Neap:

Trial Examination 2008

# VCE Mathematical Methods Units 3 & 4

## Written Examination 2

## Question and Answer Booklet

Reading time: 15 minutes
Writing time: 2 hours

Student's Name: ______________________

Teacher's Name: ______________________

**Structure of Booklet**

| Section | Number of questions | Number of questions to be answered | Number of marks |
|---|---|---|---|
| 1 | 22 | 22 | 22 |
| 2 | 5 | 5 | 58 |
| | | | Total 80 |

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved **graphics** calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.

Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

**Materials supplied**

Question and answer booklet of 28 pages with a detachable sheet of miscellaneous formulas in the centrefold.
Answer sheet for multiple-choice questions.

**Instructions**

Detach the formula sheet from the centre of this book during reading time.

Write **your name** and **teacher's name** in the space provided above on this page.

All written responses must be in English.

**At the end of the examination**

Place the answer sheet for multiple-choice questions inside the front cover of this book.

**Students are NOT permitted to bring mobile phones and/or any other electronic communication devices into the examination room.**

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2008 VCE Mathematical Methods Units 3 & 4 Written Examination 2.

## SECTION 1

**Instructions for Section 1**

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

### Question 1

The function $g : [a, b] \to R,\ g(x) = 2 - (2x + 1)^2$ has a range $[-7, 2]$.

The values of $a$ and $b$ could be

**A.** $a = -\frac{1}{2}, b = 0$

**B.** $a = -\frac{1}{2}, b = 2$

**C.** $a = -\frac{1}{2}, b = 1$

**D.** $a = 1, b = -2$

**E.** $a = 0, b = 1$

### Question 2

If $f(x + 2) = x^2 - 5x - 6$ and $g(x) = 2\log_e(x - 8)$, then $g(f(x))$ is given by

**A.** $2\log_e(x^2 - 9x + 8)$

**B.** $2\log_e(x^2 - 5x - 14)$

**C.** $2\log_e(x^2 - x - 20)$

**D.** $4(\log_e(x - 8))^2 - 18\log_e(x - 8) + 8$

**E.** $2\log_e(x^2 - 9x)$

**Question 3**

An antiderivative of $\dfrac{x^2\cos(5x) - 3x + 8}{2x^2}$ is

**A.** $\dfrac{\sin(5x) - 3x^2 + 4x}{2x^3}$

**B.** $\dfrac{\cos(5x) - 5x\sin(5x) - 3x^2 + 4x}{4x^4}$

**C.** $\dfrac{x\sin(5x) - 15x\log_e(x) - 40}{10x}$

**D.** $\dfrac{-x\sin(5x) - 15x\log_e(x) - 40}{10x}$

**E.** $\dfrac{x\sin(5x) - 15x\log_e\left(\frac{3x}{2}\right) - 40}{10x}$

**Question 4**

If $f(x) = e^{3-x^2}$ and $g(x) = \log_e(2x)$, then the rate of change with respect to $x$ of $g(f(x))$ when $x = 2$ is

**A.** 4
**B.** 2
**C.** 0
**D.** −2
**E.** −4

**Question 5**

$\displaystyle\int \sec^2(2x) - \frac{4}{3x-5}\,dx$ is equal to

**A.** $2\tan(2x) - 4\log_e|3x-5| + c$

**B.** $2\tan(2x) - 12\log_e|3x-5| + c$

**C.** $2\tan(2x) - \frac{4}{3}\log_e|3x-5| + c$

**D.** $\frac{1}{2}\tan(2x) - \frac{4}{3}\log_e|3x-5| + c$

**E.** $\frac{1}{2}\tan(2x) - 4\log_e|3x-5| + c$

**Question 6**

A function has the rule $y = \frac{1}{\left|1 + \sin\left(x - \frac{\pi}{3}\right)\right|}$, and is defined on its maximal domain.

It shares the same range with which one of the following functions, each of which is defined over its maximal domain?

