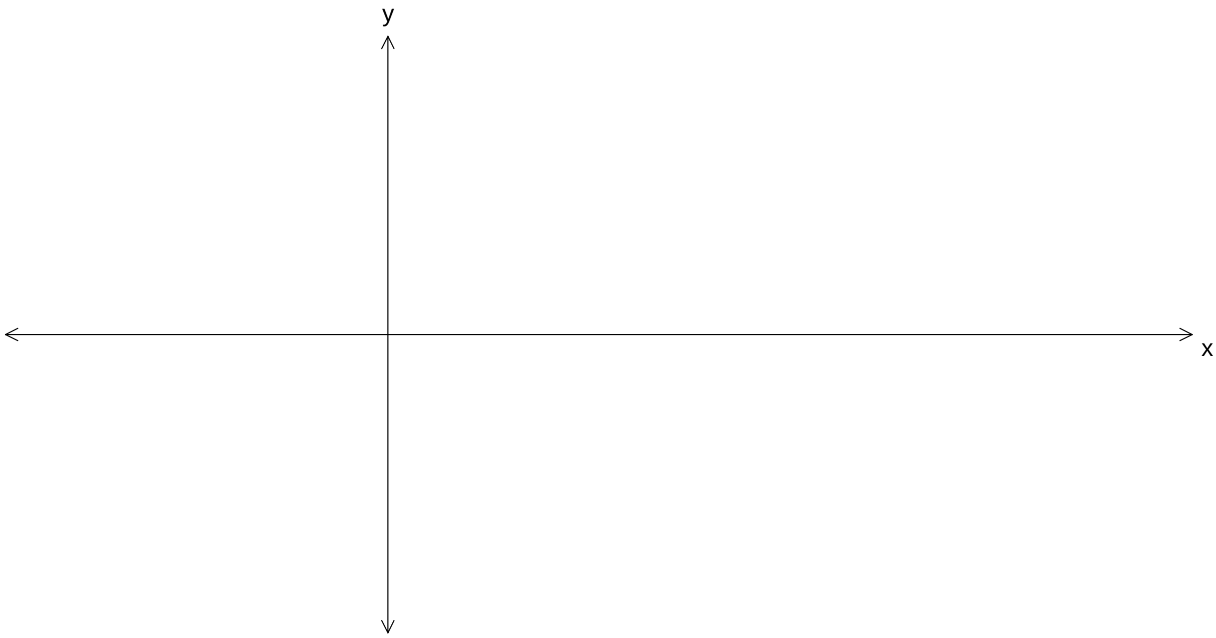
ii. Write down an equation, the solution of which gives the values of *x* for which the gradient is zero.

1 mark

iii. Solve the equation for $x \in [-\pi, 4\pi]$ to find the exact coordinates on the curve where the gradient is zero.

3 marks

iv. Sketch the graph of $y = g(x)$ on the axes below, labeling intercepts, stationary points and endpoints with exact coordinates.



2 marks
Total 17 marks

Question 4

Let $f: R \to R,\ f(x) = 9e^{-x}$.

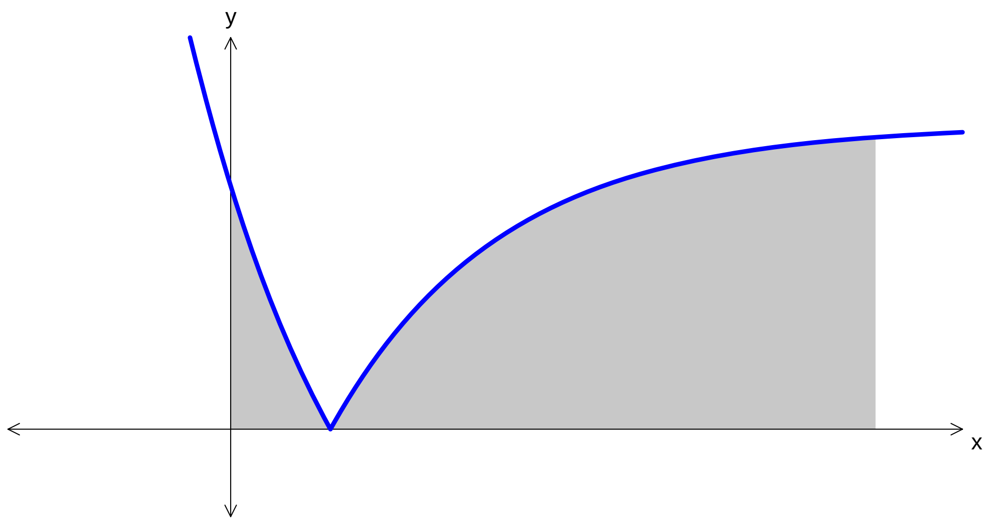
a. **i.** Show that $f(u)f(-u) = 81$, where u is a real number.

1 mark

ii. Show that $f(u+v) = \frac{1}{9}f(u)f(v)$, where u and v are real numbers.

2 marks

A panel of a stained glass window has a section of lead outline described by the graph with the equation $g(x) = \left|9e^{-x} - 5\right|$ and is shown below. The shaded area is red-coloured glass.



b. The x-intercept of the graph occurs at $x = \log_e(a)$. Find the exact value of a.

2 marks

c. The red-coloured glass is bounded by the area under the graph, the x-axis, the y-axis and the line $x = 4$. Find the exact area of the red-coloured glass.

3 marks

d. Find the value of a, correct to 3 decimal places, such that the area of the red-coloured glass enclosed by the graph , the x-axis, the y-axis and the line $x = a$ is equal to 35 square units.

2 marks

e. In producing the leadlight panel, the craftsman creates a sketch of the red-coloured panel and notices when positioning the curve that two points on the curve describing the red-coloured glass are exactly 1 unit horizontally apart. Find the coordinates of the two points, correct to 3 decimal places.

3 marks
Total 13 marks

END OF QUESTION AND ANSWER BOOK

INSIGHT

Trial Exam Paper

2009

MATHEMATICAL METHODS (CAS)

Written examination 2

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

SECTION 1

Question 1

The function $f:[0,2\pi]\to R,\ f(x)=1-3\sin(2x-\frac{\pi}{2})$ has a range and period respectively of

A. $[0,3]$ and 2

B. $[-1,3]$ and 2π

C. $\boldsymbol{[-2,4]}$ **and** $\boldsymbol{\pi}$

D. $[-3,3]$ and π

E. $[-2,4]$ and 2π

Answer is C.

Solution

- The lowest value of the function is $1-3=-2$.
- The highest value of the function is $1+3=4$.
- The period of the function is given by $\frac{2\pi}{n}=\frac{2\pi}{2}=\pi$.

Question 2

The range of the function $f(x)=e^{|x|}-3$ is

A. $\boldsymbol{[-2,\infty)}$

B. R^+

C. $(-2,0]$

D. $(-2,0)$

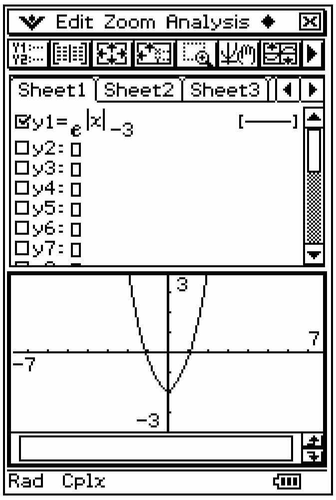
E. $(-3,\infty)$

Answer is A.

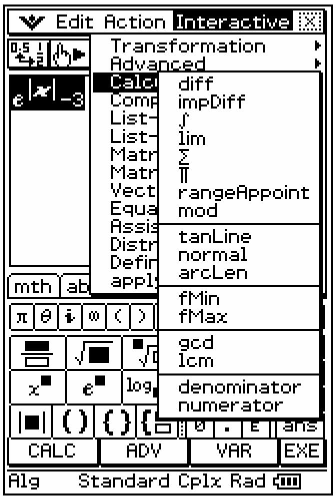
SECTION 1 – Question 2 – continued

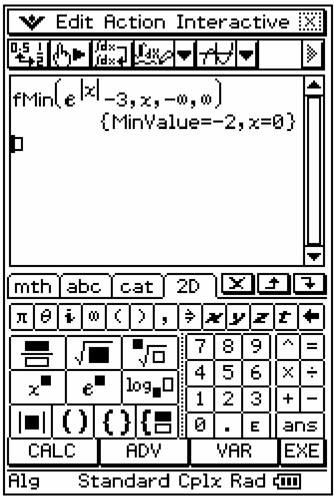
Solution

The graph of the function looks like—



and finding the minimum value of the function gives

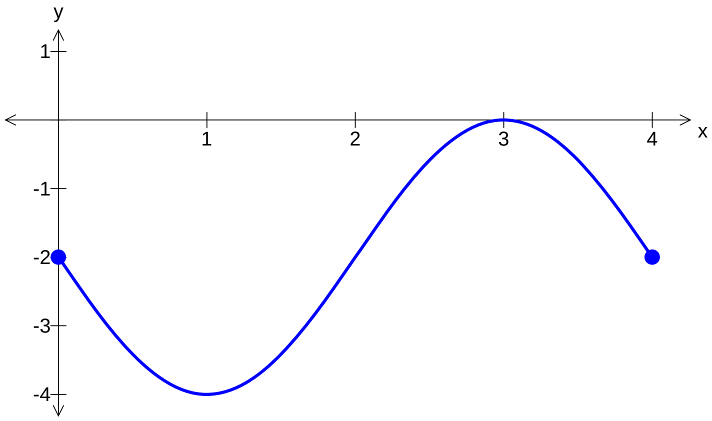




The lowest value of the function occurs at the y-intercept (0, -2), so the range is $[-2, \infty)$.

Question 3

The diagram below shows one cycle of the graph of a circular function.



A possible equation for the function whose graph is shown is

A. $y = 2 - 2\sin(4x)$

B. $y = -2 - 2\sin(4x)$

C. $y = 4 - 2\sin(4\pi x)$

D. $\boldsymbol{y = -2 - 2\sin(\frac{\pi}{2}x)}$

E. $y = -2 - 2\sin(\frac{1}{2}x)$

Answer is D.

Solution

The graph shows an upside down sin curve with an amplitude of 2, a centre value of -2 and a period of 4 units (i.e. $4 = \frac{2\pi}{n}$, so $n = \frac{\pi}{2}$).

Question 4

One cycle of the graph of the function with the equation $y = \tan(ax)$ has successive vertical asymptotes at $x = \frac{3}{8}$ and $x = \frac{5}{8}$.

A possible value for a is

A. 2
B. 2π
C. 4
D. $\boldsymbol{4\pi}$
E. 8

Answer is D.

Solution

The difference between the asymptotes is $\frac{2}{8} = \frac{1}{4}$ units, meaning the period is $\frac{1}{4}$. The period of a tan function, $y = \tan(bx)$ is given by $\frac{\pi}{b}$ so $a = 4\pi$ units.

