INSIGHT

Trial Exam Paper

2009

MATHEMATICAL METHODS/ MATHEMATICAL METHODS (CAS)

Written examination 1

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

Question 1

Solve the following for *x*.

a. $2\log_9(x-1)+\log_9 3=1$

2 marks

Solution

Using the log laws to get

$\log_9(x-1)^2+\log_9 3=1$

$\log_9 3(x-1)^2=1$

$9^1=3(x-1)^2$

$3=(x-1)^2$

$\pm\sqrt{3}=(x-1)$

$\Rightarrow x=1\pm\sqrt{3},$

$however\ \ x>1,\ so\ \ x=1+\sqrt{3}$

Mark allocation

- 1 method mark for using log laws.
- 1 answer mark for correct answer.

b. $e^{2x}-5e^x+4=0$

3 marks

Solution

Let $a=e^x$ as this forms the trinomial into a recognisable quadratic

$a^2-5a+4=0$

factorising gives $(a-1)(a-4)=0$

$so\ \ a=1\ or\ \ a=4$

$\Rightarrow e^x=1\ or\ \ e^x=4$

$\Rightarrow x=0\ or\ \ x=\log_e 4$

Mark allocation

- 1 method mark for factoring the trinomial.
- 1 answer mark for both answers for a.
- 1 answer mark for both answers for x.

Question 2

Given $f:[0,\pi]\to R,\ f(x)=|2\cos(2x)+1|$, find

a. the values of x for which $f(x)=0$.

2 marks

Solution

$f(x)=0 \Rightarrow\ 2\cos(2x)+1=0$

$2\cos(2x)=-1$

$\cos(2x)=-\frac{1}{2}$

first quadrant angle is $\frac{\pi}{3}$

$so\ 2x=\frac{2\pi}{3},\frac{4\pi}{3}$

$\Rightarrow x=\frac{\pi}{3},\frac{2\pi}{3}$

Mark allocation

- 1 mark for first quadrant angle of $\frac{\pi}{3}$.
- 1 mark for both answers.

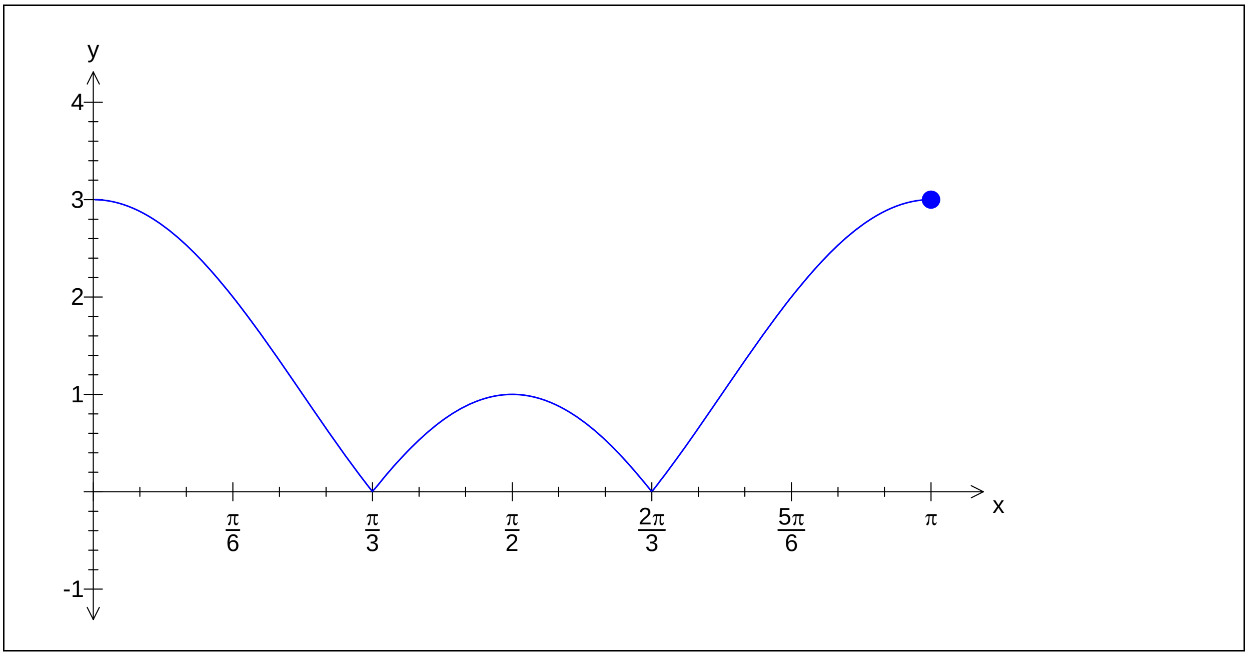
Question 2 – continued
TURN OVER

b. the exact value of $f'(x)$ when $x = \frac{\pi}{6}$.

