Year 2008

VCE

Mathematical Methods CAS Solutions Trial Examination 2

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SECTION 1

ANSWERS

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2	A	B	C	D	**E**
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SECTION 1

Question 1 **Answer C**

the average value is $\bar{y} = \dfrac{1}{b-a}\displaystyle\int_a^b y\ dx$

$$\bar{y} = \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} 3\sin^2(2x)\,dx = \frac{2}{\pi} \text{ x } \frac{3\pi}{4} = 1.5$$

Question 2 **Answer E**

Let $f(x) = \left|\sin\left(x^2\right)\right|$ then $f(x+h) = \left|\sin\left((x+h)^2\right)\right|$

$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ and $f'(k) = \lim\limits_{h\to 0}\dfrac{\left|\sin\left((k+h)^2\right)\right| - \left|\sin\left(k^2\right)\right|}{h}$

Question 3 **Answer E**

The function $f:[0,c]\to R$ $f(x) = c-2x$ is a straight line, now

$f(0) = c$ and $f(c) = -c$ so that ran $f = [-c,c] = \text{dom } f^{-1}$

interchanging x and y the inverse is f^{-1} $x = c-2y$

transposing to make y the subject $2y = c-x$ $y = f^{-1}(x) = \dfrac{c-x}{2}$

Question 4 **Answer A**

the graph of $y = \sqrt{x+3}$ when reflected in the y-axis becomes $y = \sqrt{3-x}$
then translated 2 units to the left becomes $y = \sqrt{1-x}$
then translated 3 units up becomes $y = \sqrt{1-x}+3$

Question 5 **Answer C**

$N(t) = 20e^{0.2t}$, the average rate is $\dfrac{N(3)-N(0)}{3-0}$

$$\frac{N(3)-N(0)}{3-0} = \frac{20e^{0.6}-20}{3} = 5.48$$

Question 6 **Answer B**

$f(x)=x^3 \quad g(x)=\cos(x)$ and $h(x)=\sqrt{x}$

$g(h(x))=\cos(h(x))=\cos(\sqrt{x})$

$f(g(h(x)))=f(\cos(\sqrt{x}))=(\cos(\sqrt{x}))^3=\cos^3(\sqrt{x})$

Question 7 **Answer B**

$f(x)=\sqrt[3]{x}$, and $x=-64 \quad h=0.5$ now $-63.5=x+h$

using $f(x+h)=f(x)+h\,f'(x)$

$\sqrt[3]{-63.5}=f(-64)+0.5\,f'(-64)$

Question 8 **Answer D**

this is just the chain rule, in function form

$\frac{d}{dx}(f(g(x)))=g'(x)\,f'(g(x))$

Question 9 **Answer C**

$f:[0,\pi]\to R$ where $f(x)=300\tan(3x)=\dfrac{300\sin(3x)}{\cos(3x)}$

The graph crosses the *x*-axis when $\sin(3x)=0 \Rightarrow x=0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$ Option **A.** is true

The graph has asymptotes when $\cos(3x)=0 \Rightarrow x=\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ Option **B.** is true

The range of the graph is *R*. Option **C.** is false

The period of $\tan(3x)$ is $\frac{\pi}{3}$ so the graph has three cycles in $[0,\pi]$ Option **D.** is true

The graph has a domain of $[0,\pi]$ is its restricted domain. Option **E.** is true

Question 10 **Answer B**

$f(0)=1 \quad f(a)=1$ and $f\left(\frac{a}{2}\right)=1-a\sin\left(\frac{\pi}{2}\right)=1-a$