**A.** $y = \frac{1}{1 + \cos^2(x)}$

**B.** $y = \frac{1}{1 + \sin\left(x - \frac{\pi}{3}\right)}$

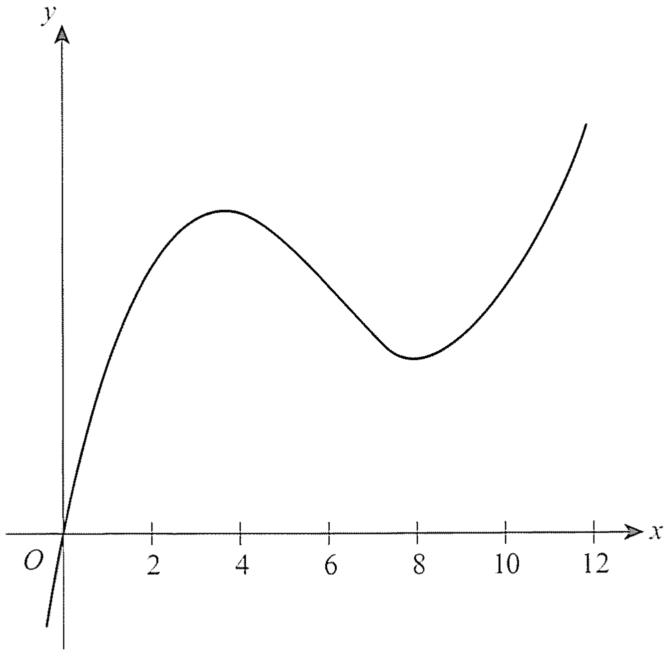
**C.** $y = \frac{1}{2} - (x + 1)^2$

**D.** $y = (x + 1)^2 - \frac{1}{2}$

**E.** $y = \left|\tan(x) + \frac{1}{2}\right|$

## Question 7

The graph of the continuous function $f$, which has exactly two stationary points where $x = 4$ and $x = 8$, is shown below.




If the function $g$ is given by $g(x) = -f\left(\frac{x+4}{2}\right)$ then the values for which $g'(x) > 0$ are

**A.** $x \in (0, 4)$

**B.** $x \in (0, 8)$

**C.** $x \in (4, 12)$

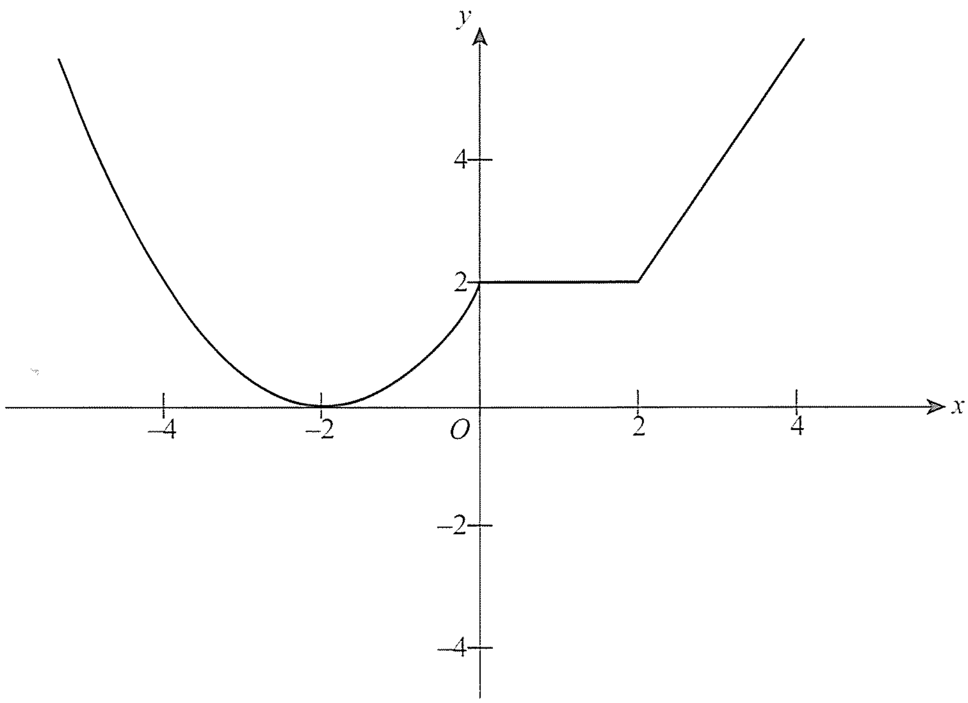
**D.** $x \in (2, 12)$

**E.** $x \in (4, 8)$

## Question 8

The function $f$ is defined on domain $R$ and has the rule $f(x) = \begin{cases} \frac{1}{2}(x+2)^2 & x < 0 \\ 2 & 0 \le x \le 2 \\ 2x - 2 & x > 2 \end{cases}$

The graph of $y = f(x)$ is shown below.




Which of the following statements **is true** about the function $f$?

**A.** It is continuous for all real values of $x$ but not anitdifferentiable.

**B.** It is not continuous for all real values of $x$ and not antidifferentiable.

**C.** It is not continuous for all real values of $x$ and not differentiable for all real values of $x$.

**D.** It is continuous for all real values of $x$ and differentiable for all real values of $x$.

**E.** It is continuous for all real values of $x$ but not differentiable for all real values of $x$.

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**Question 9**

$\int_0^{\pi} \left( \sin\left(\frac{x}{3}\right) - \frac{1}{\sqrt{x}} \right) dx$ is equal to

**A.** $\frac{3}{2} - 2\sqrt{\pi}$

**B.** $-\frac{9}{2} - 2\sqrt{\pi}$

**C.** $\frac{3}{2} + 2\sqrt{\pi}$

**D.** $-\frac{9}{2} + 2\sqrt{\pi}$

**E.** $-\frac{9}{2} - \frac{2}{\sqrt{\pi}}$

**Question 10**

If $f(x) = \sqrt[3]{(x^3 + a^3)}$, then $f^{-1}(2a)$ is equal to

**A.** $\sqrt[3]{7}a$

**B.** $a$

**C.** $\sqrt[3]{5}a$

**D.** $\sqrt[3]{9}a$

**E.** $-\sqrt[3]{5}a$

**Question 11**

The solution set of the equation $\frac{e^{3x} + 12e^{x}}{e^{x}} = 7e^{x}$ over $R$ is