2 marks

Solution

Although not directly required, the graph of $f(x)$ can assist here and in part c.
The graph of $f(x)$ is



at $x = \frac{\pi}{6}$, $f(x) = 2\cos(2x) + 1$ so $f'(x) = -4\sin(2x)$

at $x = \frac{\pi}{6}$, $f'(x) = -4\sin(\frac{2\pi}{6}) = -4 \times \frac{\sqrt{3}}{2} = -2\sqrt{3}$

Mark allocation

- 1 mark for finding $f'(x)$.
- 1 mark for correct answer.

c. the interval over which the rate of change is negative.

1 mark

Solution

The rate of change is negative when the gradient of the graph of $y = f(x)$ is negative. From the graph drawn above, it can be seen that the gradient is negative for $x \in (0, \frac{\pi}{3}) \cup (\frac{\pi}{2}, \frac{2\pi}{3})$

Mark allocation

- 1 mark for correct answer.

Question 3

The graph of the function with the rule $y = x^{\frac{2}{3}}$ is transformed as follows:

- A dilation by a factor of 2 from the y-axis
- A reflection in the x-axis
- A translation of + 4 units parallel to the x-axis
- A translation of + 1 units parallel to the y-axis

a. Write down the equation of the rule of the transformed function.

1 mark

Solution

The transformed graph is $y = -\left(\frac{x-4}{2}\right)^{\frac{2}{3}} + 1$

Mark allocation

- 1 mark for correct answer.

b. State the domain and range of the transformed function.

2 marks

Solution

domain is R

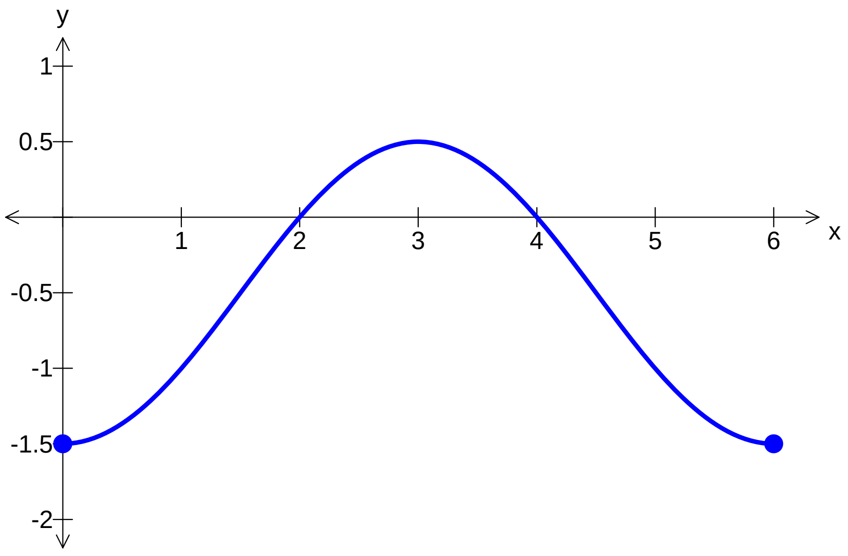
range is $(-\infty, 1]$

Mark allocation

- 1 mark for domain correct.
- 1 mark for range correct.

Question 4

The diagram below shows one cycle of the graph of a circular function.



a. State the amplitude of the function.

1 mark

Solution

Graph is centred about the line $y = -0.5$ and rises up one unit to $y = 0.5$ and down one unit to $y = -1.5$, so the amplitude is 1 unit.

Mark allocation

- 1 mark for correct answer.

b. State the equation of the function.

2 marks

Solution

Graph is an upside down cos graph so $y = -a\cos nx + \text{b}$

Graph is centred at $y = -0.5$ so $y = -a\cos nx - 0.5$

Amplitude is 1 unit so $y = -\cos nx - 0.5$

Period is 6 units so $6 = \dfrac{2\pi}{n}$, which gives $n = \dfrac{\pi}{3}$, so equation is

$$y = -\cos(\frac{\pi x}{3}) - 0.5$$

Question 5

Let $\int_2^a (e^{2x-4})\,dx = \frac{1}{2}$. Find the exact value of *a*, where $a > 2$.

