Neap:

Trial Examination 2008

VCE Mathematical Methods Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	**C**	D	E
2	A	B	C	D	**E**
3	A	B	**C**	D	E
4	A	B	C	D	**E**
5	A	B	C	**D**	E
6	A	**B**	C	D	E
7	A	B	**C**	D	E
8	A	B	C	D	**E**
9	**A**	B	C	D	E
10	**A**	B	C	D	E
11	A	B	C	**D**	E
12	A	B	C	D	**E**
13	A	B	**C**	D	E
14	A	B	C	D	**E**
15	**A**	B	C	D	E
16	A	B	C	**D**	E
17	A	**B**	C	D	E
18	**A**	B	C	D	E
19	A	B	C	D	**E**
20	A	B	**C**	D	E
21	A	B	C	**D**	E
22	A	B	**C**	D	E

SECTION 1

Question 1 **C**

The turning point maximum of g has a value of 2 when $x = -\frac{1}{2}$.

Thus $x = -\frac{1}{2}$ must be within the domain.

We require the solution to $g(x) = -7$.

$$\text{Now } 2 - (2x+1)^2 = -7$$
$$(2x+1)^2 = 9$$
$$2x + 3 = \pm 3$$
$$x = -2, 1$$

Thus if we take a domain from $x = -2$ to $x = 1$ we include a range of $[-7, 2]$.

As the turning point is part of the range, we can have a domain of $\left[-2, -\frac{1}{2}\right]$ or $\left[-\frac{1}{2}, 1\right]$

Thus $a = -\frac{1}{2}$, $b = 1$

Question 2 **E**

Given that $f(x+2) = x^2 - 5x - 6$, we deduce that $f(x) = (x-2)^2 - 5(x-2) - 6$.

Expanding gives $f(x) = x^2 - 9x + 8$.

$$\text{Thus } g(f(x)) = g(x^2 - 9x + 8)$$
$$= 2\log_e(x^2 - 9x)$$

Question 3 **C**

$$\int \frac{1}{2}\cos(5x) - \frac{3}{2x} + 4x^{-2}dx = \frac{1}{2} \times \frac{1}{5}\sin(5x) - \frac{3}{2}\log_e(x) - 4x^{-1}$$
$$= \frac{\sin(5x)}{10} - \frac{3\log_e(x)}{2} - \frac{4}{x}$$
$$= \frac{x\sin(5x) - 15x\log_e(x) - 40}{10x}$$

Question 4 **E**

$$g(f(x)) = \log_e(2(e^{3-x^2}))$$
$$= \log_e(2) + \log_e(e^{3-x^2})$$
$$= \log_e(2) + 3 - x^2$$
$$g'(f(x)) = -2x$$
$$g'(f(2)) = -2(2) = -4$$

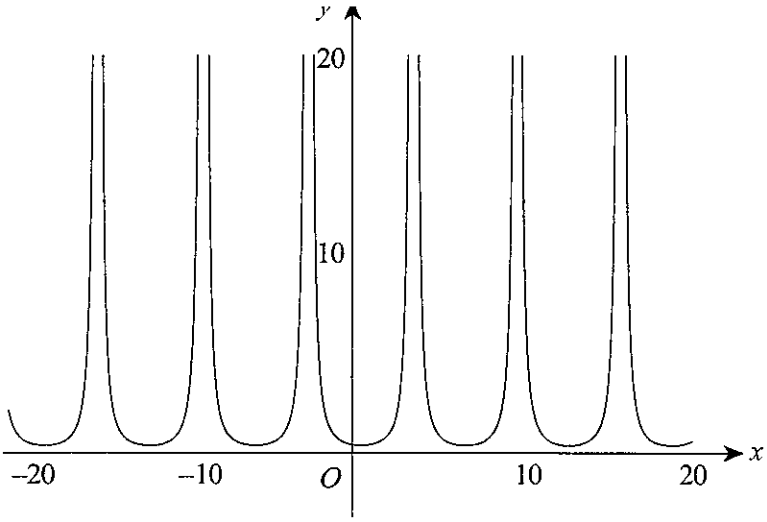
Question 5 **D**

$$\int \sec^2(kx) - 4\left(\frac{1}{ax+b}\right)dx = \frac{1}{k}\tan(kx) - \frac{4}{a}\log_e|ax+b| + c$$

Applying the formula here gives $\int \sec^2(2x) - \frac{4}{3x-5}dx = \frac{1}{2}\tan(2x) - \frac{4}{3}\log_e|3x-5| + c$

Question 6 **B**

Part of the graph of $f: R \to R,\ y = \frac{1}{\left|1 + \sin\left(x - \frac{\pi}{3}\right)\right|}$ is shown below.