**A.** $\{1, \log_e(3), \log_e(2)\}$

**B.** $\{1, \log_e(3), 2\log_e(2)\}$

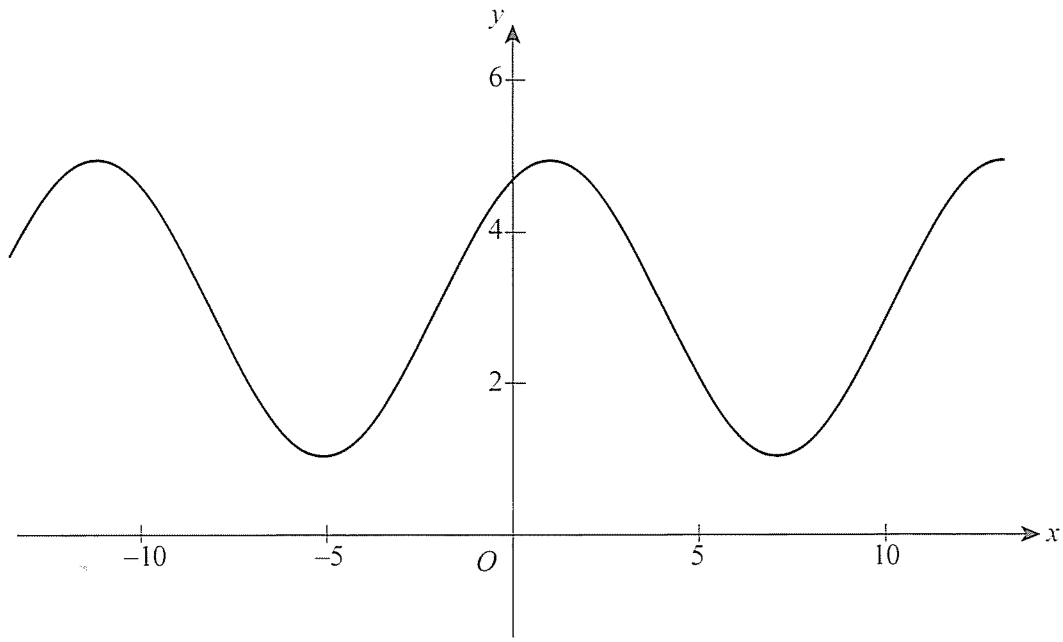
**C.** $\{\log_e(3), \log_e(2)\}$

**D.** $\{\log_e(3), 2\log_e(2)\}$

**E.** $\{0, \log_e(3), 2\log_e(2)\}$

**Question 12**

The graph of $y = f(x)$ is shown below.




Consider the following equations.

**I.** $y = 2\sin\left(\frac{\pi}{6}(x+2)\right) + 3$

**II.** $y = 3 - 2\sin\left(\frac{\pi}{6}(x-4)\right)$

**III.** $y = 2\cos\left(\frac{\pi}{6}(x-1)\right) + 3$

The equation of $y = f(x)$ could be

**A.** **I** only.

**B.** **II** only.

**C.** **I** and **III** but not **II**.

**D.** **II** and **I** but not **III**.

**E.** All three of **I, II** and **III**.

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**Question 13**

If $2x - 5y = 10$ is the equation of the normal to the graph of $y = f(x)$ at the point $(-2, 5)$, then $f'(-2)$ is equal to

**A.** $-\frac{2}{5}$

**B.** $\frac{2}{5}$

**C.** $-\frac{5}{2}$

**D.** $\frac{5}{2}$

**E.** $5$

**Question 14**

The following functions $f_1, f_2$ and $f_3$, each defined over its maximal domain, have rules as follows:

$$f_1(x) = \left|\log_e(x+2)^2\right|$$

$$f_2(x) = \log_e\left|x^2 - 4\right|$$

$$f_3(x) = (\log_e|x+2| + 4)^2$$

Which of the functions $f_1$, $f_2$ and $f_3$ have a domain of $R\backslash\{-2\}$ and a range of $R^+ \cup \{0\}$?

**A.** $f_1$ only

**B.** $f_2$ only

**C.** $f_3$ only

**D.** $f_1$ and $f_2$ only

**E.** $f_1$ and $f_3$ only

**Question 15**

The graph of the function $f$ with rule $f(x) = e^{2x}$ undergoes the following successive transformations:

- a reflection in the $y$-axis
- a translation of 1 unit to the left
- a dilation by a factor of 3 away from the $y$-axis

Which of the following equations gives the correct image of $f$ after these transformations?