Question 5

For the system of simultaneous linear equations

$$z = -3$$
$$x - y = 5$$
$$x + y = 1$$

an equivalent matrix equation is

A. $\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} z \\ y \\ x \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \mathbf{5} \\ \mathbf{1} \\ \mathbf{-3} \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$

E. $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$

Answer is B.

Solution

When matrix in B is multiplied out it gives

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$$

$$x - y = 5$$
$$x + y = 1$$
$$z = -3$$

which corresponds to the equations stated.

Question 6

The transformation $T : R^2 \to R^2$ which maps the curve with the equation $y = e^x$ to the curve with the equation $y = e^{2x-4} + 3$ could have the rule

A. $\boldsymbol{T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

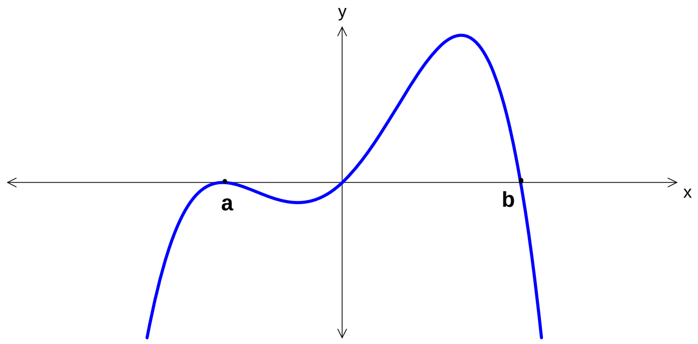
Answer is A.

Solution

Writing the final equation $y = e^{2x-4} + 3$ as $y = e^{2(x-2)} + 3$ shows the equation has undergone a dilation of factor 0.5 from the *x*-axis, and a translation of +2 units in the positive *x*-direction and +3 units in the positive *y*-direction.

The matrix $\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$ produces the dilation of factor 0.5 from the *x*-axis and the matrix $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ produces the translations.

Question 7



The graph shown could be that of a function f with the equation

A. $y = -x(x+a)^2(x-b)$

B. $y = -x(x+a)(x-b)$

C. $y = x(x-a)^2(x-b)$

D. $y = -x(x+a)^2(b-x)$

E. $\boldsymbol{y = x(x-a)^2(b-x)}$

Answer is E.

Solution

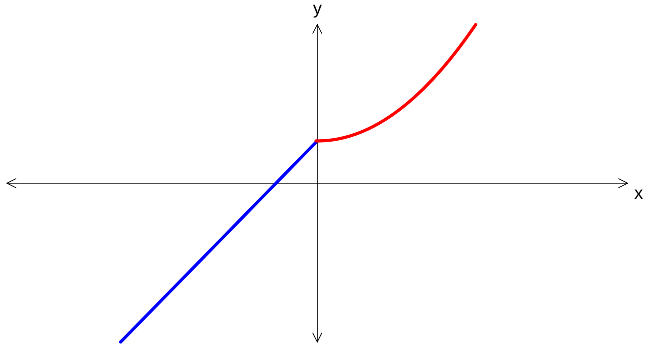
As both ends of the graph point in the same direction (in this case, down) this means the overall power of the equation is an even power, so alternative B is out.

Both ends point down so it has to be a $-x^{even}$ so alternatives C and D are out.

The graph has x-intercepts at $(a,0)$ *and* $(b,0)$ so this means that the equation must have factors of $(x-a)$ *and* $(x-b)$ $(or\ (b-x))$ — alternative E has this.

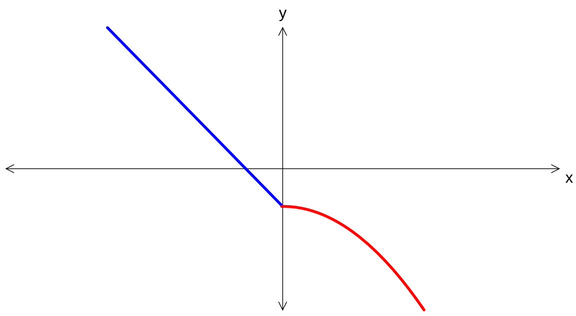
Question 8

The graph of the function with the equation $y = f(x)$ is shown below.

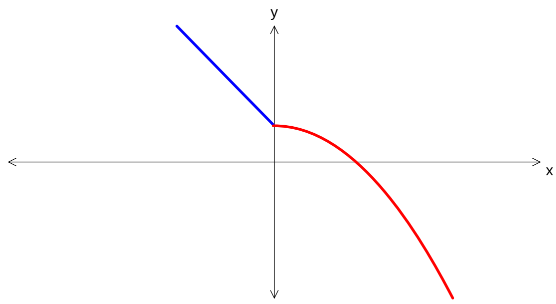


Which of the following is most likely to be the graph of the inverse function?

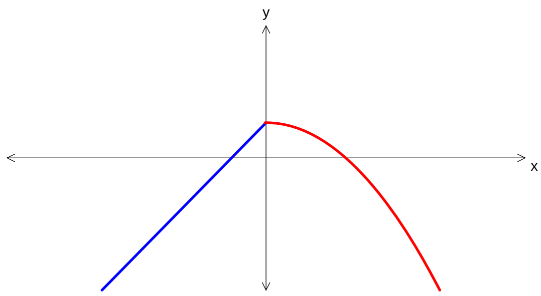
A.



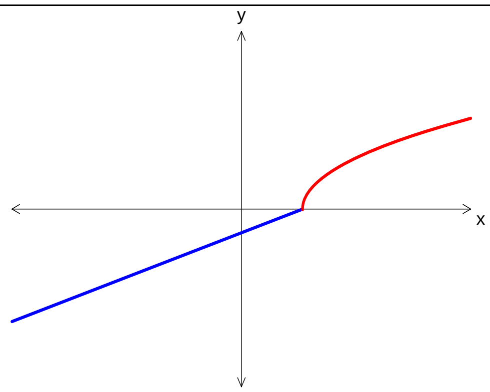
B.



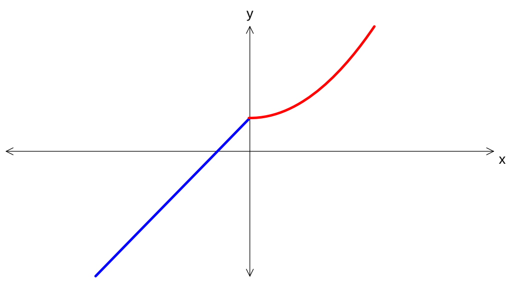
C.



D.



E.



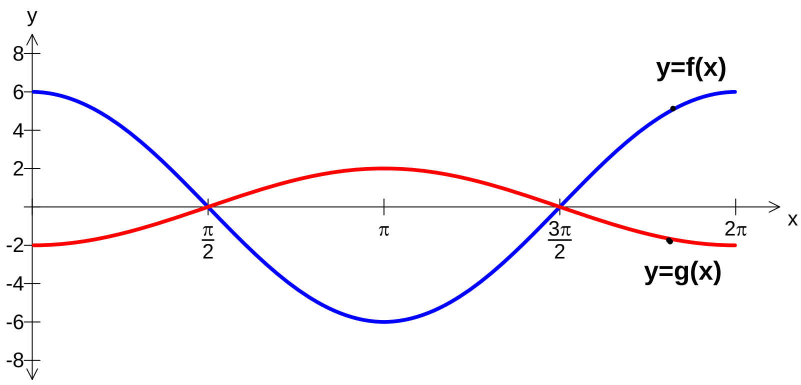
Answer is D.

Solution

The inverse function is the reflection of the graph of the function in the line $y = x$.

Question 9

The diagram below shows the graphs of two circular functions, *f* and *g*.



The graph of the function with the equation $y = f(x)$ is transformed into the graph of the function with the equation $y = g(x)$ by

A. a dilation by a scale factor of $\frac{1}{3}$ from the *y*- axis followed by a reflection in the *x*-axis.

B. a dilation by a scale factor of 3 from the *y*- axis followed by a reflection in the *x*-axis.

C. a dilation by a scale factor of 3 from the *y*- axis followed by a reflection in the *y*-axis.

D. **a dilation by a scale factor of $\frac{1}{3}$ from the *x*- axis followed by a reflection in the *x*-axis.**

E. a dilation by a scale factor of 3 from the *x*- axis followed by a reflection in the *y*-axis.

Answer is D.

Solution

Graph has been reflected in x-axis and the amplitude has decreased from 6 to 2, i.e. by a factor of $\frac{1}{3}$ in the y-direction or from the x-axis.

Question 10

The function defined by $f : A \to R,\ f(x) = e^{(x-b)^2}$, $b \in R$, will have an inverse function for all values of *b*, if its domain *A* is

A. R

B. $R \setminus \{b\}$

C. $[\boldsymbol{b}, \boldsymbol{\infty})$

D. R^+

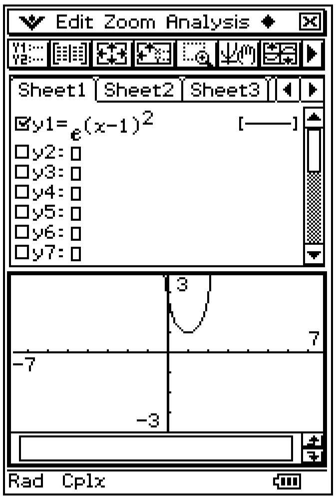
E. $[-b, \infty)$

Answer is C.

Solution

To have an inverse function, the function needs to be one-to-one.

A graph of one member of the family of curves with $b = 1$, shows



with the graph being symmetrical about $x = 1$, so the graph is one-to-one for either $(-\infty, b]$ *or* $[b, \infty)$ – therefore the answer is alternative C.

Question 11

If $f(x) = e^{2x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ then $g(f(1))$ is

A. $\dfrac{\mathbf{1}}{\boldsymbol{e}}$

B. $-e$

C. $\sqrt{e}$

D. 1

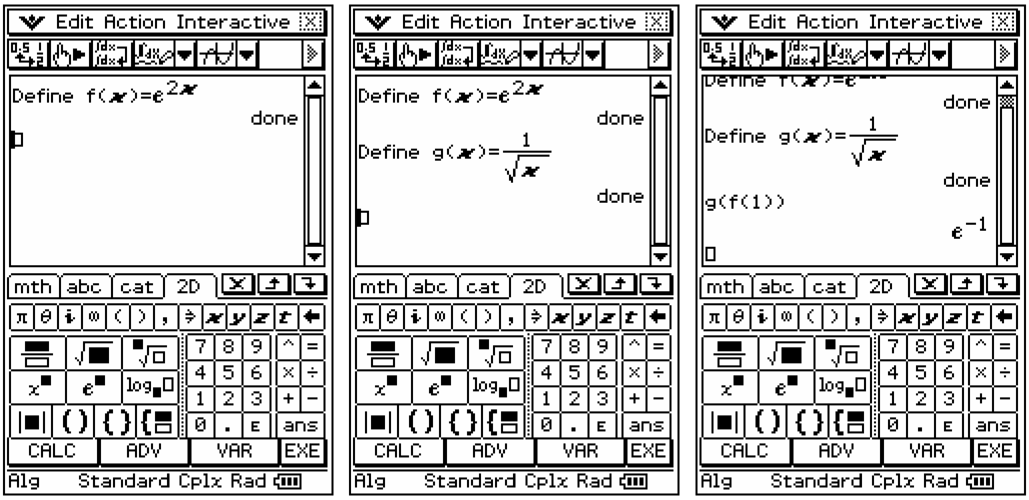
E. –1

Answer is A.