so the range is $[1,1-a]$

Question 11 **Answer E**

$y = x^4 - 4x^2$

for turning points

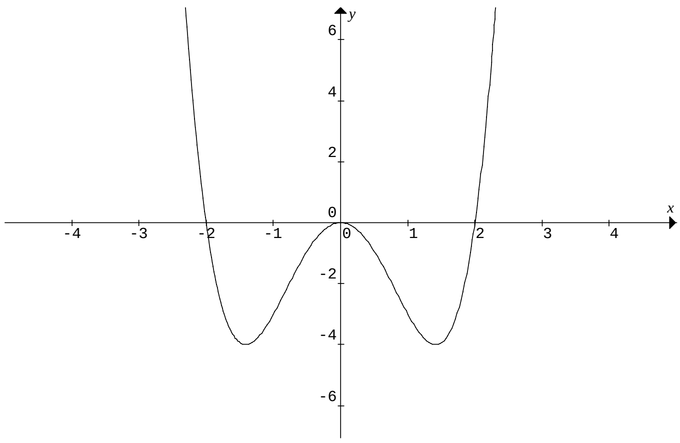
$\frac{dy}{dx} = 4x^3 - 8x = 0$

$4x\left(x^2 - 2\right) = 0$

at $x = 0 \quad x = \pm\sqrt{2}$

from the graph, the gradient is negative for

$\left(-\infty, -\sqrt{2}\right) \cup \left(0, \sqrt{2}\right)$



Question 12 **Answer D**

$(1)\ 2x - 3y = q \qquad (2)\ px + 6y = 10$

A. is true if $p = -4$, there is no unique solution.

B. is true if $p \neq -4$, there is a unique solution, the two lines intersect in a unique point.

C. is true, if $p = -4$ and $q = -5$ the two equations are the same line, so that there is an infinite number of solutions.

E. is true, if $p = -4$ and $q \neq -5$ the two equations, represent straight parallel lines, with different *y*-intercepts, so there is no solution.

D. is false.

Question 13 **Answer D**

$$\int_a^0 \left(1 - f(x)\right) dx = [x]_a^0 - \int_a^0 f(x)\,dx = (0 - a) + \int_0^a f(x)\,dx = A - a$$

Question 14 **Answer C**

$\frac{dy}{dx} = 4\sin(2x) \quad \Rightarrow \quad y = \int 4\sin(2x)\,dx = -2\cos(2x) + c$ to find *c*, use $y\left(\frac{5\pi}{3}\right) = 0$

$0 = -2\cos\left(\frac{10\pi}{3}\right) + c = 1 + c = 0 \quad \Rightarrow c = -1$

$y = -2\cos(2x) - 1$ now when $x = 0$ $y = -2\cos(0) - 1 = -3$

Question 15 **Answer C**

let $y_1 = kx$ and $y_2 = x^2 + bx + c^2$ if $y_1 = y_2$ $kx = x^2 + bx + c^2$

$x^2 + (b-k)x + c^2 = 0$ the number of roots, depends upon the discriminant

$\Delta = (b-k)^2 - 4c^2$

A. and **B.** for the graphs to touch or to be a tangent, $\Delta = 0$ $\Rightarrow b - k = \pm 2c$ $b = k \pm 2c$

both **A.** and **B.** are true.

For the graphs to intersect at two distinct points $\Delta > 0$

$\Delta > 0$ $\Rightarrow$ $(b-k)^2 > 4c^2$ $\Rightarrow b-k > 2c$ and $b-k < -2c$

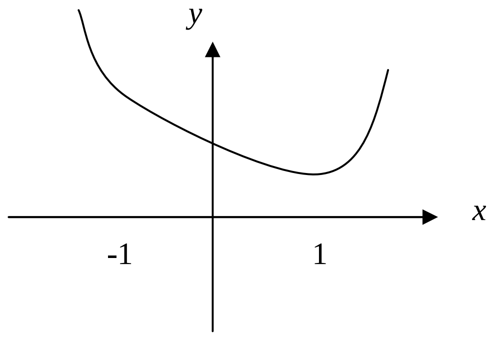
$k > b + 2c$ and $k < b - 2c$ option **D.** and **E.** are true.

For the graphs to not intersect $\Delta < 0$

$\Delta < 0$ $\Rightarrow$ $(b-k)^2 < 4c^2$ $\Rightarrow b-k < 2c$ and $b-k > -2c$ or

$k < b + 2c$ and $k > b - 2c$ or $b - 2c < k < b + 2c$ **C.** is false.

Question 16 **Answer B**



- $f'(x) = 0$ at $x = -1$ and $x = 1$
- $f'(x) < 0$ for $x < -1$ and $-1 < x < 1$
- $f'(x) > 0$ for $x > 1$

The graph has a stationary point of inflexion at $x = -1$ and a minimum at $x = 1$.

Question 17 **Answer E**

	A	$\overline{A}$	
B	$\frac{p}{2}$	$\frac{p}{2}$	p
$\overline{B}$	$\frac{p}{2}$	$1-\frac{3p}{2}$	$1-p$
	p	$1-p$	

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ this is always true.