**A.** $y = \left(\frac{1}{e^2}\right)e^{-\frac{2}{3}x}$

**B.** $y = -e^2 e^{\frac{2}{3}x}$

**C.** $y = e^2 e^{-\frac{2}{3}x}$

**D.** $y = 3e^{-2(x+1)}$

**E.** $y = \frac{1}{3}e^{-2(x+1)}$

**Question 16**

$\{x : 3\cos^2(x) = \sqrt{3}\sin(x)\cos(x)\} \cap \{x : 0 \le x \le 2\pi\}$ is equivalent to

**A.** $\left\{x : \frac{\pi}{3}, \frac{4\pi}{3}\right\}$

**B.** $\left\{x : \frac{\pi}{6}, \frac{7\pi}{6}\right\}$

**C.** $\left\{x : 0, \frac{\pi}{3}, \frac{4\pi}{3}\right\}$

**D.** $\left\{x : \frac{\pi}{3}, \frac{\pi}{2}, \frac{4\pi}{3}, \frac{3\pi}{2}\right\}$

**E.** $\left\{x : \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}\right\}$

## Question 17

The random variable $X$ has the probability density function

$$f(x) = \begin{cases} \frac{3}{64}x^2(4-x) & \text{for } x \in [0, 4] \\ 0 & \text{otherwise} \end{cases}$$

The mode of $X$ equals

**A.** $\frac{4}{9}$

**B.** $\frac{8}{3}$

**C.** $\frac{12}{5}$

**D.** 4

**E.** 0

## Question 18

A lawyer is defending his client from a charge of speeding. The lawyer's defence is based on the contention that speed cameras are inaccurate.

Suppose that when a speed camera registers 65 km/h for a randomly chosen car, the actual speed of the car is normally distributed with a mean of 63 km/h and variance of 9 km/h.

What percentage of cars are travelling less than 62 km/h when the camera registers 65 km/h?

**A.** 36.9%

**B.** 45.6%

**C.** 15.9%

**D.** 84.1%

**E.** 54.4%

## Question 19

For the discrete random variable $X$, $\text{E}(X) = 15$ and $\text{E}(X^2) = 234$.

Finding the probability that $X$ lies within two standard deviations of the mean is equivalent to finding

**A.** $\Pr(1 \le X \le 33)$

**B.** $\Pr(3 \le X \le 27)$

**C.** $\Pr(2 \le X \le 33)$

**D.** $\Pr(1 \le X \le 45)$

**E.** $\Pr(9 \le X \le 21)$

### Question 20

Medical scientists have tested a new treatment on a random sample of 30 infected rats. The results show that the probability that at least one rat is cured is 0.85. A second trial is conducted on another random group of 50 similarly infected rats.

Assuming that the treatment has the same chance of success as in the first trial, the probability that no more than two of the rats in the second trial are cured is approximately

**A.** $0.061^{50} + 50(0.939)(0.061)^{49}$

**B.** $0.005^{50} + 50(0.995)(0.005)^{49} + 1225(0.995)^2(0.005)^{48}$

**C.** $0.939^{50} + 50(0.061)(0.939)^{49} + 1225(0.061)^2(0.939)^{48}$

**D.** $0.061^{50} + 50(0.939)(0.061)^{49} + 1225(0.939)^2(0.061)^{48}$

**E.** $0.995^{50} + 50(0.005)(0.995)^{49} + 1225(0.005)^2(0.995)^{48}$

### Question 21

The largest value of $\theta$ such that $\theta < 0$, for which $-4\sin\left(2\theta + \frac{\pi}{4}\right)$ has its greatest value is

**A.** $\frac{\pi}{8}$

**B.** $\frac{3\pi}{8}$

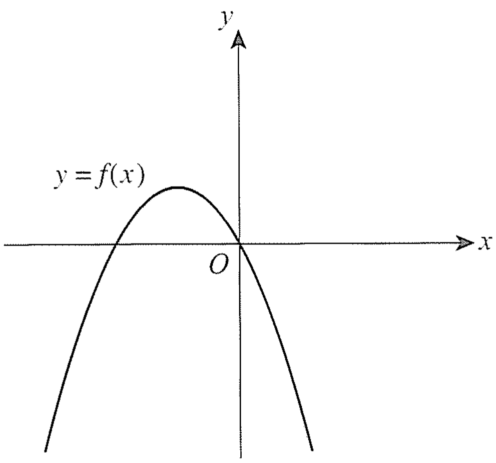
**C.** $-\frac{\pi}{8}$

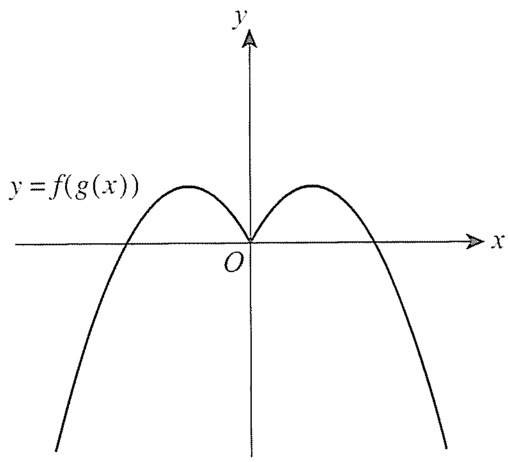
**D.** $-\frac{3\pi}{8}$

**E.** $-\frac{5\pi}{8}$

**Question 22**

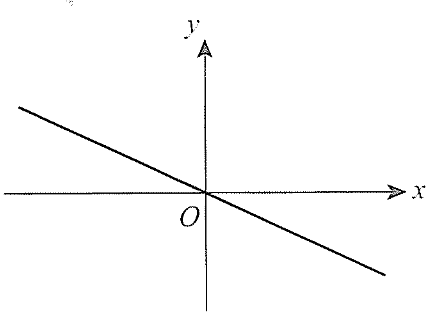
The graphs of $y = f(x)$ and $y = f(g(x))$ are shown below.