Solution

The composite function $g(f(x))$ is formed by replacing the x term in $g(x)$ with the function of $f(x)$. So $g(f(x)) = \dfrac{1}{\sqrt{e^{2x}}}$, therefore $g(f(1)) = \dfrac{1}{\sqrt{e^2}} = \dfrac{1}{e}$.

Using CAS



Question 12

If $y = \log_e(\sin(3x))$ then $\dfrac{dy}{dx}$ is equal to

A. $\mathbf{\dfrac{3\cos(3x)}{\sin(3x)}}$

B. $\dfrac{1}{\sin(3x)}$

C. $\dfrac{1}{3\cos(3x)}$

D. $3\tan(3x)$

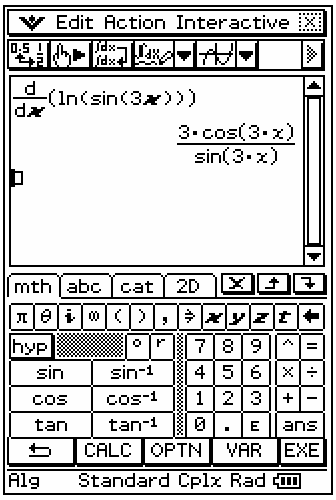
E. $-3\tan(3x)$

Answer is A.

Solution

Using the chain rule $\dfrac{d}{dx}(\log_e(\sin(3x))) = \dfrac{1}{\sin(3x)} \times 3\cos(3x) = \dfrac{3\cos(3x)}{\sin(3x)}$

Using CAS gives



Question 13

The equation of the normal to the curve with equation $y = x\cos(x)$, at the point on the curve with x-coordinate 2π is

A. $y = \dfrac{1}{x\sin(x)}$

B. $y = -x$

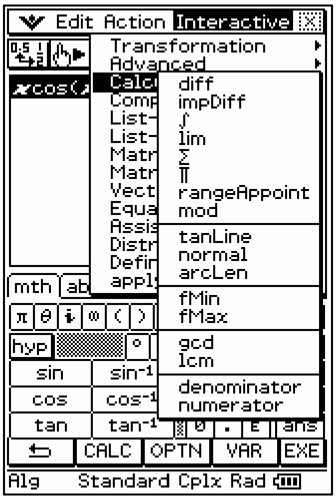
C. $y = x$

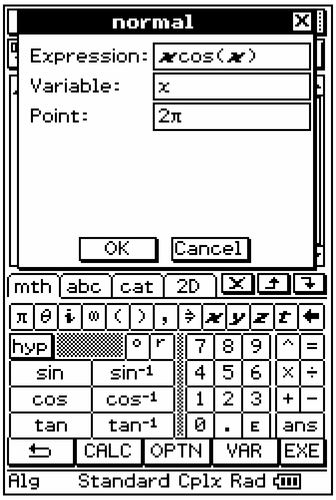
D. $\boldsymbol{y = -x + 4\pi}$

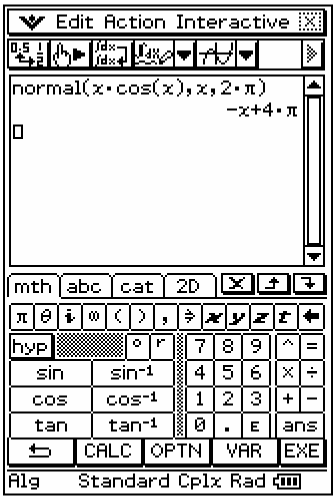
E. $y = x - 4\pi$

Answer is D.

Solution







Question 14

If $y = 2e(e^x - 1)$ then the rate of change of y with respect to x when $x = 0$ is

A. $2e^2$

B. 0

C. $2e - 1$

D. $2e^2 - 2e$

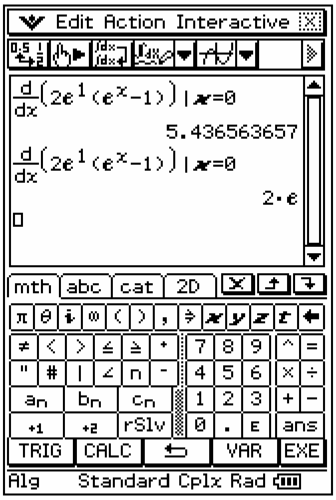
E. $\boldsymbol{2e}$

Answer is E.

Solution

The rate of change means $\dfrac{dy}{dx}$.

Using CAS gives



Therefore alternative E is equal to $\dfrac{dy}{dx}$.

Question 15

The interval $[0,5]$ is divided into n equal subintervals by the points $x_0, x_1, ...x_{n-1}, x_n$ where $0 = x_0 < x_1 < .... < x_{n-1} < x_n = 5$. Let $\partial x = x_i - x_{i-1}$ for $i = 1, 2, ..., n$.

Then $\lim\limits_{\partial x \to 0} \sum\limits_{i=1}^{n} (2x_i \partial x)$ is equal to

A. $\int_5^0 2x \, dx$

B. $\int_0^5 x^2 \, dx$

C. **25**

D. 12.5

E. 10

Answer is C.

Solution

The statement defines the limiting process for the definite integral of $f(x)$ where the interval $[a, b]$ is divided into n subintervals and this limiting process can be expressed as $\int_a^b f(x) \, dx = \lim\limits_{\partial x \to 0} \sum\limits_{i=1}^{n} (f(x_i) \partial x)$. In this case $f(x) = 2x$ and the interval is $[0, 5]$.

So $\lim\limits_{\partial x \to 0} \sum\limits_{i=1}^{n} (2x_i \partial x) = \int_0^5 (2x) \, dx = \left[x^2\right]_0^5 = 25 - 0 = 25$.

Question 16

If $f'(x) = 5e^{2x}$ and c is a real constant, then $f(x)$ is equal to

A. $\boldsymbol{\frac{5}{2} e^{2x} + c}$

B. $10e^{2x} + c$

C. $\frac{5}{2} e^{3x} + c$

D. $\frac{5}{2} e^{x^2} + c$

E. $5e^{x^2} + c$

Answer is A.

Solution

The antiderivative of e^{ax} is $\frac{1}{a} e^{ax}$. So the antiderivative of $5e^{2x}$ is $\frac{5}{2} e^{2x}$ +c.

Question 17

Let p be a function defined on the interval $[3, 5]$ and q a function such that $q'(x) = p(x)$, for all $x \in [3, 5]$

$\int_3^5 p(x)\,dx$ is equal to

A. $q(x) + c$

B. $p(5) - p(3)$

C. $q(5) - q(3)$

D. $q'(5) - q'(3)$

E. $p(x) + c$

Answer is C.

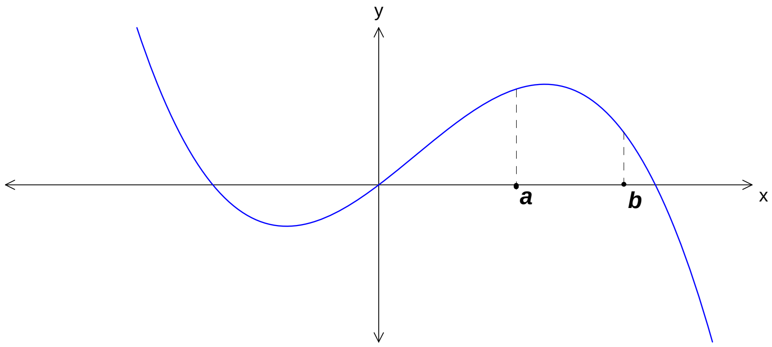
Solution

Need to replace $p(x)$ with $q'(x)$ to get

$$\begin{aligned}\int_3^5 p(x)\ dx &= \int_3^5 q'(x)\ dx \\ &= [q(x)]_3^5 \\ &= q(5) - q(3)\end{aligned}$$

Question 18

The graph of the function with equation $y = f(x)$ is shown below.



Let g be a function such that $g'(x) = f(x)$.

On the interval (a, b), the graph of g will have a

A. maximum turning point.

B. minimum turning point.

C. negative gradient.

D. **positive gradient.**

E. stationary point of inflection.

Answer is D.

Solution

The graph drawn is the graph of the derivative, i.e. the graph of the gradient function. The graph has a positive value over this domain and does not have an x-intercept or change signs (therefore no turning points and stationary points of inflexion over this interval).

Question 19

The number of defective batteries in a box of batteries ready for sale is a random variable with a binomial distribution with mean 8.1 and standard deviation 0.9.

If a battery is drawn at random from the box, the probability that it is not defective is

A. **0.1**

B. 0.9

C. 0.09

D. 0.3

E. 0.7

Answer is A.

Solution:

The distribution is binomial so the mean is given by $\mu = np$ and the standard deviation is $\sigma = \sqrt{npq}$ so

so

$8.1 = np$

$0.9 = \sqrt{npq} \quad \Rightarrow 0.81 = npq$

$\Rightarrow 0.81 = 8.1q$ replacing np with 8.1

$\Rightarrow q = 0.1$ and as $p = 1 - q,\ p = 0.9$

so the probability of a defective battery is 0.9 and a non-defective battery 0.1.

Question 20

Let X be a normally distributed random variable with mean μ and standard deviation σ. Which one of the following is not always true?

