Neap:

Trial Examination 2008

VCE Mathematical Methods Units 3 & 4

Written Examination 1

Suggested Solutions

Question 1

Let $u = \sin(x^2)$ $\quad \therefore y = u^2$ — For use of chain rule M1

$\frac{du}{dx} = ?$ $\quad \frac{dy}{du} = 2u$

let $v = x^2$ $\quad \therefore u = \sin v$

$\frac{dv}{dx} = 2x$ $\quad \frac{du}{dv} = \cos v$

$$\therefore \frac{du}{dx} = \frac{du}{dv} \times \frac{dv}{dx}$$
$$= 2x\cos(x^2)$$
$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2x\cos(x^2) \times 2\sin(x^2)$$
$$= 4x\cos(x^2)\sin(x^2)$$
A1

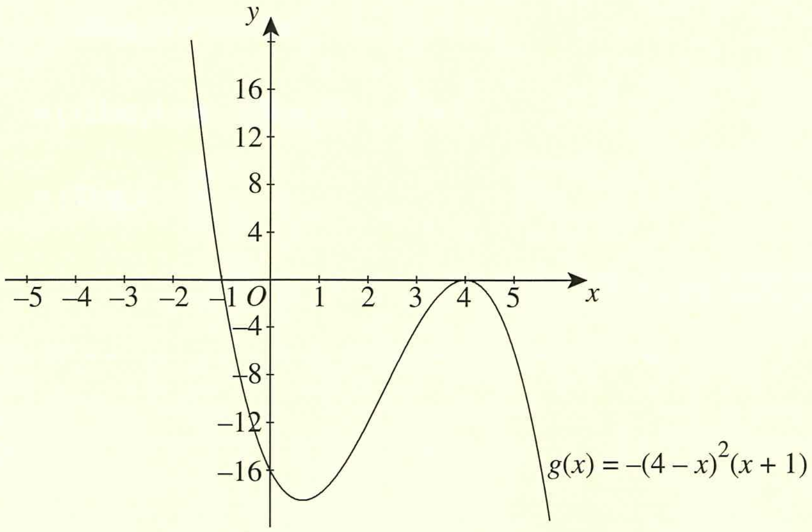
Question 2

a. $f(x) = (x+3)^2(x-2)$
$$= (x^2 + 6x + 9)(x-2)$$
$$= x^3 + 4x^2 - 3x - 18$$
$\therefore a = -4$ A1

Alternatively, $f'(x) = 3x^2 - 2ax - 3$
$= 0$ if $x = -3$

$f'(-3) = 27 + 6a - 3 = 24 + 6a$
$= 0$ if $a = -4$

b. The graph of $y = f(1-x) = f[-(x-1)]$ is obtained by reflecting $y = f(x)$ in the x-axis followed by a translation of 1 unit to the right.



M1

$\therefore \{x: f(1-x) \geq 0\} = (-\infty, -1] \cup \{4\}$ A1

Question 3

$e^{-2x} - 6e^{-x} + 8 = 0$

$(e^{-x} - 2)(e^{-x} - 4) = 0$ M1

$e^{-x} = 2$ or $e^{-x} = 4$

$\Rightarrow x = -\log_e 2$ or $x = -\log_e 4$ A1

Hence, the sum of the roots is $-\log_e(2) - \log_e(4) = -\log_e(8) = \log_e\left(\frac{1}{8}\right)$. A1

Question 4

$f'(x) = \frac{3\pi}{5}\cos\left(\frac{3\pi x}{5}\right)$

$f'(x)_{min} = -\frac{3\pi}{5}$ as minimum value of $\cos(\theta)$ is -1 A1

$\cos\left(\frac{3\pi x}{5}\right) = -1$

$\frac{3\pi x}{5} = \pi$

$x = \frac{5}{3}$ A1

Question 5

a. $g(x) = -2\left[x^2 + 4x + \frac{11}{2}\right]$

$= -2\left[(x^2 + 4x + 4) - 4 + \frac{11}{2}\right]$ M1

$= -2\left[(x+2)^2 + \frac{3}{2}\right]$

$= -2(x+2)^2 - 3$

This is a parabola with a turning point at $(-2, -3)$. A1

Alternatively, $g'(x) = -4x - 8$

$= 0$ if $x = -2$

$g(-2) = -8 + 16 - 11$

$= -3$

so turning point at $(-2,-3)$

b. Dilate $f(x) = x^2$ by a factor of 2 parallel to the y-axis to get $f_1(x) = 2x^2$. A1

Reflect $f_1(x)$ in the x-axis to get $f_2(x) = -2x^2$.

Translate $f_2(x)$ 2 units to the left and 3 units down to get $g(x)$. A1

Note that there are other possible correct solutions.