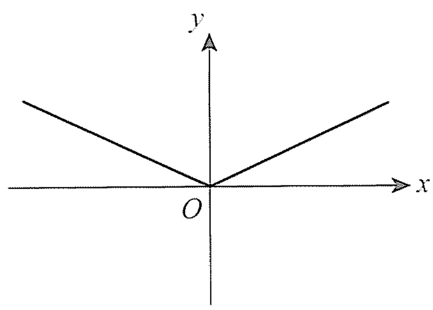




The graph of $y = g(x)$ could be represented by

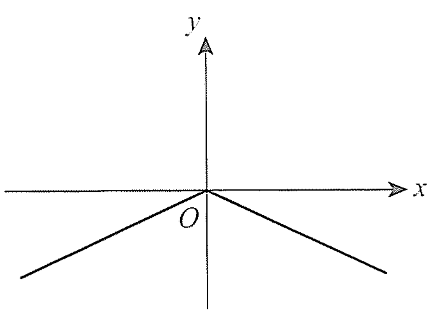
**A.**



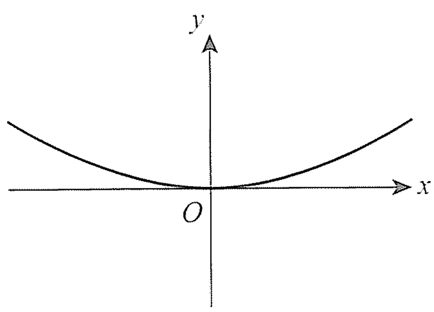

**B.**



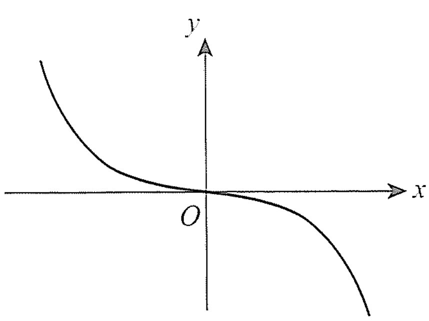

**C.**




**D.**




**E.**




**END OF SECTION 1**

## SECTION 2

**Instructions for Section 2**

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.

### Question 1

The continuous random variable $X$ represents the lifetime, in years, of a new type of electric battery to power hybrid cars. $X$ has a probability density function given by

$$f(x) = \begin{cases} -ae^{ax} + e & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

**a.** Use calculus to show that $a = 1$.

2 marks

**b.** Find the **exact** probability that the battery will last at most 6 months.

2 marks

If a battery lasts less than 3 months it is replaced under warranty. If it lasts between 3 and 6 months it is replaced at half the selling price. If it lasts in excess of 6 months the owner is totally responsible for the replacement cost.

Given that the selling price is $\$s$ per battery and the cost of production is \$450 per battery, a discrete random variable $P$, representing the profit per battery, can be defined.

**c.** Complete the probability distribution table for $P$ in terms of $s$ and the associated probability values, correct to 4 decimal places.

| | < 3 months | between 3 and 6 months | > 6 months |
|---|---|---|---|
| $P$ | | $\frac{1}{2}s - 450$ | |
| $\Pr(P = p)$ | | | |

4 marks

**d.** Find an expression in terms of $s$ for the mean profit per battery and find the minimum selling price, correct to the nearest dollar, to ensure the profit per battery exceeds \$250.00.

2 marks

**e.** The batteries are believed to be viable for small trucks but each small truck requires three batteries.

Given that batteries fail independently of one another, find the probability, correct to 3 decimal places, that at the end of 6 months more than one of the three batteries has failed.

3 marks
Total 13 marks

**Question 2**

Jane Blonde is a secret service agent for the Military Intelligence Agency, MIA. As she is an elite athlete and holder of a Masters in Engineering she has been commissioned to design a fitness track. The path of the track can be modelled by a function of the form

$$H(x) = \begin{cases} mx & 0 \le x < 10 \\ 5\sin\left(\frac{\pi}{15}(x-10)\right) + 20 & 10 \le x \le 85 \\ -\frac{\pi}{3}x + \frac{60 + 85\pi}{3} & 85 < x \le A \end{cases}$$

where $H$ is the height, in metres, above the horizontal ground level and $x$ is the distance, in metres, from a point $O$ where the track will start and $m$ and $A$ are real constants.