3 marks

Solution

$$\int_2^a (e^{2x-4})\,dx = \left[\frac{e^{2x-4}}{2}\right]_2^a = \frac{e^{2a-4}}{2} - \frac{1}{2}$$

so $\frac{e^{2a-4}}{2} - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \frac{e^{2a-4}}{2} = 1$

$\Rightarrow e^{2a-4} = 2$

$\Rightarrow 2a - 4 = \log_e 2$

$\Rightarrow a = \frac{\log_e 2 + 4}{2}$

Mark allocation

- 1 answer mark for correct integral of e^{2x-4}.
- 1 method mark for substituting in terminals and obtaining expression of the form $e^{2a-4} = 2$.
- 1 mark for correct answer.

TURN OVER

Question 6

The random variable X is normally distributed with mean 50 and standard deviation 5.

The random variable Z is normally distributed with mean 0 and standard deviation 1.

If $\Pr(Z < -2) = 0.0228$, find

a. $\Pr(X < 40)$

1 mark

Solution

Converting to standard normal gives

$$\begin{aligned}\Pr(X < 40) &= \Pr(Z < \frac{40-50}{5}) \\ &= \Pr(Z < -2) \\ &= 0.0228\end{aligned}$$

Mark allocation

- 1 mark for correct answer.

b. $\Pr(X < 60 \mid X > 50)$

2 marks

Solution

Using the conditional probability rule gives

$$\begin{aligned}\Pr(X < 60 \mid X > 50) &= \frac{\Pr(X < 60 \cap X > 50)}{\Pr(X > 50)} \\ &= \frac{\Pr(50 < X < 60)}{\Pr(X > 50)}\end{aligned}$$

again converting to standard normal gives

$$\frac{\Pr(50 < X < 60)}{\Pr(X > 50)} = \frac{\Pr(0 < Z < 2)}{\Pr(Z > 0)}$$

Using the symmetry of the normal distribution curve, $\Pr(z < -2) = \Pr(z > 2)$

So the $\Pr(0 < Z < 2) = 0.5 - 0.0228 = 0.4772$ and $\Pr(Z > 0) = 0.5$,

Therefore $\Pr(X < 60 \mid X > 50) = \dfrac{0.4772}{0.5} = 0.4772 \times 2 = 0.9544$

Mark allocation

- 1 mark for using conditional probability.
- 1 mark for correct answer.

Question 7

The life of a battery, in hours, can be modelled by the random variable X with probability density function $f(x) = \begin{cases} \dfrac{c}{x^2} & if \quad x > 20 \\ 0 & if \quad x \le 20 \end{cases}$

a. Find the value of *c*.

2 marks

Solution

If the function describes a pdf then $\int_{20}^{\infty} \frac{c}{x^2}\, dx = 1$.

$$\int_{20}^{\infty} \frac{c}{x^2}\, dx = \left[\frac{-c}{x}\right]_{20}^{\infty}$$

$$= -c\left(\lim_{x\to\infty}\frac{1}{x} - \frac{1}{20}\right)$$

$$= -c\left(0 - \frac{1}{20}\right)$$

$$= c \times \frac{1}{20}$$

so $\quad c \times \frac{1}{20} = 1 \quad \Rightarrow c = 20$

Mark allocation

- 1 mark for setting pdf = 1.
- 1 mark for correct answer—integral must be correct and must have *dx* on end.

Question 7 – continued
TURN OVER

b. Find the median life of a battery according to this model.

2 marks

Solution

Let the median life be *m*. Median is such that $\int_{20}^{m} \frac{20}{x^2}\, dx = 0.5$. Antidifferentiating gives

$$\int_{20}^{m} \frac{20}{x^2}\, dx = 0.5$$

$$\left[\frac{-20}{x}\right]_{20}^{m} = 0.5$$

$$\frac{-20}{m} - \frac{-20}{20} = 0.5$$

$$\frac{-20}{m} + 1 = 0.5$$

$$\frac{-20}{m} = -0.5$$

$$m = 40$$

Mark allocation

- 1 mark for setting up the integral equal to 0.5 (must include *dx*).
- 1 mark for correct answer.