A. $\Pr(X > \mu) = 0.5$

B. $\Pr(X > a) = 1 - \Pr(X < a)$

C. $\Pr(\mu - \sigma < X < \mu + \sigma) \approx 0.68$

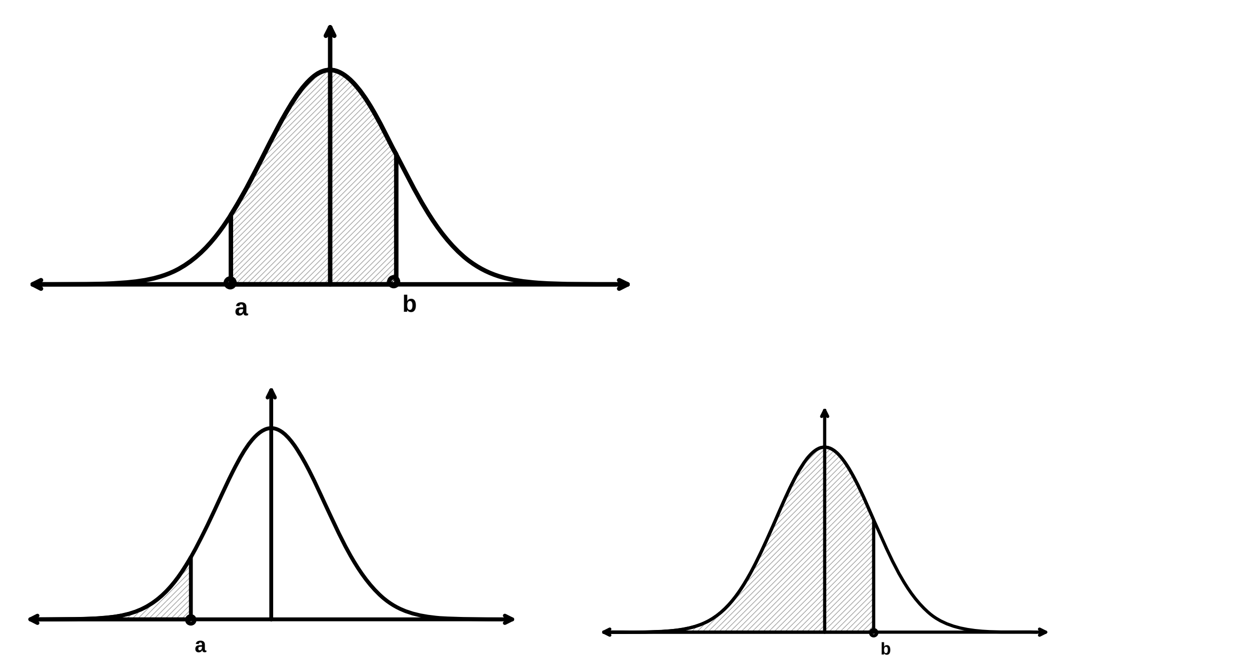
D. $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$

E. $\boldsymbol{\Pr(a < X < b) = \Pr(X > b) - \Pr(X > a)}$

Answer is E.

Solution

$\Pr(a < X < b) = \Pr(X < b) - \Pr(X < a)$ and this can be seen from a series of normal distribution diagrams.



Question 21

The random variable X has the following probability distribution.

x	1	2	3
$\Pr(X=x)$	a	b	0.1

If the mean of X is 1.3, then the values of a and b respectively are—

A. 0.5, 0.4
B. **0.8, 0.1**
C. 0.1, 0.8
D. 0.3, 0.7
E. 0.4, 0.5

Answer is B.

Solution

Since the distribution is a pdf the sum of the probabilities is 1.

i.e.,

$a + b + 0.1 = 1$

$a + b = 0.9$

The mean of the pdf

$E(X) = \sum xp(x) = 1\times a + 2\times b + 3\times 0.1 = a + 2b + 0.3 = 1.3$

$so \;\; a + 2b = 1$

solving simultaneously gives $b = 0.1 \; and \; a = 0.8$

SECTION 1 – continued

Question 22

Juicy Giant orange juice is packed in small glass bottles labelled as containing 250 *ml*. The packing process produces bottles that are normally distributed with a standard deviation of 3 *ml*. In order to guarantee that only 1% of bottles are under-volume, the actual mean volume, in *ml*, would be required to be closest to

A. 256

B. **257**

C. 258

D. 250.01

E. 247

Answer is B.

Solution

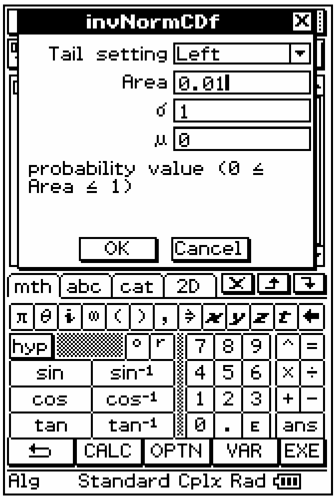
$X \sim N(\mu, \sigma = 3)$

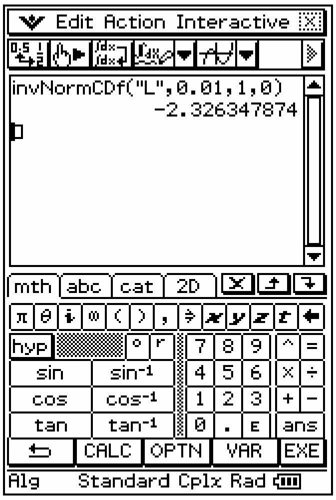
$\Pr(X < 250) = 0.01$

converting to standard normal gives

$$\Pr(Z < \frac{250-\mu}{3}) = 0.01$$

Using CAS find the invnorm of the standard normal distribution to get—





This means—

$$\frac{250-\mu}{3} = -2.326$$

$250 - \mu = -6.978$

$\mu = 256.979$

END OF SECTION 1
TURN OVER

SECTION 2

Question 1

Grace shops each day at one of two supermarkets, Costless and Spendway. Her choice of supermarket each day depends only on which supermarket she has shopped at on the previous day. If she shops at Costless one day then the chance of her shopping at Costless the next day is $\frac{3}{5}$.

The transition matrix for the probabilities of Grace shopping at either supermarket given the supermarket she shopped at on the previous day is

$$\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}$$

a. If she shops at Spendway one day, what is the chance she shops at Costless the next day?

1 mark

Solution

The transition matrix shows

$$\begin{array}{c c} & \begin{array}{cc} C & S \end{array} \\ \begin{array}{c} C \\ S \end{array} & \begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix} \end{array} \longleftarrow \text{Current state}$$

Next state $\longrightarrow$

so the $\Pr(S \text{ then } C) = \frac{1}{3}$.

Mark allocation

- 1 answer mark for answer of $\frac{1}{3}$.

b. Suppose she shops at Costless one Friday.

What is the exact probability she shops at Costless the next Monday?

3 marks

Solution

Using a matrix method here will be easier than using a tree diagram.

$$T^3\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\frac{3}{5} & \frac{1}{3}\\ \frac{2}{5} & \frac{2}{3}\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix}$$

$$=\begin{bmatrix}\frac{523}{1125}\\ \frac{602}{1125}\end{bmatrix}$$

so the $\Pr(\textit{costless on Mon})=\frac{523}{1125}$

Mark allocation

- 1 method mark for using matrices and setting up.
- 1 answer mark for the resulting matrix $\begin{bmatrix}\frac{523}{1125}\\ \frac{602}{1125}\end{bmatrix}$
- 1 answer mark for $\frac{523}{1125}$.

c. In the long term, what percentage of days, to the nearest percent, will Grace shop at Spendway?

2 marks

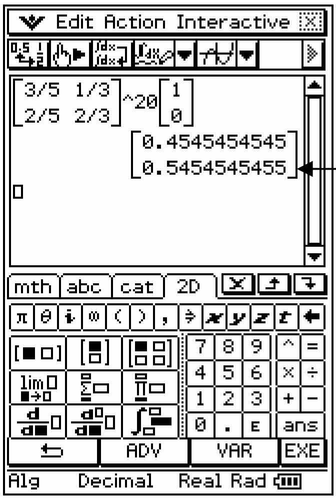
Solution

In a two-state markov chain with a transition matrix of $\begin{bmatrix}1-a & b\\ a & 1-b\end{bmatrix}$

the steady state matrix is given by $\begin{bmatrix}\frac{b}{a+b}\\ \frac{a}{a+b}\end{bmatrix}$.

In this case $\Pr(\textit{long-term Spendway})=\frac{a}{a+b}=\frac{\frac{2}{5}}{\frac{2}{5}+\frac{1}{3}}=\frac{6}{11}=54.54\%=55\%$.

Alternatively consider $\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and let n become large, say 20. This gives



Pr(*spendway*)

Mark allocation

- 1 method mark for valid method.
- 1 answer mark for correct answer.

The time, *t* in minutes, she spends in the supermarket is independent of the type of supermarket she shops at and is a random variable with probability density function

$$f(t) = \begin{cases} at(60-t) & if \;\; 0 \le t \le 60 \\ 0 & otherwise \end{cases}$$

d. Show that $a = \dfrac{1}{36000}$

2 marks

Solution

To be a probability density function the area under the graph needs to equal 1.

i.e. $\int_0^{60} at(60-t)\ dt \ = 1$

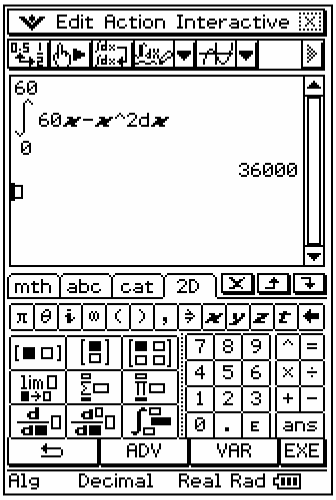
so set

$$\int_0^{60} at(60-t)\ dt \ = 1$$

$$LHS = \int_0^{60} at(60-t)\ dt \ = a\int_0^{60} t(60-t)\ dt$$

$$= a\int_0^{60} (60t - t^2)\ dt$$

$\int_0^{60} (60t - t^2)\ dt \ = 36000$ using CAS



so

$$\Rightarrow a\int_0^{60} (60t - t^2)\ dt \ = a \times 36000$$

$$36000a = 1$$

$$\Rightarrow a = \frac{1}{36000}$$

Mark allocation

- 1 method mark for setting the integral with correct terminals and dt equal to 1.
- 1 answer mark for steps leading to a correct answer (note: no need to show the antidifferentiating steps).

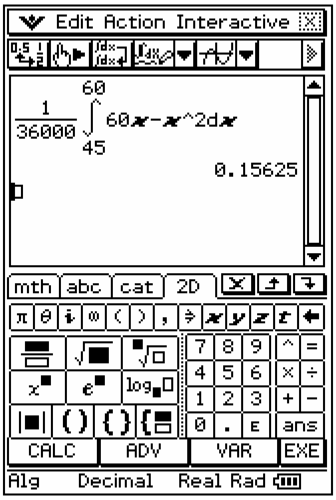
e. What is the probability, correct to 3 decimal places, that Grace spends longer than 45 minutes shopping in the supermarket?

2 marks

Solution

$$\Pr(\textit{more than } 45 \text{ min} s) = \int_{45}^{60} \frac{1}{36000} t(60-t)\, dt$$

using the CAS calculator this integral equals $0.15625 = 0.156$ correct to 3 decimal places.



Mark allocation

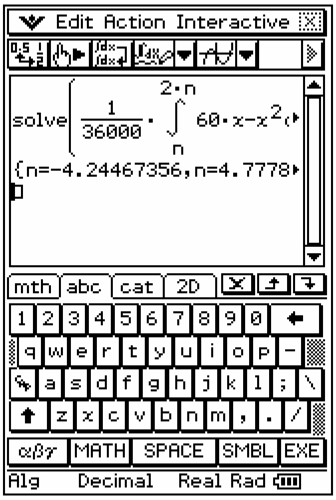
- 1 method mark for setting up the integral (terminals correct and dt).
- 1 answer mark for correct answer.

f. On 5% of occasions Grace spends between n and $2n$ minutes shopping. Find the value of n, correct to 3 decimal places.

2 marks

Solution

Set up $\dfrac{1}{36000}\displaystyle\int_{n}^{2n} x(60-x)\,dx = 0.05$ and solve for n using CAS



this gives 3 solutions—only one is valid within the domain of [0, 60].