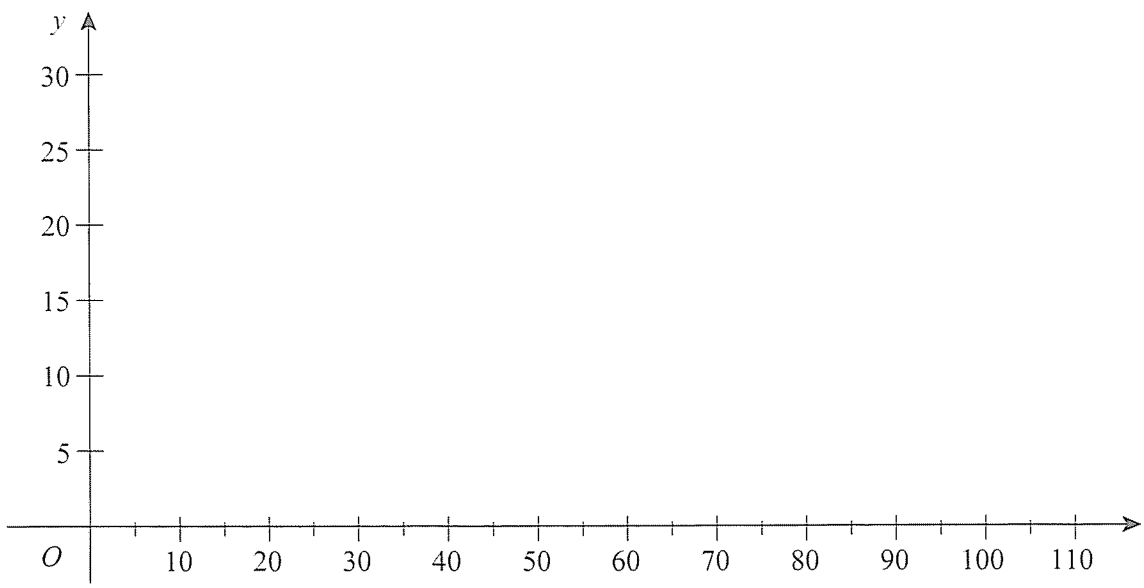
**a.** If $H$ is continuous on $(0, A)$, find the value of $m$.

1 mark

**b.** Given that $H(A) = 0$, find the exact value of $A$.

1 mark

**c.** On the axes below, sketch the graph of $H$ for $0 \le x \le A$.




3 marks

**d.** State the maximum height reached by the track and find the value(s) of $x$ when this maximum is reached.

2 marks

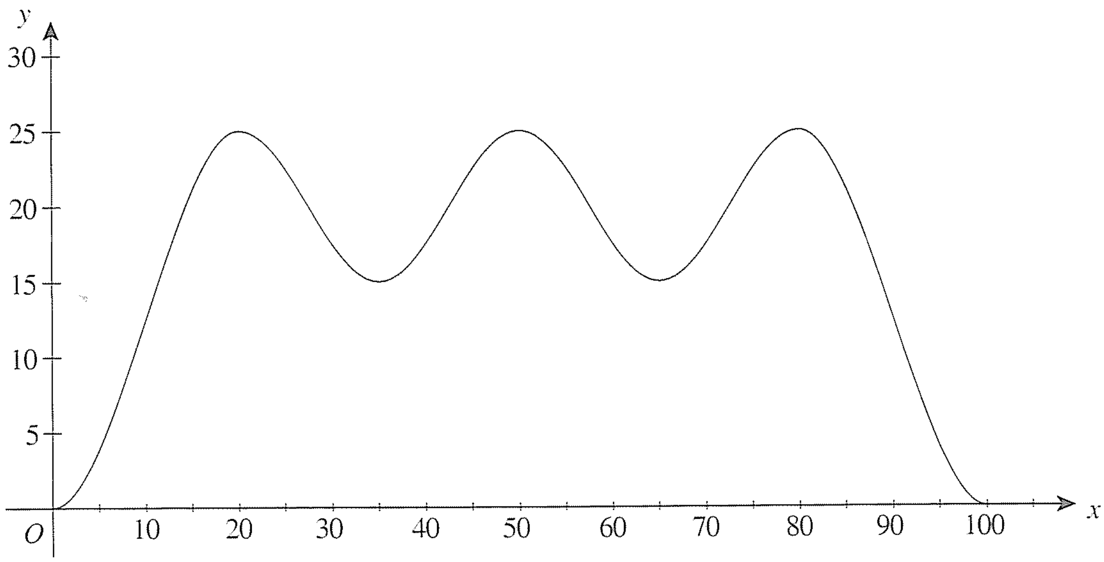
**e.** Use calculus to show that the graph of $H$ is not smooth at $x = 10$ and determine whether or not it is smooth at $x = 85$.

3 marks

Jane has reviewed her model and has changed it to

$$H(x) = \begin{cases} \dfrac{3x^2}{16} - \dfrac{x^3}{160} & 0 \le x \le 20 \\ 5\cos\left(\dfrac{\pi}{15}(x-20)\right) + 20 & 20 < x \le 80 \\ \dfrac{x^3}{160} - \dfrac{27x^2}{16} + 150x - 4375 & 80 < x \le 100 \end{cases}$$

The graph of $H$ in this case is smooth, as shown below.