Question 6

a. $f\left(g\left(\frac{\pi}{3}\right)\right) = \tan\left(2\left(\frac{\pi}{3} + \frac{\pi}{2}\right)\right)$

$= \tan\left(\frac{5\pi}{3}\right)$

$= -\tan\left(\frac{\pi}{3}\right)$

$= -\sqrt{3}$ A1

b. $f(g(x)) = -1 \Rightarrow \tan\left(2\left(x + \frac{\pi}{2}\right)\right) = -1$

$2x + \pi = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, 4\pi - \frac{\pi}{4}$

$2x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$ M1

$x = -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}$

As $x \in [0, 2\pi]$, $x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$ A1

Question 7

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$ M1

$V = 2\pi^2(30r^2 + r^3)$

$\frac{dV}{dr} = 2\pi^2(60r + 3r^2)$

$\therefore \frac{dr}{dt} = \frac{1}{2\pi^2(60r + 3r^2)} \times 20$ A1

$= \frac{1}{2\pi^2(60(2) + 3(2)^2)} \times 20$, when $r = 2$

$= \frac{5}{66\pi^2}$ cm/s A1

Question 8

$\int 2x\log_e(2x) + x\,dx = x^2\log_e(2x)$

$\int 2x\log_e(2x)dx = x^2\log_e(2x) - \int xdx$ M1

$\int x\log_e(2x)dx = \frac{1}{2}\left(x^2\log_e(2x) - \frac{1}{2}x^2\right)$ A1

Question 9

a. By appropriate substitution we obtain the probability values:

$\Pr(X=0)=-4a$ $\quad\Pr(X=1)=-9a$ $\quad\Pr(X=2)=-10a$ $\quad\Pr(X=3)=-7a$ M1

As $\sum \Pr(X=x)=1$ we have $-4a-9a-10a-7a=1$

$a=-\frac{1}{30}$ A1

b. As $E(X)=\sum \Pr(X=x)$ we have

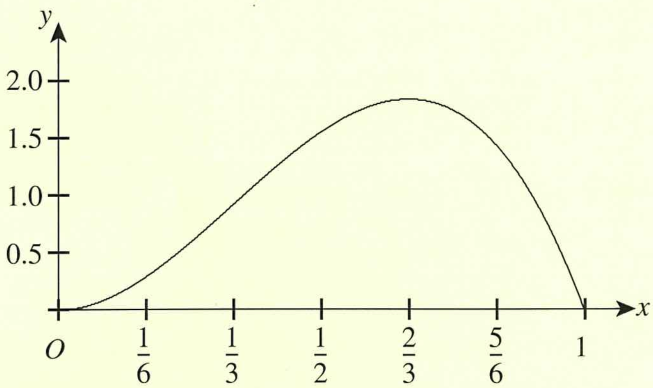
$E(X)=0\times-4a+1\times-9a+2\times-10a+3\times-7a=-50a=-50\times-\frac{1}{30}=\frac{5}{3}$ M1, A1

So we require $\Pr\left(X>\frac{5}{3}\right)$

$=\Pr(X\geq 2)=-17a=\frac{17}{30}$ A1

Question 10

a.



A1

b. $\Pr\left(\frac{1}{2}\leq X\leq 1\right)$

$=\int_{\frac{1}{2}}^{1}(12x^2-12x^3)dx$ M1

$=[4x^3-3x^4]_{\frac{1}{2}}^{1}$

$=(4-3)-\left(\frac{4}{8}-\frac{3}{16}\right)$

$=1-\frac{5}{16}$ A1

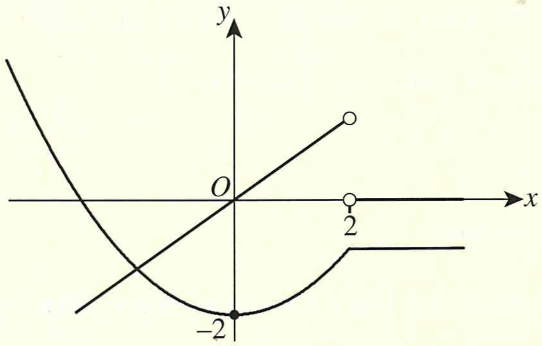
$=\frac{11}{16}$

c. The median is the value of X such that $\Pr(X\leq m)=\frac{1}{2}$

Thus $\int_{0}^{m}12x^2(1-x)dx=\frac{1}{2}$ A1

Question 11

a.



Parabolic with turning point (0,–2) A1

Right-hand end of parabola could end above x-axis

Continuous at $x = 2$ A1

Horizontal line for $x > 2$ A1

b. range of $f(x)$; $y \in [-2, \infty)$ A1

Question 12

a. $f(0) = \frac{12}{(0-2)} - 3$

$= 3$

$f'(x) = -12(x-2)^{-2}$ M1

$f'(0) = -\frac{12}{(0-2)^2}$

$= -3$

$\therefore$ The equation of the tangent is $y = -3x + 3$ A1

b. $\text{Area} = \int_0^2 \frac{12}{x+2} - 3dx - \int_0^1 -3x + 3dx$ Integration of two regions M1

$= [12\log_e(x+2) - 3x]_0^2 - \left[-\frac{3x^2}{2} + 3x\right]_0^1$ M1

$= (12\log_e 4 - 6 - 12\log_e 2) - \left(-\frac{3}{2} + 3\right)$

$= 12\log_e 2 - \frac{15}{2}$ A1