$\Pr(A \cap B) = \dfrac{p}{2}$ then $\Pr(\overline{A} \cap \overline{B}) = 1 - \dfrac{3p}{2} = \dfrac{2-3p}{2}$

Option **A**. is true

If A and B are mutually exclusive then $\Pr(A \cap B) = 0$

so that $\Pr(A \cup B) = \Pr(A) + \Pr(B) = 2p$ Option **B.** is true.

If A and B are independent then $\Pr(A \cap B) = \Pr(A).\Pr(B)$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) = 2p - p^2 = p(2-p)$ Option **C.** is true.

If A and B are independent then $\Pr(A / B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{\Pr(A).\Pr(B)}{\Pr(B)} = p$ Option **D.** is true.

If A and B are mutually exclusive then $\Pr(A / B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = 0$ Option **E.** is false.

Question 18 **Answer D**

Since it is a discrete random variable, the probabilities add to one, so that $a+b=1$

$E(X)=\sum x\Pr(X=x)=(-1)a+(1)b=b-a$

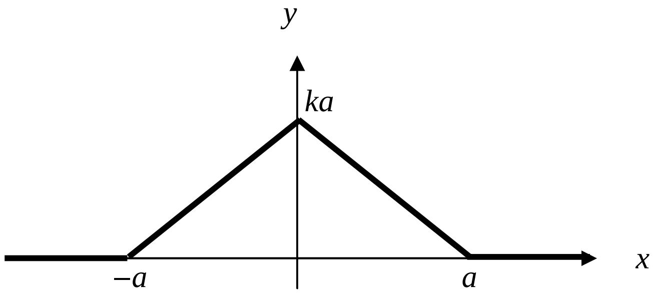
$E(X^2)=\sum x^2\Pr(X=x)=(-1)^2a+(1)^2b=a+b$

$VAR(X)=E(X^2)-(E(X))^2=(a+b)-(b-a)^2$ since $a+b=1$ and $b=1-a$

$VAR(X)=1-(1-2a)^2=1-(1-4a+4a^2)$

$VAR(X)=4a-4a^2=4a(1-a)$

Question 19 **Answer A**



$$f(x)=\begin{cases} k(a-x) & \text{for } 0\le x\le a \\ k(a+x) & \text{for } -a\le x\le 0 \\ 0 & \text{elsewhere}\end{cases}$$

The total area under the two triangles is one.

$A=2\left(\frac{1}{2}a.ka\right)=ka^2=1$ $\qquad k=\frac{1}{a^2}$

Question 20 **Answer A**

Given that $\Pr(|Z|<c)=\Pr(-c<Z<c)=a$

$\Pr(0<Z<c)=\Pr(-c<Z<0)=\frac{a}{2}$

$\Pr(Z\ge -c)=\Pr(Z\le c)=\frac{a}{2}+0.5=\frac{a+1}{2}$

Question 21 **Answer A**

The higher probabilities $\Pr(X=8)\approx 0$ $\Pr(X=9)\approx 0$ $\Pr(X=10)\approx 0$ are small, and the graph is right (or positively skewed) so the probability of a success on any one trial p is very small, $p\ll 0.5$, $p=0.3$ is the correct choice.

Question 22 **Answer D**

want *BR* or *RB*

$\Pr(BR+RB)=\frac{b}{(b+r)}\text{x}\frac{r}{(b+r-1)}+\frac{r}{(b+r)}\text{x}\frac{b}{(b+r-1)}=\frac{2br}{(b+r)(b+r-1)}$

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a. $f(x)=x^2 \quad f(2)=4$, A is the point $(2,4)$,

also on $y=ax^3+bx^2+cx+d$ substitute

$(1) \quad 4=8a+4b+2c+d$ A1

C is the point $(4,0)$

$(2) \quad 0=64a+16b+4c+d$ A1

at A the join is smooth, the gradients are equal, so that

$\frac{dy}{dx}=2x=3ax^2+2bx+c$ at $x=2$

$(3) \quad 4=12a+4b+c$ A1

at B, we have a maximum $\frac{dy}{dx}=0$ at $x=3$

$(4) \quad 0=27a+6b+c$ A1

b. the four equations become the one matrix equation $AX=C$ where

$A=\begin{bmatrix}8&4&2&1\\64&16&4&1\\12&4&1&0\\27&6&1&0\end{bmatrix} \quad X=\begin{bmatrix}a\\b\\c\\d\end{bmatrix} \quad C=\begin{bmatrix}4\\0\\4\\0\end{bmatrix}$ the solution for X is M1