**f.** **i.** The curve with equation $y = \dfrac{3x^2}{16} - \dfrac{x^3}{160}$ undergoes the following transformations.

- reflection in the $y$-axis
- translation of 100 units parallel to the $x$-axis

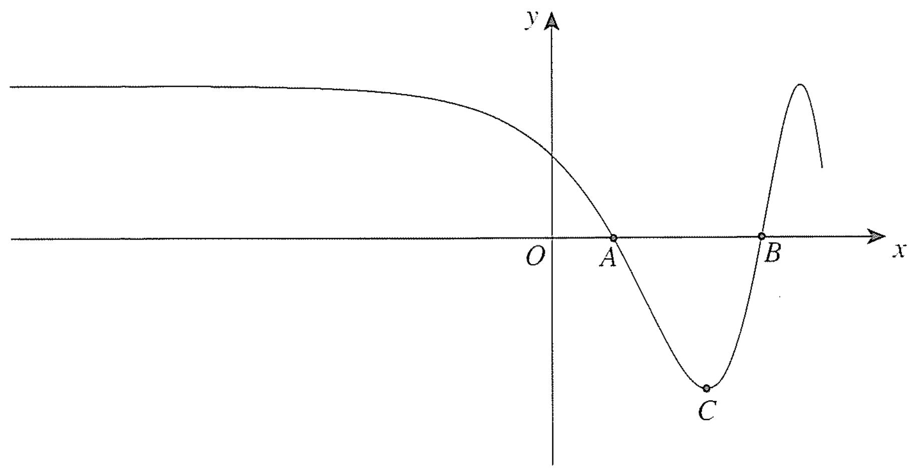
Show that it becomes $y = \dfrac{x^3}{160} - \dfrac{27x^2}{16} + 150x - 4375$.

**ii.** Write down definite integrals in terms of $\frac{3x^2}{16} - \frac{x^3}{160}$ and $5\cos\left(\frac{\pi}{15}(x-20)\right) + 20$ that would determine the area between the graph of $H$ and the $x$-axis and calculate this area.

2 + 2 = 4 marks
Total 14 marks

## Question 3

Part of the graph of the function $f : R \rightarrow R, f(x) = \cos(e^x)$ is shown below.




**a.** Explain why the graph of $f$ has a horizontal asymptote at $y = 1$.

1 mark

**b.** Find the exact $x$-intercepts for points $A$ and $B$.

2 marks

**c.** On the diagram on page 21, point $C$ is the first stationary point for which $x > 0$.

Find the exact coordinates of $C$.

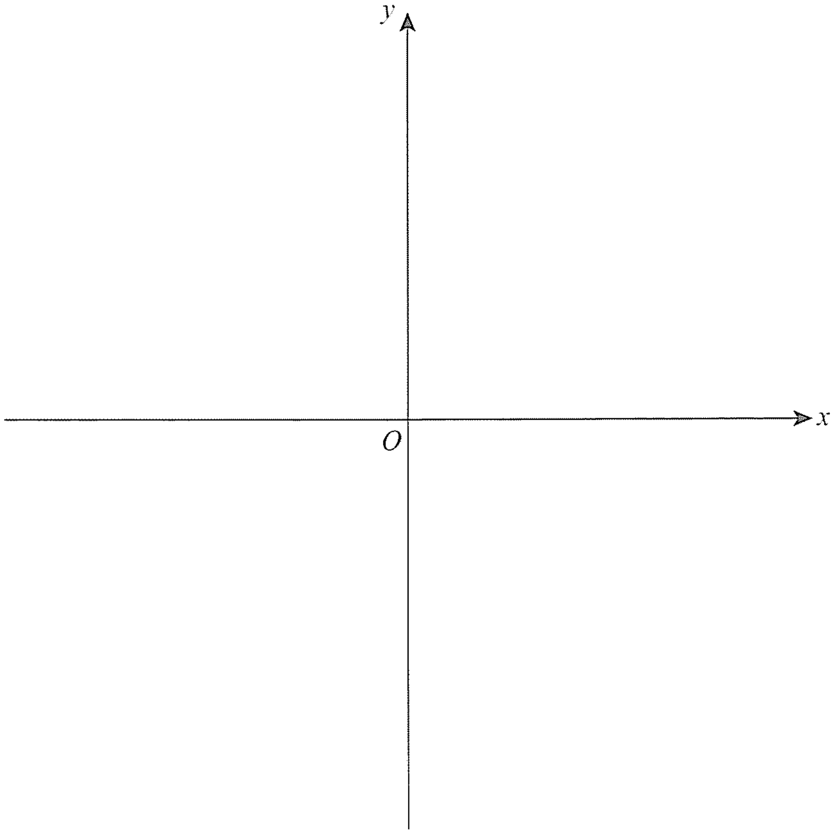
2 marks

Let $g$ be the function $g : [-1, k] \rightarrow R$, where $g(x) = f(x)$.