The minimum turning point can be found using a calculator or by recognising that $\left|1 + \sin\left(x - \frac{\pi}{3}\right)\right|$ has a maximum value of 2 and hence $f(x)$ will have a minimum value of 0.5. So we are looking for the alternative whose function is such that its range equals $[0.5, \infty)$.

The correct response is **B**, as $1 + \sin\left(x - \frac{\pi}{3}\right)$ is never smaller than zero, so

$1 + \sin\left(x - \frac{\pi}{3}\right) = \left|1 + \sin\left(x - \frac{\pi}{3}\right)\right|$ for all values of x.

Question 7 **C**

Given $f'(4) = 0\ \therefore \frac{x+4}{2} = 4$

$x = 4 \Rightarrow g'(4) = 0$

and $f'(8) = 0\ \therefore \frac{x+4}{2} = 8$

$x = 12 \Rightarrow g'(12) = 0$

and $f'(x) < 0\ \ x \in (4, 8)$

$\therefore g'(x) > 0\ \ x \in (4, 12)$

Question 8 **E**

$f(x)$ is clearly continuous as it can be drawn without the pen leaving the page.

$f(x)$ is **not** differentiable at $x = 0$ or $x = 2$.

Question 9 **A**

$$\int_0^{\pi}\left(\sin\left(\frac{x}{3}\right) - x^{-\frac{1}{2}}\right)dx = \left[-3\cos\left(\frac{x}{3}\right) - 2x^{\frac{1}{2}}\right]_0^{\pi}$$

$$= \left(-3\left(\frac{1}{2}\right) - 2\sqrt{\pi}\right) - (-3(1) - 0)$$

$$= \frac{3}{2} - 2\sqrt{\pi}$$

Question 10 **A**

Let $f^{-1}(2a) = y$

$\therefore f(y) = 2a$

Now $f(x) = \sqrt[3]{(x^3 + a^3)}$, so $f(y) = \sqrt[3]{(y^3 + a^3)} = 2a$

Cubing both expressions for $f(y)$ gives $y^3 + a^3 = (2a)^3$

$$y^3 = 7a^3$$

$$y = \sqrt[3]{7}a$$

Question 11 **D**

$$\frac{e^{3x} + 12e^x}{e^x} = 7e^x$$

Multiplying both sides by e^x gives

$$e^{3x} + 12e^x = 7e^{2x}$$

$$e^{3x} - 7e^{2x} + 12e^x = 0$$

$$e^x(e^{2x} - 7e^x + 12) = 0$$

$$e^x(e^x - 3)(e^x - 4) = 0$$

$$e^x \neq 0,\ e^x = 3 \text{ or } e^x = 4$$

$$x = \log_e(3) \text{ or } \log_e(4)$$

$$= \log_e(3) \text{ or } 2\log_e(2)$$

Question 12 **E**

All have a period of $\left(\frac{2\pi}{\pi/6}\right) = 12$.

All have an amplitude of 2.

All are translated up 3 units.

All have a y-intercept of $3 + \sqrt{3}$.

Alternatively, checking on graphic calculator confirms **E**.

Question 13 C

The gradient of a normal with the equation $2x - 5y = 10$ is $\frac{2}{5}$.

The gradient of the tangent must therefore equal $-\frac{5}{2}$.

So $f'(-2) = -\frac{5}{2}$

Question 14 E

Let us consider the domains first.
For f_1 we must have $(x+2)^2 > 0$, which is satisfied by $R\backslash\{-2\}$.
For f_2 we must have $|x^2 - 4| > 0$, which is satisfied by $R\backslash\{\pm 2\}$.
For f_3 we must have $|x+2| > 0$, which is satisfied by $R\backslash\{-2\}$.
So f_1 and f_3 both have the desired domain.
Close inspection shows that both also have the desired range of $R^+ \cup \{0\}$.