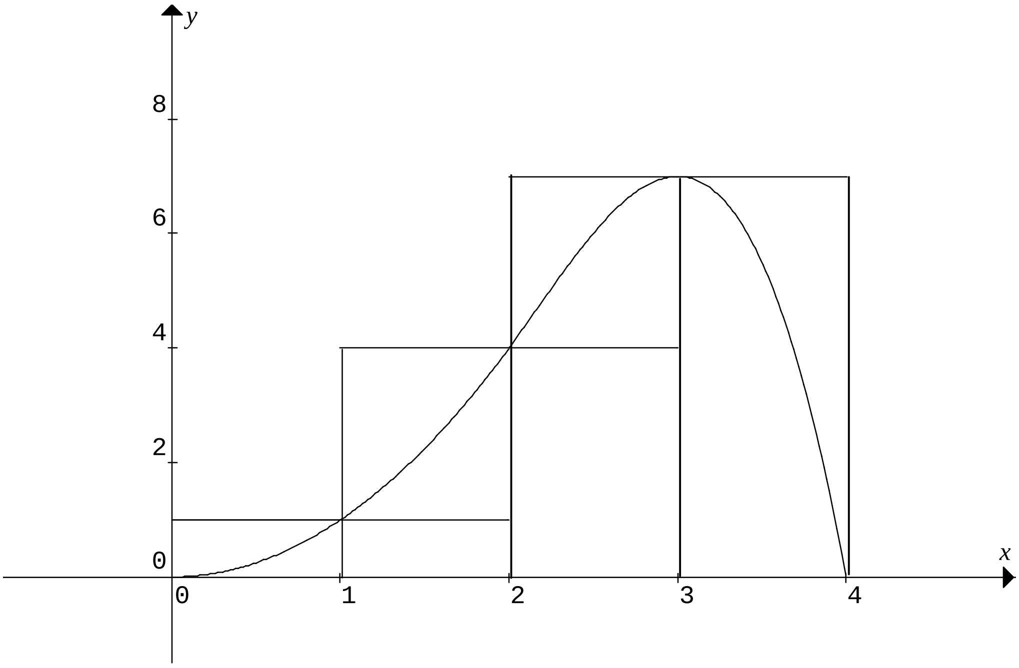
$$X=A^{-1}C=\begin{bmatrix}8&4&2&1\\64&16&4&1\\12&4&1&0\\27&6&1&0\end{bmatrix}^{-1}\begin{bmatrix}4\\0\\4\\0\end{bmatrix}=\begin{bmatrix}-2\\13\\24\\16\end{bmatrix}$$ A1

so that $a=-2 \quad b=13 \quad c=-24 \quad d=16$

c. $f(x)=\begin{cases}x^2 & \text{for } 0\le x\le 2\\ -2x^3+13x^2-24x+16 & \text{for } 2\le x\le 4\end{cases}$

at B $x=3$ $f(3)=7$, completing the table of values A1

x	0	1	2	3	4
y	0	1	4	7	0



using left (end-point) rectangles $L = h\left(y_0 + y_1 + y_2 + y_3\right) = 1\left(0+1+4+7\right) = 12$

using right (end-point) rectangles $R = h\left(y_1 + y_2 + y_3 + y_4\right) = 1\left(1+4+7+0\right) = 12$

both give the area of the grassed region as $12\,\text{m}^2$ A1

the area of the rectangular region is 28 m^2, so that,

$16\,\text{m}^2$ is the area of the trees and shrubs, the ratio of areas $12:16$ or $3:4$ A1

d.i. $A = \int_0^2 x^2 dx + \int_2^4 \left(-2x^3 + 13x^2 - 24x + 16\right) dx$ A1

ii. $A = 13\frac{1}{3}\ \text{m}^2$ A1

e.i. the point *P* is on $y = x^2$ and $0 < p < 2$, so that *P* has coordinates $P\left(p, p^2\right)$ and D is the point $D(3,0)$ A1

Let s be the distance from *P* to *D*, the path length

$s = d\left(PD\right) = \sqrt{(3-p)^2 + p^4}$ A1

ii. for the path length *s*, to be a minimum $\frac{ds}{dp} = \frac{4p^3 - 2(3-p)}{2\sqrt{(3-p)^2 + p^4}} = 0$ M1