Question 8

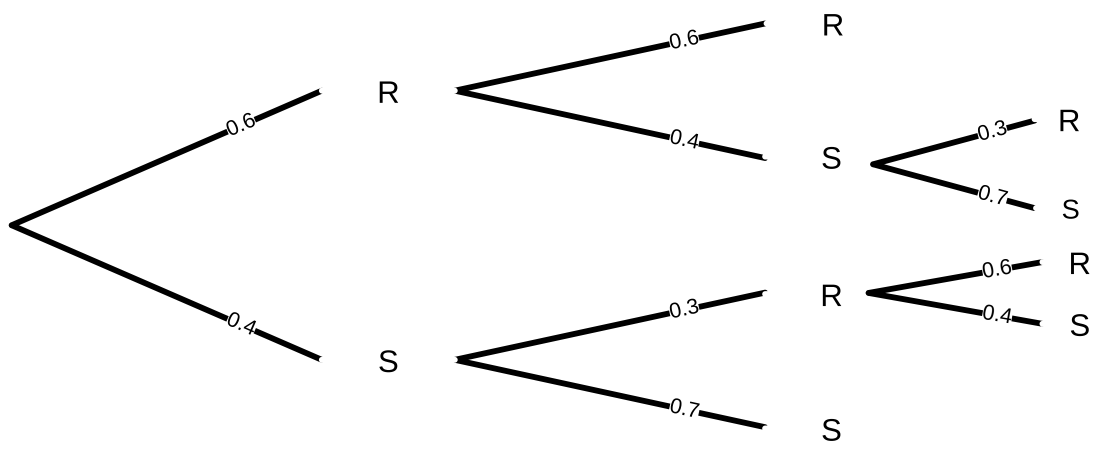
The Roosters and the Swans are rival handball teams. When they play against each other the probability of winning is dependent upon the result of their previous match. If the Roosters have won the previous match, then the probability that they will win the next match is 0.6. If the Swans have won the previous match, then the probability that they will win the next match is 0.7.

These two teams are about to start a "best of three" finals series which is played until either team has won 2 matches. The probability that the Roosters win the first match is 0.6.

a. Draw a tree diagram to show the possible outcomes.

1 mark

Solution



Mark allocation

- 1 mark for correct tree diagram—must have correct number of branches and have probabilities labelled.

b. What is the probability that the Swans win the first two games?

1 mark

Solution

$\Pr(SS) = 0.4 \times 0.7 = 0.28$

Mark allocation

- 1 mark for correct answer.

Question 8 – continued
TURN OVER

c. Find the probability that the Swans win the finals series, i.e. win two games.

2 marks

Solution

$$\begin{aligned}\Pr(\mathit{swans\ win\ the\ series}) &= \Pr(SS) + \Pr(RSS) + \Pr(SRS)\\ &= 0.4\times 0.7 + 0.6\times 0.4\times 0.7 + 0.4\times 0.3\times 0.4\\ &= 0.28 + 0.168 + 0.048\\ &= 0.496\end{aligned}$$

Mark allocation

- 1 method mark for selecting three outcomes each involving the swans winning twice.
- 1 mark for correct answer.

Question 9

Find the coordinates of the points on the curve $y = 4x^2 + 5$ at which the tangents drawn to the curve pass through the point (0, 4).

4 marks

Solution

The gradient of the tangent is equal to the gradient of the curve, i.e. $\frac{dy}{dx}$

$$\frac{dy}{dx} = 8x$$

Two points on the tangent line are $(0,4)$ *and* $(x, 4x^2 + 5)$ and the gradient of the tangent line is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4x^2 + 5 - 4}{x - 0} = \frac{4x^2 + 1}{x}$

Let $m = \frac{dy}{dx}$ to get

$$\frac{4x^2 + 1}{x} = 8x$$

$$4x^2 + 1 = 8x^2$$

$$1 = 4x^2$$

$$\frac{1}{4} = x^2$$

$$x = \frac{\pm 1}{2}$$

so points are $\left(\frac{1}{2}, 6\right)$ *and* $\left(\frac{-1}{2}, 6\right)$

Question 9 – continued

Mark allocation

- 1 mark for finding $\frac{dy}{dx}$.
- 1 mark for finding two points on tangent line.
- 1 mark for finding $x = \pm\frac{1}{2}$.
- 1 mark for two correct points.

Question 10

a. Find $\frac{d}{dx}(\log_e(3x^2+1))$.

2 marks

Solution

Using the chain rule $\frac{d}{dx}(\log_e(3x^2+1)) = \frac{6x}{3x^2+1}$

Mark allocation

- 1 method mark for evidence of the derivative of the log term becoming $\frac{1}{3x^2+1}$
- 1 answer mark for correct answer.