**d.** State the maximum value of $k$ such that the inverse function $g^{-1}$ exists.

1 mark

**e.** **i.** On the axes below sketch graphs of $g$ and $g^{-1}$, carefully labelling any axis intercepts and end points with their exact values.




**ii.** Given $a < 1$, find the value of $a$ for which $g(a) = g^{-1}(a)$ giving your answer correct to two decimal places.

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$3 + 1 = 4$ marks
Total 10 marks

**Question 4**

**a.** The cubic $2x^3 + 6x^2 - 2x - 6$ can be expressed in the form $2(x + a)(x + 1)(x - 1)$. Find $a$.

1 mark

**b.** The graph of $y = 2x^3 + 6x^2 - 2x - 6$ is reflected in the $y$-axis, then reflected in the $x$-axis and then translated 2 units in the positive $x$ direction to give the graph of $y = h(x)$.

Show that $h(x) = 2(x - 5)(x - 3)(x - 1)$.

3 marks

**c.** Prove that the graphs of $y = 2x^3 + 6x^2 - 2x - 6$ and $y = h(x)$ meet, and locate the coordinates of any intersection points.

2 marks

**d.** The graph of $y = 2x^3 + 6x^2 - 2x - 6$ may be obtained from the graph of $y = h(x)$ using a single transformation.

State this transformation.

1 mark

**e.** For the graph of $y = 2x^3 + 6x^2 - 2x - 6$, let $p$ be the horizontal distance between the maximum and minimum turning points and let $q$ be the vertical distance between the maximum and minimum turning points. The graph undergoes a dilation of factor 0.5 away from the $x$-axis followed by a dilation of factor 3 away from the $y$-axis.

Let $d$ be the distance between the maximum and minimum turning points on the transformed graph. Express $d$ in terms of $p$ and $q$.

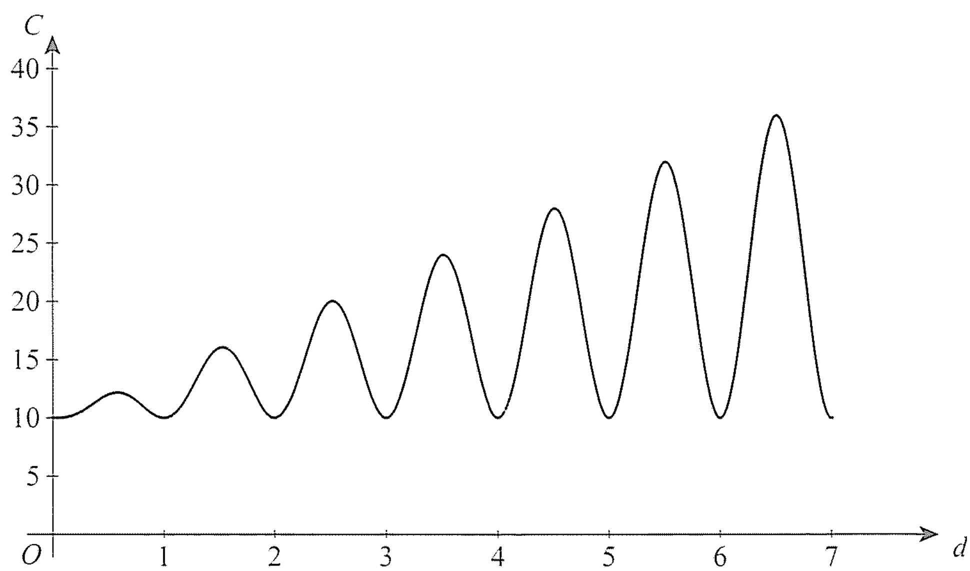
2 marks
Total 9 marks

### Question 5

A local petrol service station has been monitoring the number of cars per hour visiting the service station over a seven day cycle. The management has analysed the data and developed a mathematical model for $C$ (the number of cars per hour) as a function of $d$ (time in days).

$$C(d) = 2(d - d\cos(2\pi d)) + 10, \text{ where } d \in [0,7]$$

Take $d = 0$ as corresponding to 12:00 am Monday, i.e. the beginning of Monday.

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**a.** Given that the minimum value of $C$ each day equals 10, show using algebra that this value of $C$ occurs at the same time each day.

3 marks

**b.** When more than 30 cars/hour visit the service station, the management needs to increase staffing levels.

Find the number of hours, correct to 1 decimal place, over the 7 day cycle when more than 30 cars/hour are visiting the service station.

2 marks

**c.** **i.** Show that if $f(x) = x\sin(2\pi x)$, then $f'(x) = \sin(2\pi x) + 2\pi x\cos(2\pi x)$.

**ii.** Hence, find an antiderivative of $x\cos(2\pi x)$.

1 + 3 = 4 marks

Over any one-day period the area under the curve of $C(d)$ represents the **average** number of cars per hour that visit the service station during that period.

**d.** Using your answer to **part c. ii.** determine the number of cars, to the nearest car, that visit the service station on Wednesday.

3 marks

Total 12 marks

**END OF QUESTION AND ANSWER BOOKLET**