$4p^3 - 6 - 2p = 0$ since $0 < p < 2$

$p = 1$ A1

$s_{\min} = \sqrt{5}$ A1

Question 2

a. i. $f'(x) = 2 + 4\cos\left(\frac{x}{2}\right)$ A1

ii. maximum value of the gradient is 6 and occurs when

$\cos\left(\frac{x}{2}\right) = 1$ M1

$x = 0\ ,\ 4\pi$ now, $f(0) = 0$ and $f(4\pi) = 8\pi$ the coordinates are

$(0,0)$ and $(4\pi, 8\pi)$ A1

b. for stationary points $f'(x) = 0$

$4\cos\left(\frac{x}{2}\right) = -2$

$\cos\left(\frac{x}{2}\right) = -\frac{1}{2}$

$x = \frac{4\pi}{3}\ ,\ \frac{8\pi}{3}$

$f\left(\frac{4\pi}{3}\right) = \frac{8\pi}{3} + 8\sin\left(\frac{2\pi}{3}\right) = \frac{8\pi}{3} + 4\sqrt{3}$

the maximum coordinate is $\left(\frac{4\pi}{3}, \frac{8\pi}{3} + 4\sqrt{3}\right)$ A1

$f\left(\frac{8\pi}{3}\right) = \frac{16\pi}{3} + 8\sin\left(\frac{4\pi}{3}\right) = \frac{16\pi}{3} - 4\sqrt{3}$

the minimum coordinate is $\left(\frac{8\pi}{3}, \frac{16\pi}{3} - 4\sqrt{3}\right)$ A1