$n = 4.778$

Mark allocation

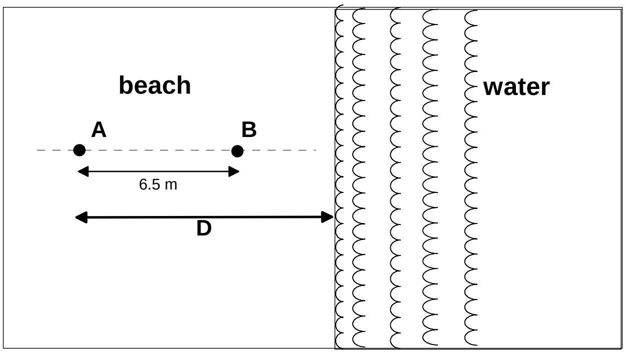
- 1 method mark for setting up the integral.
- 1 answer mark for the correct value of n, correct to 3 decimal places.

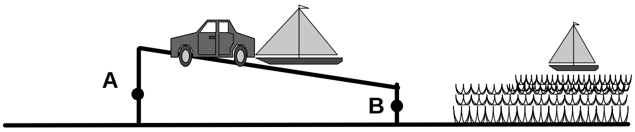
Total 12 marks

Question 2

Keith likes to holiday at Point Roadknight beach where the waves roll on to the beach at regular intervals.

The diagrams below show the beach with two markers, *A* and *B* at the start and end of the boat ramp, where *A* is 6.5 metres further up the beach than *B*. The line *AB* is perpendicular to the water's edge.



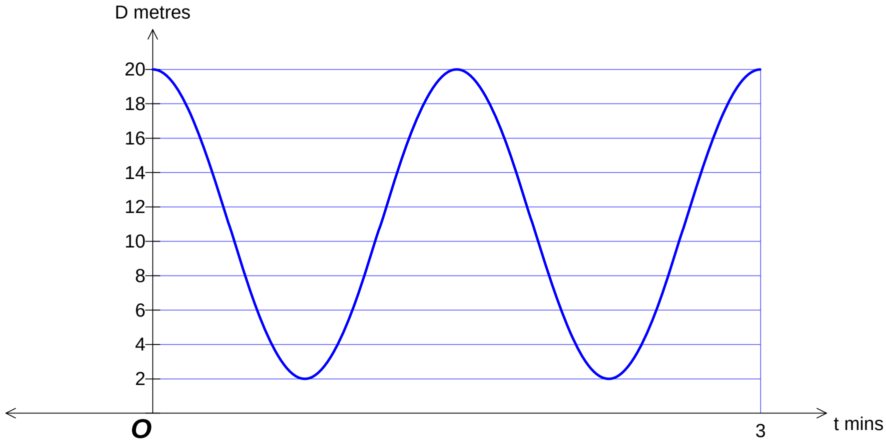


View from above **Side view**

Keith records the distance of the water's edge from the top marker *A*. He calculates that on one particular day the distance *D* metres of the water's edge from the top marker *A* is a function of time *t* (in minutes from when he starts to observe the waves). It can be modelled exactly by the equation

$$D = a\cos(bt) + c$$

where a, b and c are positive constants. The graph of *D* as a function of time *t* is shown below:



a. State the maximum and minimum distances of the water's edge from marker *A* on this day.

2 marks

Solution

The maximum distance from A is 20 metres and the minimum is 2 metres.

Mark allocation

- 1 answer mark for maximum.
- 1 answer mark for minimum.

b. Find the number of waves that hit the beach in one hour on this day.

2 marks

Solution

There are 2 waves in 3 minutes, so in

$$\begin{aligned} 60 \ mins &= 20 \times 3 \ mins \\ &= 20 \times 2 \ waves \\ &= 40 \ \ waves \end{aligned}$$

Mark allocation

- 1 method mark for a recognizing that there are 2 waves in 3 minutes or calculating the period.
- 1 mark for correct answer.

c. Find the values of *a*, *b* and *c*.

3 marks

Solution

Graph is centred around 11, so $c = 11$.

The amplitude is 9 units so $a = 9$.

There are 2 waves in 3 minutes so the period is $\frac{3}{2} = \frac{2\pi}{n}$, so $b = \frac{4\pi}{3}$.

Mark allocation

- 1 answer mark for a.
- 1 answer mark for b.
- 1 answer mark for c.

d. **i.** Write down an equation, the solution of which gives the times when the marker at *B* is in the water on this day.

1 mark

Solution

Marker *B* is in the water when the water is less than 6.5 metres from *A* i.e. when $D < 6.5$, so $9\cos\frac{4\pi t}{3} + 11 < 6.5$ is the equation that gives the times when marker *B* is in the water.

Mark allocation

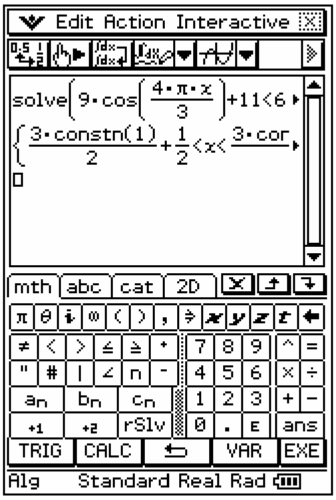
- 1 answer mark for answer.

ii. Find the exact percentage of time that the marker at B is in the water.

3 marks

Solution

Solve $9\cos\frac{4\pi t}{3} + 11 < 6.5$ using CAS.



Interpreting the general solutions gives $\frac{1}{2} < t < 1$ for the first cycle.

So the marker is in the water for between $\frac{1}{2} < t < 1$ minutes, or for $\frac{1}{2}$ minutes in total.

$\frac{1}{2}$ minutes each cycle means $\frac{\frac{1}{2}}{\frac{3}{2}} \times 100\% = 33\frac{1}{3}\%$.

Mark allocation

- 1 method mark for finding $\frac{1}{2} < t < 1$.
- 1 answer mark for $\frac{1}{2}$ minutes in total per cycle.
- 1 answer mark for correct percentage.

Due to winds, tides and currents, on some days the waves come further up the beach and are closer together. Keith observes that on such a day the distance of the water's edge from marker A can be described by the equation

$D = (8+S)\cos(\frac{8\pi t}{3}) + 11$ where S metres is the seasonal tidal factor which varies with the factors described above. S is normally distributed with a mean of 2 and a standard deviation of 0.3.

e. On a particular day the waves just reach the top of the boat ramp at marker A. Find the value of S on this day.

2 marks

Solution

To just reach the marker at A, $D = 0$ so this means the amplitude $8 + S = 11$ which gives $S = 3$.

Mark allocation

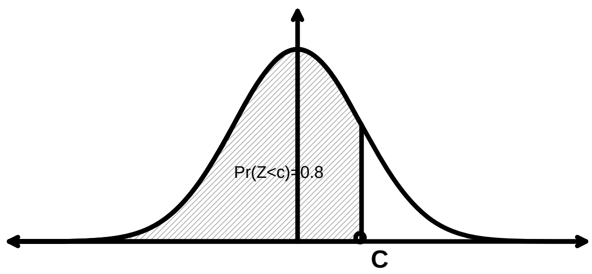
- 1 method mark for recognizing amplitude equals 11.
- 1 answer mark for correct answer.

f. Marcus decides to go fishing from the boat ramp at Point Roadknight beach, however he does not know the value of S. How many metres up the boat ramp from marker B should he stand so that he has an 80% chance of not getting wet from the waves? Answer correct to two decimal places.

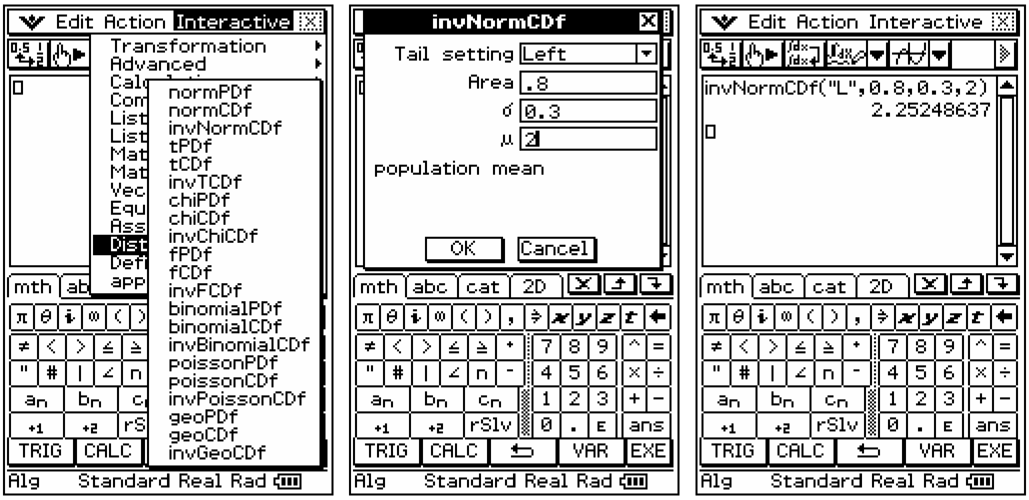
3 marks

Solution

$\mu = 2$ *and* $\sigma = 0.3$



Using $invnorm(0.8,\ 2,\ 0.3)$ gives $S = 2.2525$



with $S = 2.2525$ the water goes to within 0.7475 metres of marker A, so Marcus should position himself 5.7525 = 5.75 metres up from marker B.

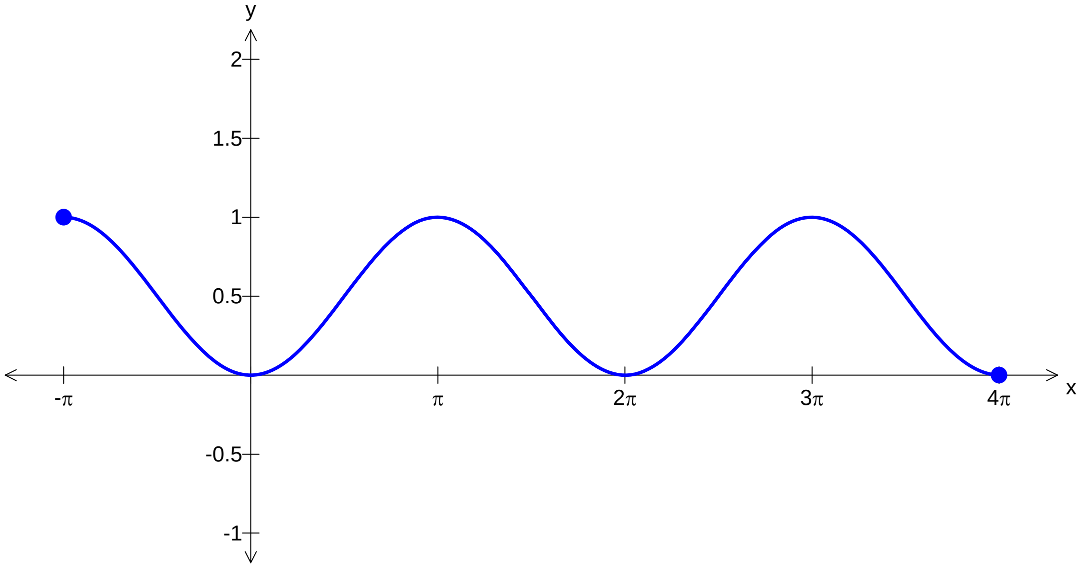
Mark allocation

- 1 method mark for using invnorm.
- 1 answer mark for 0.7475 m from marker A.
- 1 answer mark for 5.75 m from marker B.