Question 15 A

Starting with $y = e^{2x}$ and reflecting in the y-axis gives $y = e^{-2x}$.
A translation of 1 unit left gives $y = e^{-2(x+1)}$.
A dilation by a factor of 3 away from the y-axis is achieved by replacing x with $\frac{x}{3}$.

We get $y = e^{-2\left(\frac{1}{3}x + 1\right)} = e^{\frac{-2}{3}x - 2} = e^{-2}e^{\frac{-2}{3}x} = \left(\frac{1}{e^2}\right)e^{\frac{-2}{3}x}$

Question 16 D

$0 = 3\cos^2(x) - \sqrt{3}\sin(x)\cos(x)$

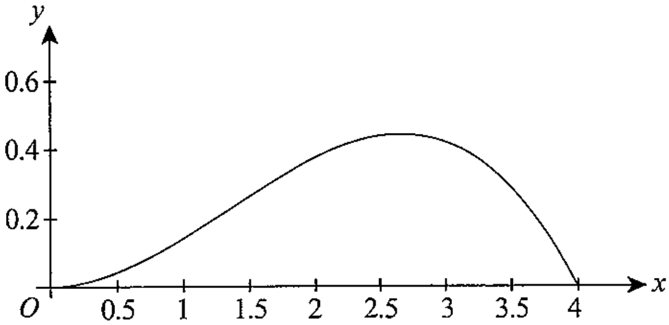
$0 = \cos(x)(3\cos(x) - \sqrt{3}\sin(x))$

$\cos(x) = 0 \quad$ or $\quad \tan(x) = \frac{3}{\sqrt{3}} = \sqrt{3}$

$x = \frac{\pi}{2}$ or $\frac{3\pi}{2} \qquad x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

Question 17 B

The mode of X is the value of x for which the graph of $y = \frac{3}{64}x^2(4-x)$ has a maximum value in the domain $[0, 4]$.



Using the calculator, the maximum turning point is at $(2.66667, 0.44444)$,

i.e. $x = 2\frac{2}{3} = \frac{8}{3}$ is the mode.

Alternatively, using calculus

$$\frac{dy}{dx} = \frac{3}{64}(8x - 3x^2)$$

$$\frac{dy}{dx} = 0, \text{ if } x = 0 \text{ or } \frac{8}{3}$$

Hence the mode is $x = \frac{8}{3}$ (where the maximum turning point occurs).

Question 18 A

Let X be a random variable representing the true speed of cars which the speed camera registers as travelling 65 km/h. Thus X is normally distributed with a mean of 63 and a standard deviation of 3.

We require $\Pr(X < 62)$.

Now $\Pr(X < 62) = \Pr\left(Z < \frac{62-63}{3}\right) = \Pr\left(Z < -\frac{1}{3}\right) = 0.3694$.

Alternatively, use normcdf (–9999, 62, 63, 3)

The percentage of cars registered as travelling 65 km/h that are actually travelling less than 62 km/h is 36.9%.

Question 19 E

$$\begin{aligned}\text{var}(X) &= \text{E}(X^2) - [\text{E}(X)]^2 \\ &= 234 - 15^2 \\ &= 9\end{aligned}$$

Thus $\sigma = 3$

We require $\Pr(\text{E}(X) - 2\sigma \le X \le \text{E}(X) + 2\sigma)$

$$= \Pr(15 - 2 \times 3 \le X \le 15 + 2 \times 3)$$

$$= \Pr(9 \le X \le 21)$$

Question 20 C

Let X represent the number of cured rats from the first sample of 30. X is a binomial random variable with probability of success p.

$$\Pr(X \geq 1) = 0.85$$
$$1 - \Pr(X = 0) = 0.85$$
$$\Pr(X = 0) = 0.15$$
$$\Rightarrow (1 - p)^{30} = 0.15$$
$$1 - p = 0.15^{\frac{1}{30}}$$
$$p = 0.0613$$

Let Y represent the number of rats cured in the sample of 50. We require $\Pr(Y \leq 2)$.

$$= \Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2)$$
$$= 0.939^{50} + 50(0.061)(0.939)^{49} + 1225(0.061)^2(0.939)^{48}$$

Question 21 D

$-4\sin\left(2\theta + \frac{\pi}{4}\right)$ will have its greatest value when $\sin\left(2\theta + \frac{\pi}{4}\right) = -1$.