c. at $x = 2\pi$ $f(2\pi) = 4\pi + \sin(\pi) = 4\pi$

$f'(2\pi) = 2 + 4\cos(\pi) = -2$ A1

the equation of the tangent is

$y - 4\pi = -2(x - 2\pi)$

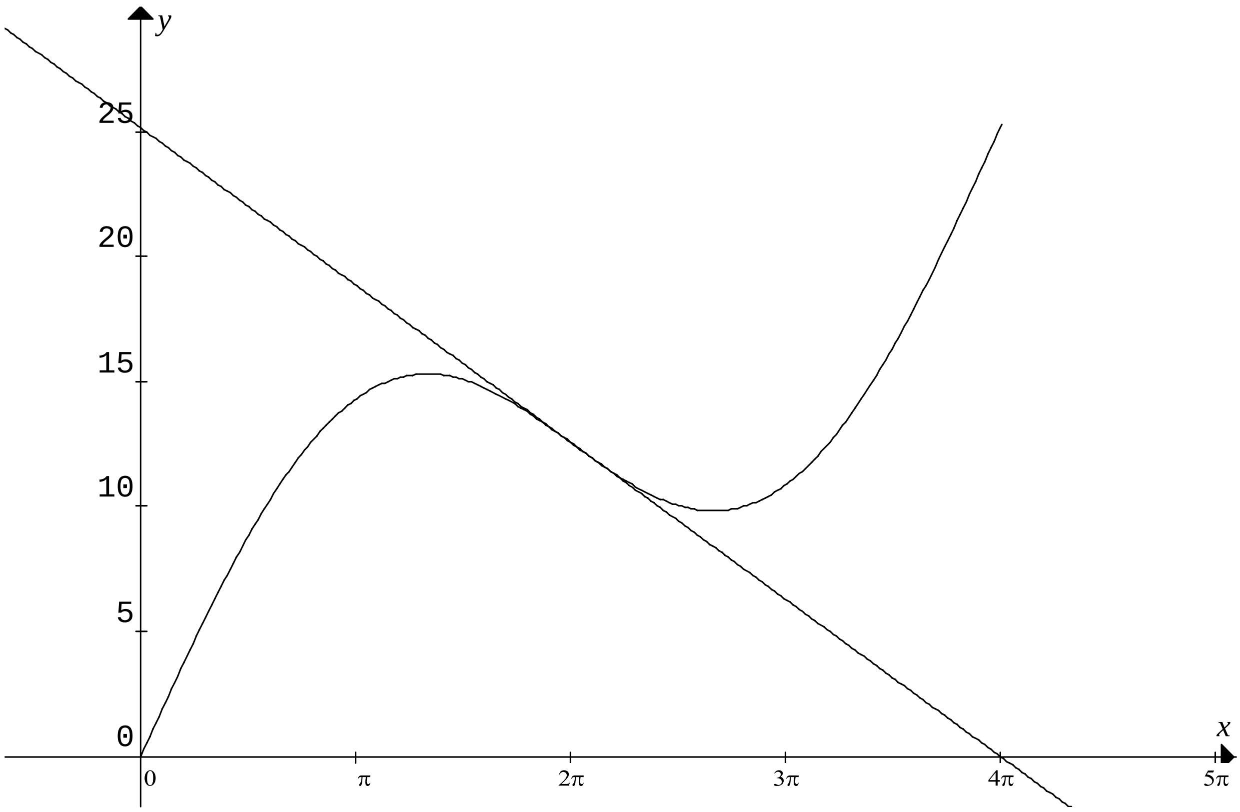
$y = -2x + 8\pi$ A1

d. correct graph, shape, restricted domain $[0, 4\pi]$,

maximum at $\left(\frac{4\pi}{3}, \frac{8\pi}{3} + 4\sqrt{3}\right) \approx (4.2, 15.3)$

minimum at $\left(\frac{8\pi}{3}, \frac{16\pi}{3} - 4\sqrt{3}\right) \approx (8.4, 9.8)$ A1

correct tangent, on either side of the curve and passing through $(4\pi, 0)$ A1



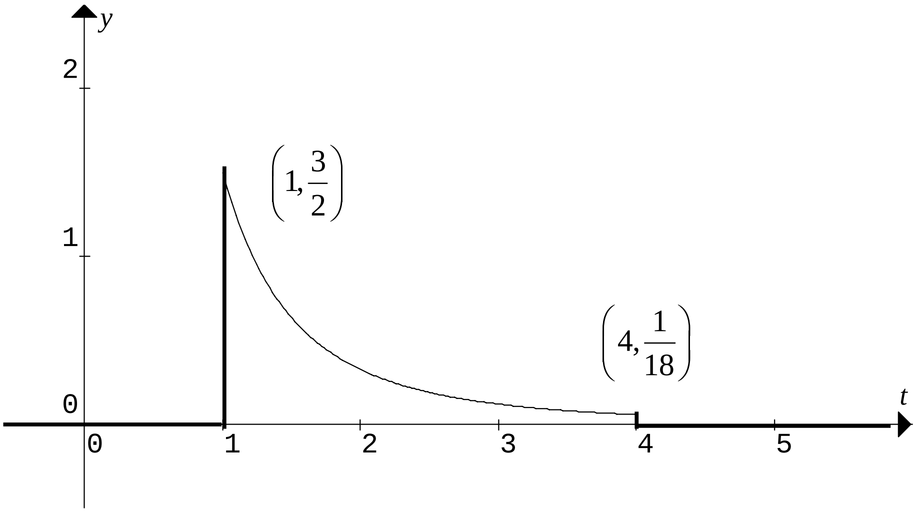
e. $\cos\left(\frac{x}{2}\right) = -\frac{1}{2}$

$\frac{x}{2} = 2k\pi \pm \cos^{-1}\left(-\frac{1}{2}\right) = 2k\pi \pm \frac{2\pi}{3}$ A1