Total 16 marks

Question 3

The graph of $f:[-\pi,4\pi]\to R,\ f(x)=\sin^2\left(\frac{x}{2}\right)$ is shown below.

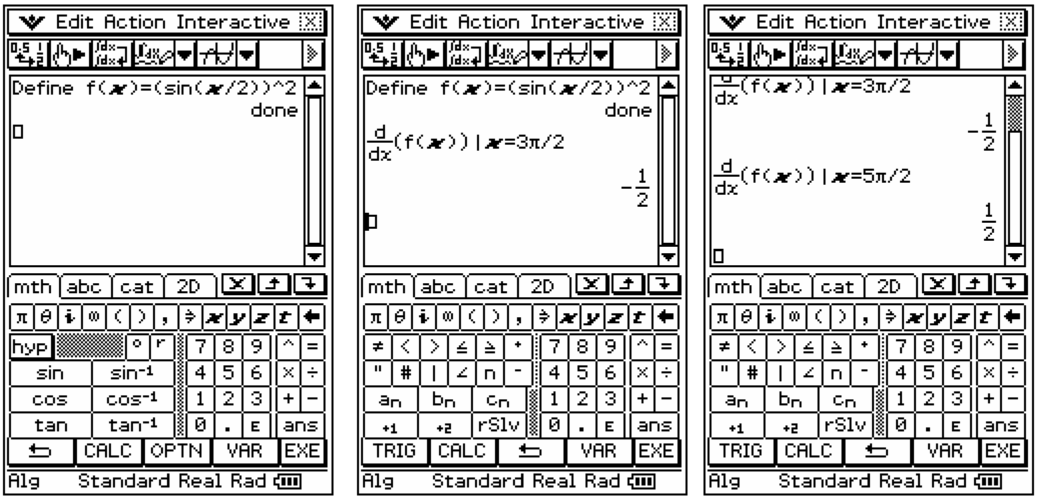


a. Find $f'(\frac{3\pi}{2})$ and $f'(\frac{5\pi}{2})$.

2 marks

Solution

Define the function $f(x)=\sin^2\left(\frac{x}{2}\right)$ and then using CAS find $f'(\frac{3\pi}{2})$ and $f'(\frac{5\pi}{2})$.



Mark allocation

- 1 answer mark for $f'(\frac{3\pi}{2})$.
- 1 answer mark for $f'(\frac{5\pi}{2})$.

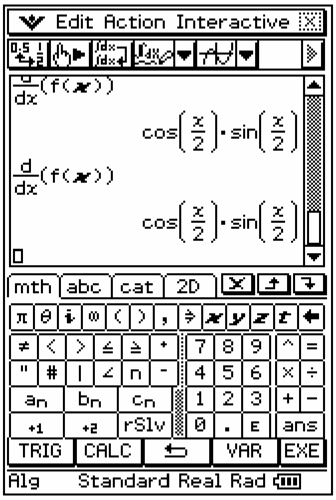
b. i. Find the gradient of the normal to the curve at any point *x*.

2 marks

Solution

$$m_{normal} = \frac{-1}{m_{tangent}} = \frac{-1}{f'(x)}$$

Using CAS



so $m_{normal} = \dfrac{-1}{\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)}$

Mark allocation

- 1 answer mark for $f'(x)$.
- 1 answer mark for gradient of normal.

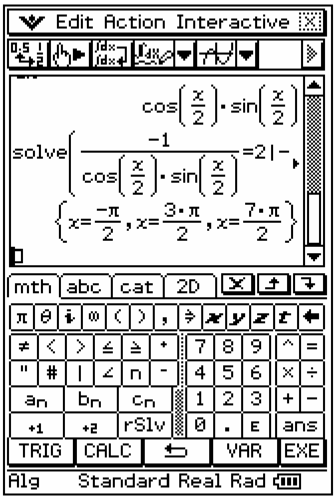
ii. Find exact coordinates of the points on the curve $y = f(x)$ where the gradient of the normal to the curve is equal to 2.

2 marks

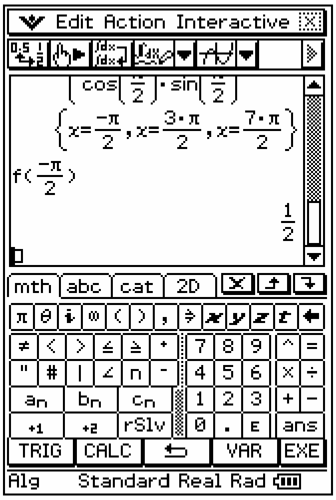
Solution

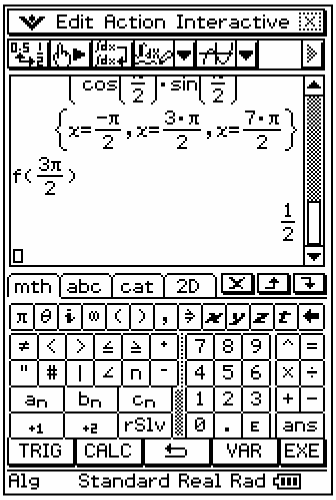
Using CAS solve

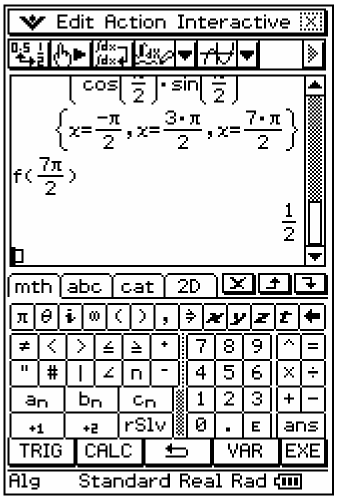
$$\frac{-1}{\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)} = 2 \quad \text{for the domain } x \in [-\pi, 4\pi]$$



then use CAS to find the corresponding *y*-values.







so the points are $\left(\frac{-\pi}{2},\frac{1}{2}\right),\left(\frac{3\pi}{2},\frac{1}{2}\right),\left(\frac{7\pi}{2},\frac{1}{2}\right)$.

Mark allocation

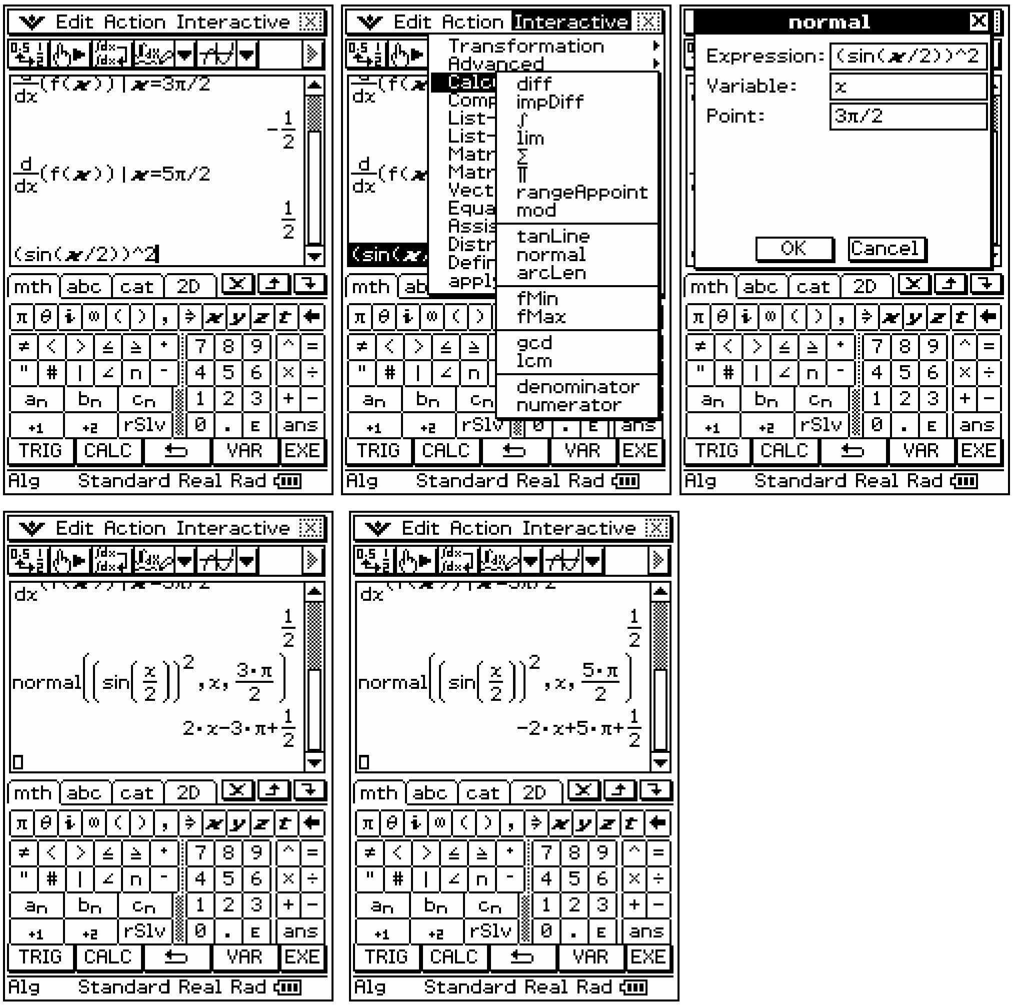
- 1 answer mark for the *x*-values.
- 1 answer mark for the exact coordinates.

iii. Find the exact equations of the **normals** to the curve at the points where $x = \dfrac{3\pi}{2}$ and $x = \dfrac{5\pi}{2}$.

2 marks

Solution

Using CAS gives



$$y = 2x - 3\pi + \frac{1}{2}$$

$$y = -2x + 5\pi + \frac{1}{2}$$

Mark allocation

- 1 mark for finding equation of normal for $x = \dfrac{3\pi}{2}$.
- 1 mark equation of normal for $x = \dfrac{5\pi}{2}$.

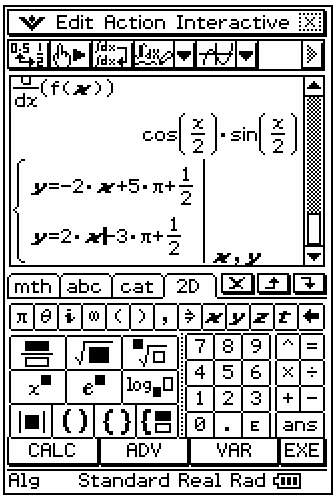
c. The graph of $y = f(x)$ is transformed to give the graph of $y = f(x) + b$. Find the exact value of b, such that the graph of the normal at $x = \frac{3\pi}{2}$ and the graph of the normal at $x = \frac{5\pi}{2}$ intersect on the x-axis.