$\left(2\theta + \frac{\pi}{4}\right) = -\frac{\pi}{2}$, given $\theta < 0$

Thus $2\theta = -\frac{3\pi}{4}$

$\theta = -\frac{3\pi}{8}$

Question 22 C

As $f(g(x)) = f(x)$ for $x < 0$

$\therefore g(x) = x$, for $x < 0$

Also $f(g(x)) = f(-x)$ for $x > 0$

$\therefore g(x) = -x$, for $x > 0$

SECTION 2

Question 1

a. As f is a probability function $\int_0^1 (-ae^{ax} + e)\ dx = 1$.

Thus $[-e^{ax} + ex]_0^1 = 1$ M1

$(-e^a + e) - (-e^0 + 0) = 1$

$-e^a + e + 1 = 1$

$\therefore\ e^a = e$

$a = 1$ A1

b. As X is measured in years, we require $\Pr(X \le 0.5)$.

$\Pr(X \le 0.5) = \int_0^{0.5} (-e^x + e)dx$

$= [-e^x + ex]_0^{0.5}$ M1

$= (-e^{0.5} + 0.5e) - (-e^0 + 0)$

$= 1 + \frac{e}{2} - \sqrt{e}$ A1

c. We require the associated probabilities which relate to the battery lasting less than 3 months, between 3 and 6 months and greater than 6 months.

Now using the given result in **part b.** $\Pr(X > 0.5) = 1 - \Pr(X \le 0.5)$

$= 1 - \left(\frac{e}{2} - \sqrt{e} + 1\right) = \sqrt{e} - \frac{e}{2} = 0.2896$ A1

To find the probability the battery lasts less than 3 months we require $\Pr(X \le 0.25)$.

$\Pr(X \le 0.25) = \int_0^{0.25} (-e^x + e)dx = 0.3955$ using calculator. A1

$\Pr(0.25 \le X \le 0.5) = 1 - (0.3955 + 0.2896)$

$= 0.3149$ A1

P	-450	$\frac{1}{2}s - 450$	$s - 450$
$\Pr(P = p)$	0.3955	0.3149	0.2896

A1

d. The mean profit per battery is E(P), where

$E(P) = -450 \times 0.3955 + \left(\frac{1}{2}s - 450\right) \times 0.3149 + (s - 450) \times 0.2896$

$= 0.44705s - 450$

For $E(P) > 250$, $0.44705s - 450 > 250$, A1

so $s > 1565.82$

For the profit to exceed \$250 we set the selling price at \$1566. A1

e. Define the random variable X as the number of batteries lasting less than 6 months.

X is a binomial random variable with $n = 3$ and $p = 0.3955 + 0.3149 = 0.7104$ M1