$x = 4k\pi \pm \frac{4\pi}{3}$ where $k \in J$

Question 3

a. correct graph and $y = 0$ for $(-\infty, 1]$ and $[4, \infty)$ A1



b. want $\Pr(T > 3 | T > 2) = \dfrac{\Pr(T > 3)}{\Pr(T > 2)}$ M1

$$\Pr(T > 2) = \int_2^4 \frac{81}{2(2t+1)^3}\,dt = \frac{7}{25}$$ A1

$$\Pr(T > 3) = \int_3^4 \frac{81}{2(2t+1)^3}\,dt = \frac{4}{49}$$

$$\Pr(T > 3 | T > 2) = \frac{100}{343}$$ A1

c. $Y \sim \text{Bi}\left(n = 4, p = \dfrac{7}{25}\right)$

$\Pr(Y \geq 1) = 1 - \Pr(Y = 0)$ M1

$$\Pr(Y \geq 1) = 1 - \left(1 - \frac{7}{25}\right)^4$$

$\Pr(Y \geq 1) = 0.731$ A1

d. $$E(T) = \int_1^4 \frac{81t}{2(2t+1)^3}\,dt$$ A1

$E(T) = 1.75$ A1

e. $\int_1^m \frac{81}{2(2t+1)^3}\,dt = \frac{1}{2}$ A1

$\left[-\frac{81}{8(2t+1)^2}\right]_1^m = \frac{1}{2}$

$\frac{1}{(2m+1)^2} - \frac{1}{9} = -\frac{4}{81}$ since $1 < m < 4$ M1

$m = \frac{1}{2}\left(\frac{9}{\sqrt{5}} - 1\right)$

$m = 1.5$ years A1

f. $X \sim \mathrm{N}\left(\mu = ?, \sigma^2 = ?^2\right)$ times in months

$\Pr(X < 25) = 0.18$

$\frac{25-\mu}{\sigma} = -0.915$ M1

$(1) \quad -0.915\,\sigma = 25 - \mu$

$\Pr(X > 57) = 0.04$

$\frac{57-\mu}{\sigma} = 1.75$ M1

$(2) \quad 1.75\sigma = 57 - \mu$

now subtract equations $(2) - (1)$

$2.665\,\sigma = 32$

$\sigma = 12$ months A1

substituting gives $\mu = 57 - 1.75 \text{ x } 12 = 36$ months A1

Question 4

a. we require $1+\frac{x}{2}>0 \quad \Rightarrow x>-2$

domain $D=(-2,\infty)$ A1

b. $y=3\log_e\left(1+\frac{x}{2}\right) \qquad \frac{dy}{dx}=\frac{3}{2+x}$

since $\frac{dy}{dx}\neq 0 \qquad \Rightarrow$ no turning points A1

c. $f(x)=3\log_e\left(\frac{1}{2}(x+2)\right)$

- dilation by a factor of 3 parallel to the *y*-axis (or away from the *x*-axis) A1
- dilation by a factor of 2 parallel to the *x*-axis (or away from the *y*-axis) A1
- translation by 2 units to the left parallel to the *x*-axis (or away from the *y*-axis) A1

d.i. $f(u-2)=3\log_e\left(1+\frac{u-2}{2}\right)=3\log_e\left(\frac{u}{2}\right)$ and

$f(v-2)=3\log_e\left(1+\frac{v-2}{2}\right)=3\log_e\left(\frac{v}{2}\right)$

$f(u-2)+f(v-2)=3\log_e\left(\frac{u}{2}\right)+3\log_e\left(\frac{v}{2}\right)=3\log_e\left(\frac{uv}{4}\right)$ M1

since $u>0$ and $v>0$

$f(auv+b)=3\log_e\left(1+\frac{auv+b}{2}\right)=3\log_e\left(\frac{uv}{4}\right)$

$1+\frac{b}{2}+\frac{auv}{2}=\frac{uv}{4}$

$b=-2 \quad a=\frac{1}{2}$ A1

ii. $f(u)+f(-u)=3\log_e\left(1+\frac{u}{2}\right)+3\log_e\left(1-\frac{u}{2}\right)$

$f(u)+f(-u)=3\log_e\left(\left(1+\frac{u}{2}\right)\left(1-\frac{u}{2}\right)\right)=3\log_e\left(1-\frac{u^2}{4}\right)=f\left(-\frac{u^2}{4}\right)$ M1