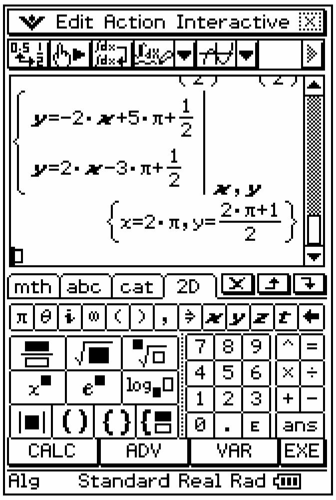
2 marks

Solution

Need to find the point of intersection of the two normals.

Using simultaneous equations on CAS, find the point of intersection.





If the graph is moved down $\frac{1}{2} + \pi$ units then the normals will intersect on the x-axis, so $b = -\frac{1}{2} - \pi$.

Mark allocation

- 1 method mark for finding point of intersection.
- 1 answer mark for value of *b*.

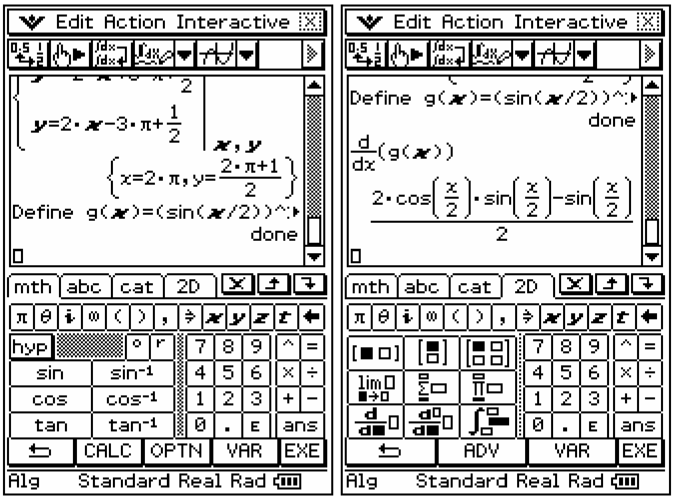
Let $g:[-\pi, 4\pi] \to R,\ g(x) = \sin^2\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) + 1$.

d. **i.** Find $g'(x)$.

1 mark

Solution

Define the function using CAS then find $g'(x)$.



Mark allocation

- 1 answer mark for correct derivative.

ii. Write down an equation, the solution of which gives the values of x for which the gradient is zero.

1 mark

Solution

$$\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) - \frac{1}{2}\sin\left(\frac{x}{2}\right) = 0$$

Mark allocation

- 1 answer mark for correct answer.

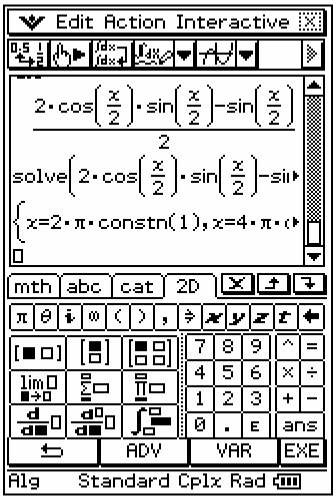
iii. Solve the equation for $x \in [-\pi,\ 4\pi]$ to find the exact coordinates on the curve where the gradient is zero.

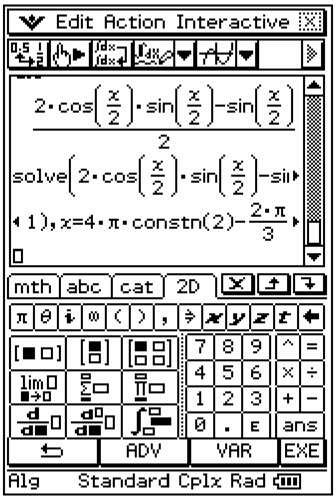
3 marks

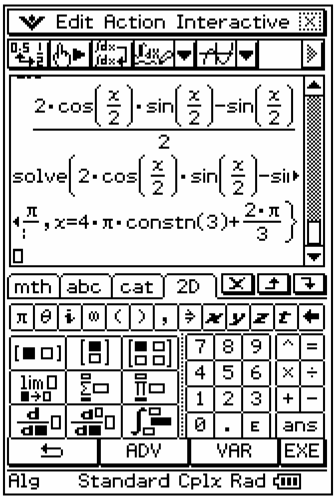
Solution

Using CAS gives

let $g'(x) = 0$ and solving for x gives

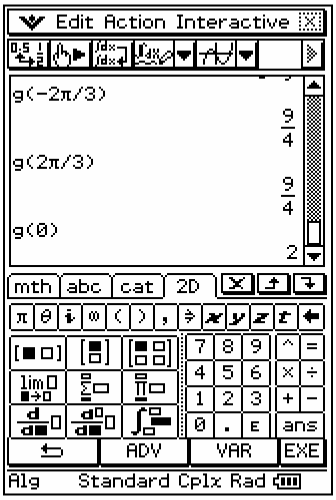


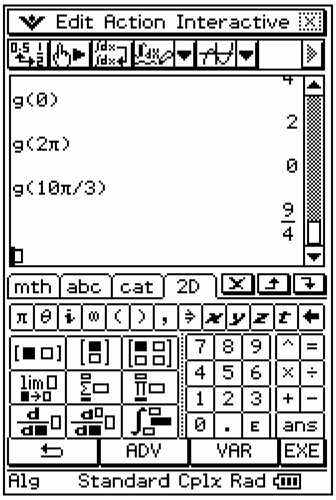




Using the given domain the solutions are at $x = 0, \dfrac{-2\pi}{3}, \dfrac{2\pi}{3}, 2\pi, \dfrac{10\pi}{3},\ 4\pi$

Use CAS to find *y*-coordinates.



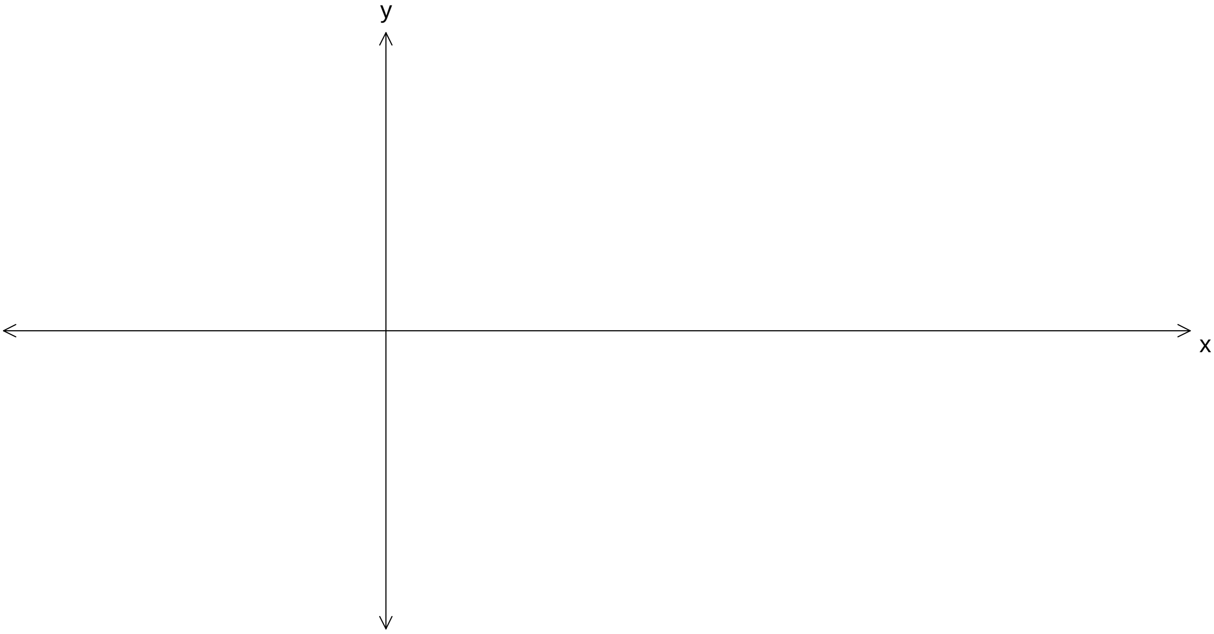


so the coordinates are $(0, 2), (\dfrac{-2\pi}{3}, \dfrac{9}{4}), (\dfrac{2\pi}{3}, \dfrac{9}{4}), (2\pi, 0), (\dfrac{10\pi}{3}, \dfrac{9}{4}),\ (4\pi, 2)$.

Mark allocation

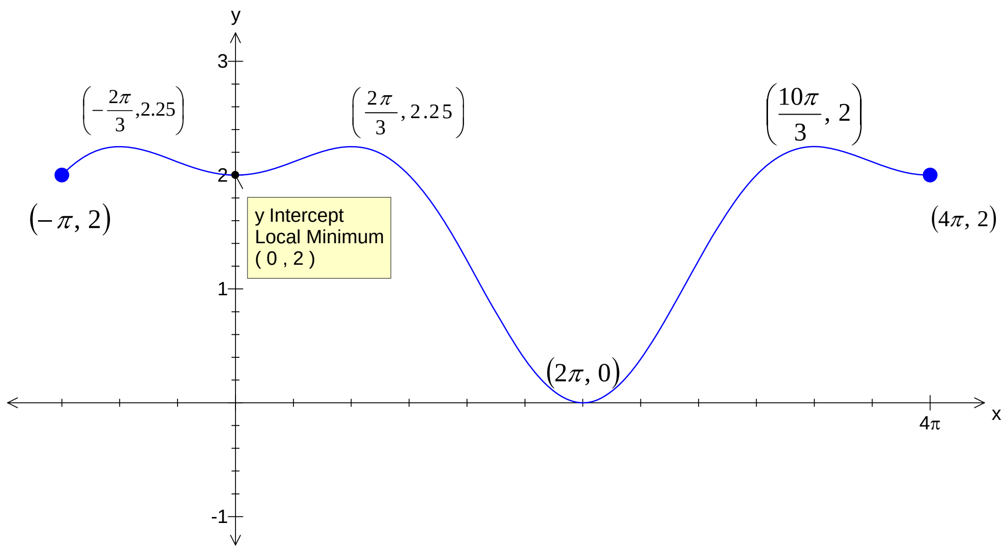
- 1 method mark for solving equation equal to zero and giving one solution.
- 1 answer mark for $x = 0,\ 2\pi, 4\pi\ , \dfrac{2\pi}{3}, \dfrac{10\pi}{3}, \dfrac{-2\pi}{3}$.
- 1 answer mark for exact coordinates.

iv. Sketch the graph of $y = g(x)$ on the axes below, labeling intercepts, stationary points and endpoints with exact coordinates.



2 marks

Solution



Mark allocation

- 1 mark for correct shape over correct domain.
- 1 mark for all points labeled and correct.

Total 17 marks

Question 4

Let $f : R \to R,\ f(x) = 9e^{-x}$.

a. **i.** Show that $f(u)f(-u) = 81$, where u is a real number.