We require $\Pr(X > 1) = \Pr(X = 2) + \Pr(X = 3)$ M1

$$= \binom{3}{2}(0.7104)^2(0.2906) + \binom{3}{3}(0.7104)^3$$

$$= 0.797$$ A1

Alternatively, using calculator

$1 - \text{Bincdf}(3, 0.7104, 1)$

$= 0.797$

Question 2

a. $H(10) = 5\sin(0) + 20$

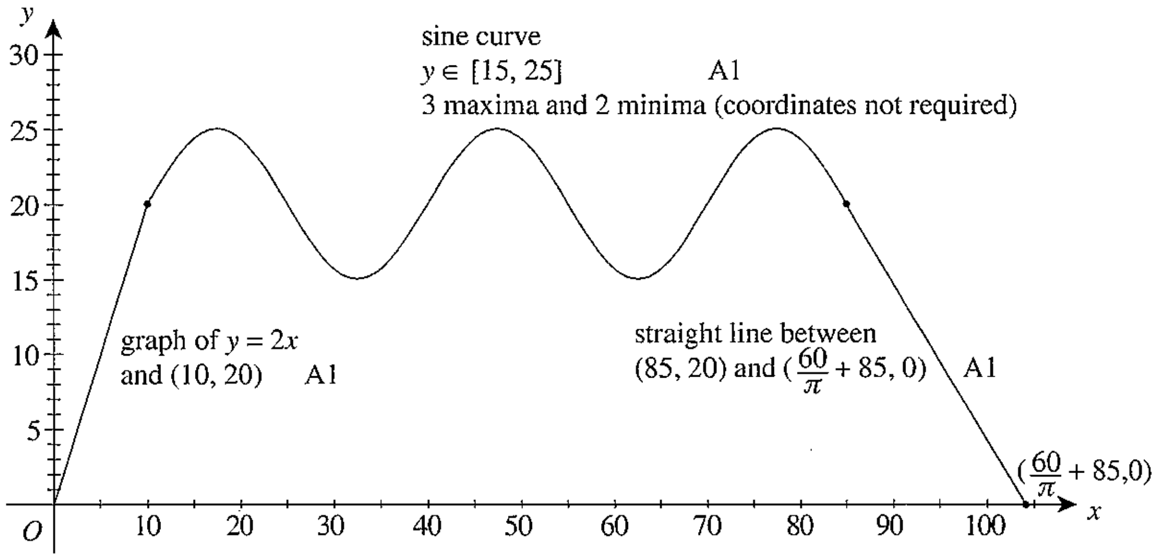
$= 20$

$\therefore\ m = \dfrac{20}{10} = 2$ A1

b. $-\dfrac{\pi}{3}A + \dfrac{60 + 85\pi}{3} = 0$

$A = \dfrac{60}{\pi} + 85$ A1

c.



d. $H_{\max} = 25$ m A1

$5\sin\left(\dfrac{\pi}{15}(x - 10)\right) + 20 = 25$

$\sin\left(\dfrac{\pi}{15}(x - 10)\right) = 1$

$\dfrac{\pi}{15}(x - 10) = \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}$

$x - 10 = \dfrac{15}{2}, \dfrac{75}{2}, \dfrac{135}{2}$ or use graphics calculator.

$x = \dfrac{35}{2}, \dfrac{95}{2}, \dfrac{155}{2}$ A1

e. $x < 10 \quad m = 2$

If $y = 5\sin\left(\frac{\pi}{15}(x-10)\right) + 20$,

$$y' = \frac{\pi}{3}\cos\left(\frac{\pi}{15}(x-10)\right).$$ M1

at $x = 10$

$$y' = \frac{\pi}{3}$$

So the gradients are different, and therefore **not smooth.** A1

$x > 85 \quad m = -\frac{\pi}{3}$

If $y = 5\sin\left(\frac{\pi}{15}(x-10)\right)$

$$y' = \frac{\pi}{3}\cos\left(\frac{\pi}{15}(x-10)\right)$$

at $x = 85$

$$y' = \frac{\pi}{3}\cos(5\pi) = -\frac{\pi}{3}$$

So the gradients are equal, and therefore **smooth.** A1

f. **i.** For the reflection in the y-axis

$$y = \frac{3(-x)^2}{16} - \frac{(-x)^3}{160}$$

$$= \frac{3x^2}{16} + \frac{x^3}{160}$$ M1

For the translation of 100 units parallel to the x-axis

$$y = \frac{3(x-100)^2}{16} + \frac{(x-100)^3}{160}$$

$$= \frac{3x^2 - 600x + 300}{16} + \frac{x^3 - 300x^2 + 30\,000x + 1\,000\,000}{160}$$

$$= \frac{x^3}{160} - \frac{27x^2}{16} + 150x - 4375$$ A1

ii. $$A = 2\int_0^{20}\left(\frac{3x^2}{16} - \frac{x^3}{160}\right)dx + 2\int_{20}^{50} 5\cos\left(\frac{\pi}{15}(x-20)\right) + 20\ dx$$ A1

$$= 2 \times 250 + 2 \times 600$$

$$= 1700 \text{ m}^2$$ A1

Question 3

a. As $x \to -\infty$, $e^x \to 0$, and so $\cos(e^x) \to \cos(0)$.
Thus $f(x) \to 1$ which means that $y = 1$ is a horizontal asymptote. A1

b. Solving $f(x) = 0$ for x intercepts gives $\cos(e^x) = 0$, and so $e^x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. M1