provided that $1-\frac{u^2}{4}>0$ or $u^2<4$

$|u|<2$ or $u\in(-2,2)$ A1

e. $f \quad y = 3\log_e\left(1+\frac{x}{2}\right)$ interchanging *x* and *y*

$f^{-1} \quad x = 3\log_e\left(1+\frac{y}{2}\right)$ rearranging for *y* M1

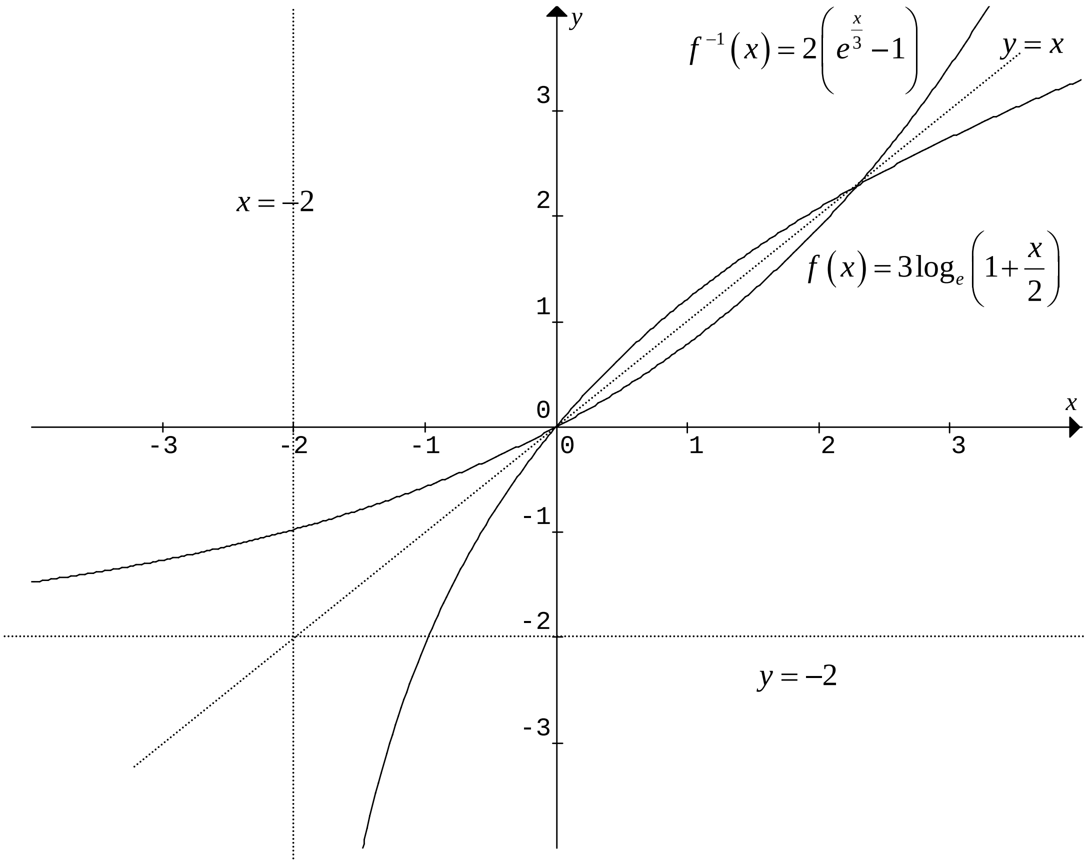
$f^{-1}(x) = 2\left(e^{\frac{x}{3}} - 1\right)$ A1

dom $f^{-1} =$ ran $f = R$ must give domain since a function is required

f. both graphs pass through the origin $(0,0)$, the graphs are reflection in the line $y = x$

the graph of f has $x = -2$ as a vertical asymptote A1

the graph of f^{-1} has $y = -2$ as a horizontal asymptote A1



g. i. the coordinate is $(2.288, 2.288)$

since p satisfies $f^{-1}(x) = f(x) = x$ or

$$3\log_e\left(1+\frac{p}{2}\right) = 2\left(e^{\frac{p}{3}} - 1\right) = p$$

so that $p = 2.288$ A1

g. ii. let the area bounded by the graph of f, the x-axis and the line $x = p$ be

$$A_1 = \int_0^p f(x)\,dx = \int_0^p 3\log_e\left(1+\frac{x}{2}\right)dx$$

let the area bounded by the graph of f^{-1}, the x-axis and the line $x = p$ be

$$A_2 = \int_0^p f^{-1}(x)\,dx = \int_0^p 2\left(e^{\frac{x}{3}} - 1\right)dx$$

but $A_1 + A_2 = p^2$ A1

(the area of the square of side length p)

$$A_1 = p^2 - A_2 = p^2 - \int_0^p f^{-1}(x)\,dx$$

$$A_1 = p^2 - 2\int_0^p \left(e^{\frac{x}{3}} - 1\right)dx$$ A1

$$A_1 = p^2 - 2\left[\left(3e^{\frac{p}{3}} - p\right) - 3\right]$$

$$A_1 = p^2 - 6e^{\frac{p}{3}} + 2p + 6$$

$$A_1 = p^2 - 3\left(2\left(e^{\frac{p}{3}} - 1\right)\right) + 2p$$ A1

$$A_1 = p^2 - 3p + 2p$$

$$A_1 = p^2 - p$$

END OF SECTION 2 SUGGESTED ANSWERS