1 mark

Solution

$f(u)f(-u) = 9e^{-u} \times 9e^{u} = 81e^{-u+u} = 81e^{0} = 81$

Mark allocation

- 1 method mark for valid steps leading to the given answer—must be logical and clear.

ii. Show that $f(u+v) = \frac{1}{9}f(u)f(v)$, where u and v are real numbers.

2 marks

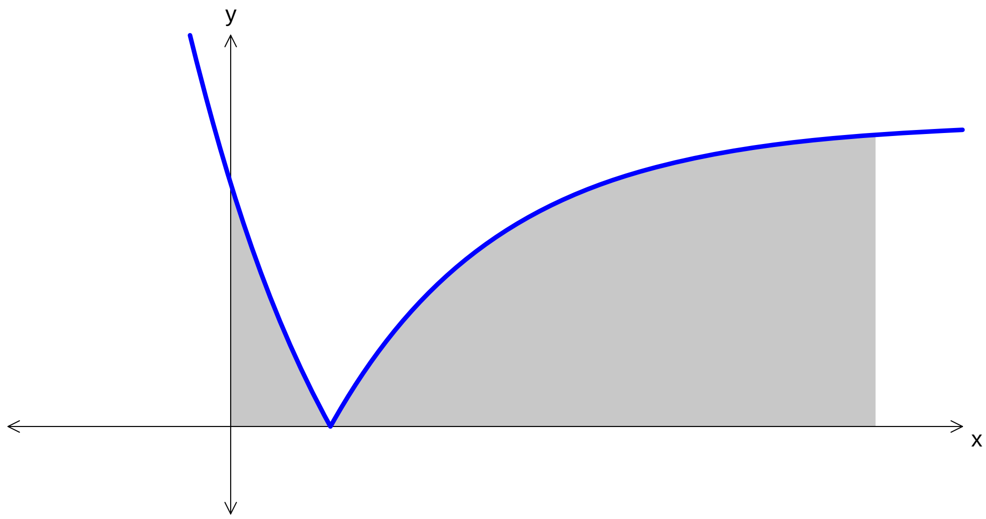
Solution

$f(u+v) = 9e^{-(u+v)} = 9e^{-u}e^{-v} = \frac{1}{9} \times 9e^{-u} \times 9e^{-v} = \frac{1}{9}f(u)f(v)$

Mark allocation

- 1 answer mark for forming $f(u+v)$.
- 1 method mark for valid steps that lead to correct answer—again must be clear and logical.

A panel of a stained glass window has a section of lead outline described by the graph with the equation $g(x) = \left|9e^{-x} - 5\right|$ and is shown below. The shaded area is red-coloured glass.



b. The x-intercept of the graph occurs at $x = \log_e(a)$. Find the exact value of a.

2 marks

Solution

x-intercept occurs when $y = 0$.

$9e^{-x} - 5 = 0$

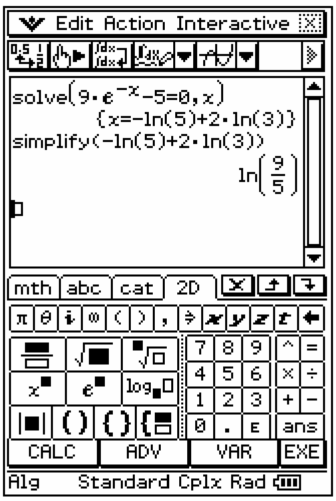
$9e^{-x} = 5$

$e^{-x} = \frac{5}{9}$

$-x = \log_e(\frac{5}{9})$

$x = \log_e(\frac{9}{5})$

using CAS we have



Mark allocation

- 1 method mark for setting equation equal to 0. (Note – must show a working step as question is worth 2 marks.)
- 1 answer mark for answer.

c. The red-coloured glass is bounded by the area under the graph, the x-axis, the y-axis and the line $x = 4$. Find the exact area of the red-coloured glass.

3 marks

Solution

To handle the absolute value function the integral is separated into two functions.

$$area = \int_0^{\log_e(\frac{9}{5})} 9e^{-x} - 5\ dx + \int_{\log_e(\frac{9}{5})}^{4} -(9e^{-x} - 5)\ dx \quad ^{***}$$

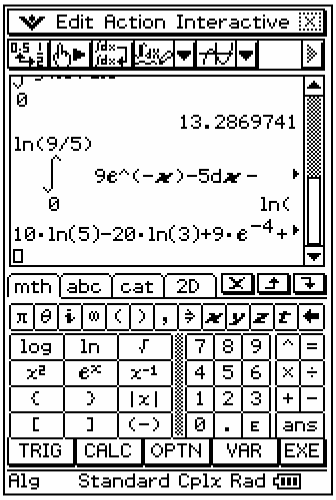
$$= \left[-9e^{-x} - 5x\right]_0^{\log_e(\frac{9}{5})} + \left[9e^{-x} + 5x\right]_{\log_e(\frac{9}{5})}^{4}$$

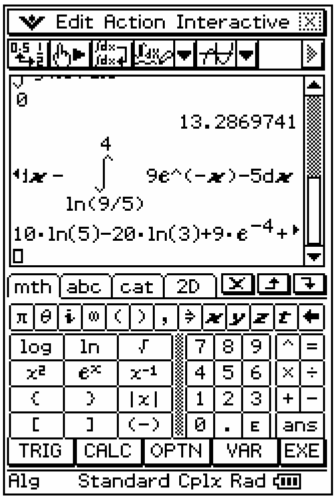
$$= (-9\left(\frac{5}{9}\right) - 5\log_e(\frac{9}{5})) - (-9 - 0) + (9e^{-4} + 20) - (9\left(\frac{5}{9}\right) + 5\log_e(\frac{9}{5}))$$

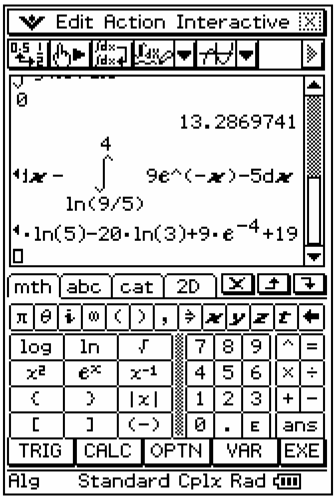
$$= -5 - 5\log_e(\frac{9}{5}) + 9 + 9e^{-4} + 20 - 5 - 5\log_e(\frac{9}{5})$$

$$= 19 - 10\log_e(\frac{9}{5}) + 9e^{-4} \quad ^{***}$$

using CAS it is also best to separate the absolute value function into two integrals.







Mark allocation

- 1 method mark for forming correct integral with correct terminals.
- 1 method mark for splitting the integral into two parts.
- 1 answer mark for correct answer.

Note

- *Question does not ask for calculus to be shown—only the setting up step and the answer are required for full marks—see *** labels.*

d. Find the value of a, correct to 3 decimal places, such that the area of the red-coloured glass enclosed by the graph , the x-axis, the y-axis and the line $x = a$ is equal to 35 square units.

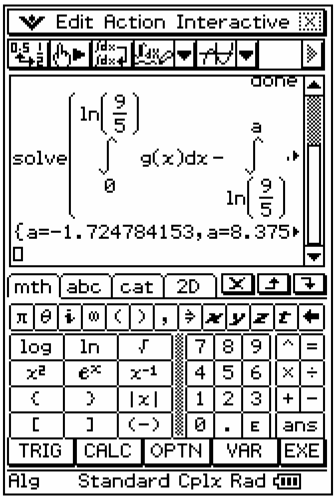
2 marks

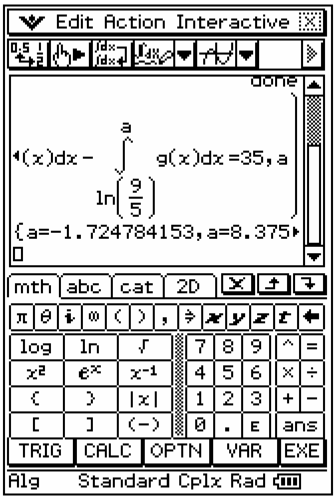
Solution

$$\int_0^{\log_e(\frac{9}{5})} 9e^{-x} - 5 \, dx = 1.06107$$

$$\text{so } let \int_{\log_e(\frac{9}{5})}^{a} 5 - 9e^{-x} \, dx = 35 - 1.06107 = 33.93893$$

using CAS and solving $\int_0^{\log_e(\frac{9}{5})} 9e^{-x} - 5 \, dx + \int_{\log_e(\frac{9}{5})}^{a} 5 - 9e^{-x} \, dx = 35$ for a gives





so $a = 8.375$ gives an area of 35 units.

Mark allocation

- 1 method mark for writing equation as integral with a,
$$\int_0^{\log_e(\frac{9}{5})} 9e^{-x} - 5 \, dx + \int_{\log_e(\frac{9}{5})}^{a} 5 - 9e^{-x} \, dx = 35 .$$
- 1 answer mark for correct answer.

Note

- *Again only the setting up step and the answer are required—showing calculus is not required.*

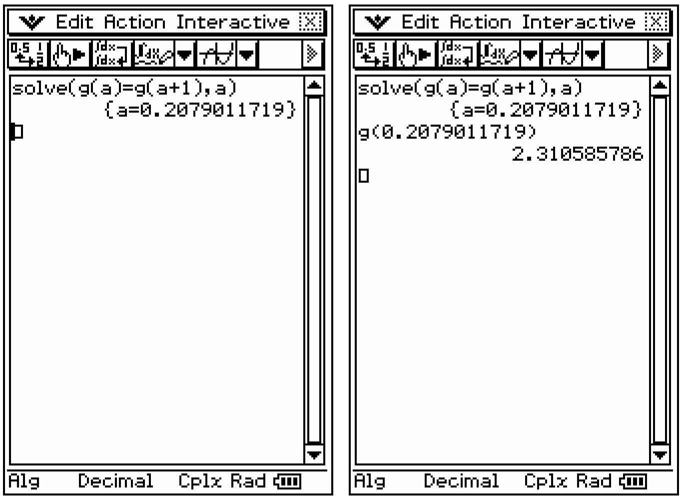
e. In producing the leadlight panel, the craftsman creates a sketch of the red-coloured panel and notices when positioning the curve that two points on the curve describing the red-coloured glass are exactly 1 unit horizontally apart. Find the coordinates of the two points, correct to 3 decimal places.

3 marks

Solution

Using CAS set up equations describing the two points.

If one point has the coordinates $(a, g(a))$ the other will have the coordinates $(a+1, g(a+1))$. As the two points are horizontally positioned $g(a) = g(a+1)$. This equation can be solved using CAS.



The coordinates of the points are $(0.208,\ 2.311)$ $(1.208,\ 2.311)$.

Mark allocation

- 1 method mark for setting up equations $g(a) = g(a+1)$.
- 1 answer mark for finding the value of *a*.
- 1 answer mark for both points as coordinates.

Total 13 marks

END OF SOLUTIONS BOOK