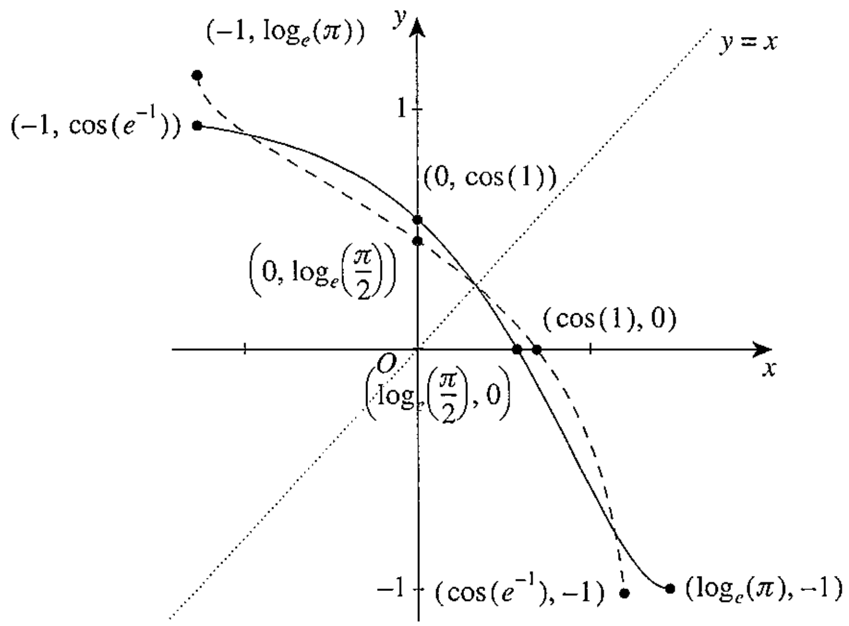
Thus $x = \log_e\left(\frac{\pi}{2}\right)$ or $\log_e\left(\frac{3\pi}{2}\right)$, and so

point A has an x-intercept $\log_e\left(\frac{\pi}{2}\right)$ and point B has an x-intercept $\log_e\left(\frac{3\pi}{2}\right)$. A1

c. Point C will be the first positive solution to the equation $f'(x) = 0$.
Now $f'(x) = -e^x \sin(e^x) = 0$, so $\sin(e^x) = 0$. M1
Thus the first positive solution is when $e^x = \pi$ and $x = \log_e(\pi)$.
Now $f(\log_e(\pi)) = \cos(e^{\log_e(\pi)}) = \cos(\pi) = -1$,
so, point C has coordinates $(\log_e(\pi), -1)$. A1

d. g^{-1} exists if g is a one to one function. Thus the largest value possible for k is the x coordinate of C.
Thus $k = \log_e(\pi)$. A1

e. **i.**



$g^{-1}(x)$ reflection about $y = x$ M1
axial intercepts A1
terminal points A1

ii. $g(a) = g^{-1}(a)$ on three occasions as shown on the graph above. These coordinates are $(-0.767, 0.894)$, $(1, 1)$ and $(0.894, -0.767)$.
Thus if $a < 1$, the required value of a is -0.77, correct to 2 decimal places. A1

Question 4

a. $2x^3 + 6x^2 - 2x - 6$

$= 2x^2(x+3) - 2(x+3)$

$= 2(x+3)(x^2-1)$

$= 2(x+3)(x-1)(x+1)$

Hence, $a = 3$ A1

Alternatively, in the expansion of $2(x+a)(x+1)(x-1)$, the constant term is $2(a)(1)(-1) = -2a$. Hence, $-2a = -6$, giving $a = 3$.

b. After reflection in the y-axis we have

$y = 2(-x+3)(-x+1)(-x-1) = -2(x-3)(x-1)(x+1)$ A1

A reflection in the x-axis now gives $y = 2(x-3)(x-1)(x+1)$. A1

A translation of 2 units in the positive x direction requires $x \to (x-2)$ so that we now have

$h(x) = 2[(x-2)-3][(x-2)-1][(x-2)+1]$

$= 2(x-5)(x-3)(x-1)$ A1

c. Points of intersection occur when

$2x^3 + 6x^2 - 2x - 6 = 2(x-5)(x-1)(x-3)$

$2x^3 + 6x^2 - 2x - 6 = 2x^3 - 18x^2 + 46x - 30$ M1

$24x^2 - 48x + 24 = 0$

$x^2 - 2x + 1 = 0$

This has a double root at $x = 1$ indicating the graphs touch at $x = 1$.