Question 10 – continued
TURN OVER

b. Hence find $\int_0^4 \frac{x}{3x^2+1}\,dx$.

2 marks

Solution

'Hence' means it has to be obvious that the answer above has been used in the calculation—in particular we will use the 'reverse' of the equation formed in part a.

$\frac{d}{dx}(\log_e(3x^2+1)) = \frac{6x}{3x^2+1}$ so it follows that

$\int \frac{6x}{3x^2+1}\,dx = \log_e(3x^2+1)+c$ — this result will be used in the calculation.

So

$$\int_0^4 \frac{x}{3x^2+1}\,dx = \frac{1}{6}\int_0^4 \frac{6x}{3x^2+1}\,dx$$

$$= \frac{1}{6}[\log_e(3x^2+1)]_0^4 \quad \text{using result from above}$$

$$= \frac{1}{6}[\log_e(49) - \log_e(1)]$$

$$= \frac{1}{6}\log_e(49)$$

1 method mark for establishing $\int \frac{6x}{3x^2+1}\,dx = \log_e(3x^2+1)+c$ and attempting to use it.

Mark allocation

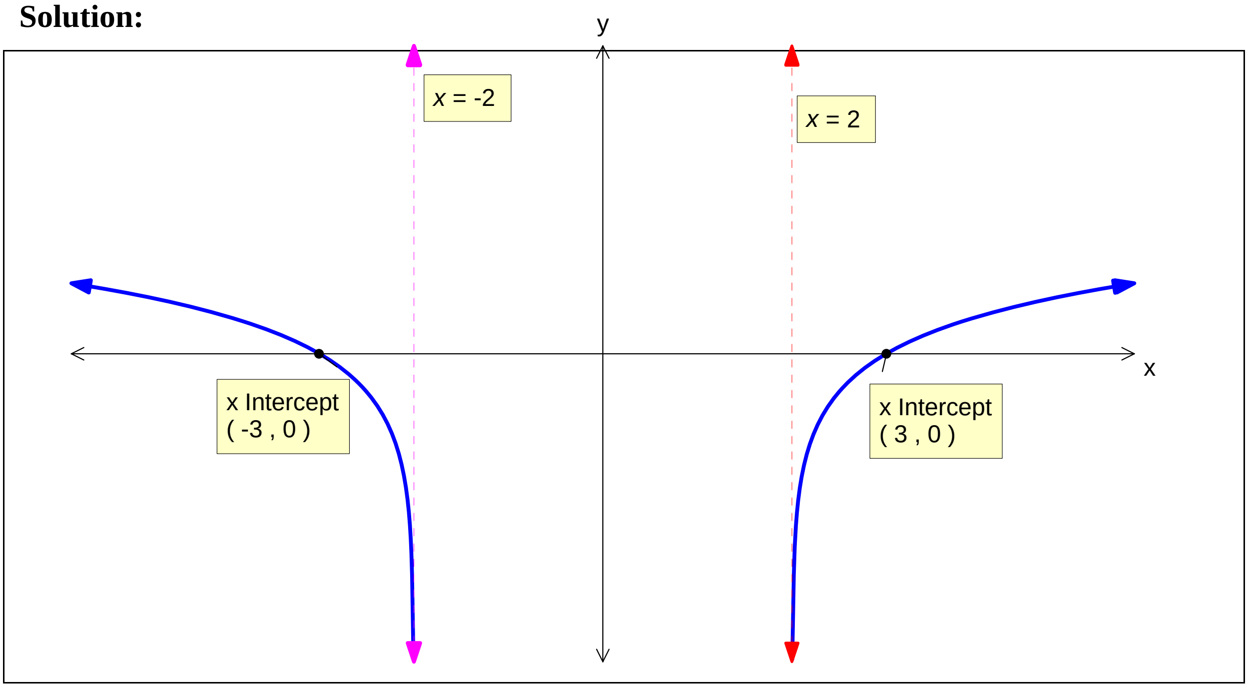
- 1 answer mark for correct answer.

Question 11

On the axes below, sketch the graph of the function with the rule $f(x) = \log_e(|x| - 2)$. Label any asymptotes with the equation and intercepts as coordinates.

2 marks

Solution:



Mark allocation

- 1 method mark for having one branch correct with asymptote and intercept labelled.
- 1 answer mark for having both branches correct and asymptotes and intercepts labelled.

END OF SOLUTIONS BOOK