$h(1) = 2 \times -4 \times -2 \times 0 = 0.$ A1

The graphs touch at (1, 0).

d. Comparing the factored form of both functions

$h(x) = 2(x-5)(x-1)(x-3)$ and $y = 2(x+3)(x-1)(x+1)$ it can be seen that changing x to $x+4$ in $h(x)$ will result in $y = 2(x-1)(x+3)(x+1)$. Thus a translation of 4 units to the left is required. A1

e. A dilation of factor 0.5 away from the x-axis means that the vertical distance q will be reduced to $0.5q$. A dilation of factor 3 away from the y-axis means that the horizontal distance p will be increased to $3p$.

Thus $d = \sqrt{(3p)^2 + (0.5q)^2}$ M1

$= \sqrt{9p^2 + \frac{1}{4}q^2}$ A1

Question 5

a. $2d - 2d\cos(2\pi d) + 10 = 10$ M1

$2d - 2d\cos(2\pi d) = 0$

$2d(1 - \cos(2\pi d)) = 0$

$d = 0$ or $\cos(2\pi d) = 1$

$2\pi d = 2k\pi \quad k = 0, 1, 2, \ldots$

$d = 0, 1, 2, 3, 4, 5, 6, 7.$ A1

$\therefore$ minimum occurs at midnight each day. A1

b. Plotting $y = 30$ together with the graph of $y = C(d)$ on a graphics calculator gives intersections (5.411, 30), (5.607, 30), (6348, 30) and (6.667, 30). M1

Total time for which $C > 30$, is $(5.607 - 5.411) + (6.667 - 6.348)$

$= 0.515$ days

$= 0.515 \times 24$ hours $= 12.4$ hours A1

c. **i.** $f(x) = u \times v$

$f'(x) = u'v + uv'$

$= 1\sin(2\pi x) + x(2\pi\cos(2\pi x))$ A1

$= \sin(2\pi x) + 2\pi x\cos(2\pi x)$

ii. $\int(\sin(2\pi x) + 2\pi x\cos(2\pi x))dx = x\sin(2\pi x)$

$\therefore \int 2\pi x\cos(2\pi x)dx = x\sin(2\pi x) - \int\sin(2\pi x)dx$ M1

$2\pi\int x\cos(2\pi x)dx = x\sin(2\pi x) + \frac{1}{2\pi}\cos(2\pi x)$ M1

$\therefore \int x\cos(2\pi x)dx = \frac{1}{2\pi}\left(x\sin(2\pi x) + \frac{1}{2\pi}\cos(2\pi x)\right)$ A1

d. Wednesday $d = 2$ to $d = 3$

$\therefore \int_2^3 (2d - 2\cos(2\pi d) + 10)dd$

$= \left[d^2 - 2\left(\frac{1}{2\pi}(d\sin(2\pi d)) + \frac{1}{2\pi}\cos(2\pi d)\right) + 10d\right]_2^3$ M1

$= \left[\left(3^2 - 2\left(\frac{1}{2\pi}(3\sin(6\pi)) + \frac{1}{2\pi}\cos(6\pi)\right) + 10(3)\right) - \left(2^2 - 2\left(\frac{1}{2\pi}(2\sin(4\pi)) + \frac{1}{2\pi}\cos(4\pi)\right) + 10(2)\right)\right]$

$= \left(9 - 2\left(0 + \frac{1}{2\pi}\right) + 30\right) - \left(4 - 2\left(0 + \frac{1}{2\pi}\right) + 20\right)$

$= 15$ cars/hour M1

$\therefore$ number of cars $= 15 \times 24 = 360$ cars. A1