INSIGHT

Trial Exam Paper

2008

MATHEMATICAL METHODS

Written examination 2

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

Question 1

The range of the function $f(x) = x^3 - 5x^2 + 3x + 9,\ x \in (0,5]$ is

A. (0,5]
B. (9,24)
C. [9,24]
D. (0,24]
E. [0,24]

Answer is E

Solution:

Answer is E—need to examine the graph of the function over the domain stated—it is not enough to just substitute in the endpoints. Graph has a minimum value of 0 and a maximum value of 24.

Question 2

The equations of the asymptotes of the graph with the rule

$y = \frac{-3x-1}{x+1}$ are

A. $x = -1,\ y = -3$
B. $x = 1,\ y = -3$
C. $x = -1,\ y = 0$
D. $x = 1,\ y = -1$
E. $x = -1,\ y = -\frac{1}{3}$

Answer is A

Solution:

The equation $y = \frac{-3x-1}{x+1}$ can be rewritten to give $y = \frac{2}{x+1} - 3$ using re-expression or long division.

In this form it can be seen that the asymptotes are at $x = -1$ *and* $y = -3$

Question 3

The number of solutions to the equation $(x^2 - a)(x^3 - b^3)(x + c) = 0$ where $a, b, c \in R^+$ is

A. 6
B. 5
C. 4
D. 3
E. 2

Answer is C

Solution:

The solutions arise from the number of linear factors so we need to look to see whether the existing factors can be further reduced to linear factors.

$(x^2 - a)(x^3 - b^3)(x + c) = 0$

$(x - \sqrt{a})((x + \sqrt{a})(x - b)(x^2 + xb + b^2)(x + c) = 0$

so a total of 4 linear factors, therefore 4 solutions.

Question 4

Given that $\log_2 c + \log_2 5 = 3\log_2 3$, then c is equal to

A. 22
B. 1.8
C. $2^{5.4}$
D. 5.4
E. 4

Answer is D

Solution:

Using log laws $\log_2 5c = \log_2 27$

$so\ 5c = 27$

$c = \dfrac{27}{5} = 5.4$

Section 1 continues
TURN OVER

Question 5

The curve with equation $f(x) = x^3 - bx^2 - 9x + 7$ has a stationary point when $x = -1$. The value of b is

A. -3

B. 3

C. $-\frac{1}{2}$

D. 2

E. 6

Answer is B

Solution:

Stationary point occurs when $f'(x) = 0$

So let $3x^2 - 2bx - 9 = 0$ *when* $x = -1$

this gives

$3 + 2b - 9 = 0$

$2b = 6$

$b = 3$

Question 6

The maximal domain, D, of the function $f : D \to R$ with the rule $f(x) = \cos(\sqrt{(2x-3)}$ is

A. $R \setminus \left\{\frac{3}{2}\right\}$

B. $R \setminus \{3\}$

C. R

D. $(\frac{3}{2}, \infty)$

E. $[\frac{3}{2}, \infty)$

Answer is E

Solution:

For the function to be defined inside the square needs to be greater than or equal to 0.

So

$2x-3\geq 0$

$x\geq \frac{3}{2}$

Question 7

The period and amplitude of the function $f(x)=1-2\sin(\frac{\pi}{4}-2x)$ are respectively

A. $\frac{\pi}{2}$, 2

B. $\frac{\pi}{4}$, -2

C. π, 2

D. 2π, 2

E. π, -2

Answer is C

Solution:

amplitude is 2 (not negative!)

period is $\frac{2\pi}{2}=\pi$

Question 8

$\left|p^2-5p\right|=6$ for

A. $p=-1$ only

B. $p=6$ only

C. $p=-1$ or $p=6$ only

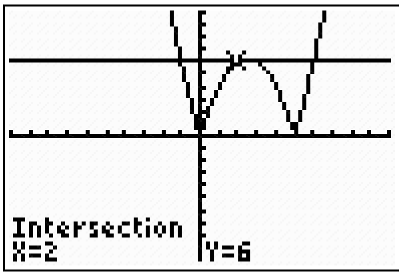
D. $p=2$ or $p=3$ only

E. $p=-1$ or $p=6$ or $p=2$ or $p=3$

Answer is E

Solution:

This is best done graphically; use a graphics calculator and graph $y_1 = abs(x^2 - 5x)$ and $y_2 = 6$ and find the points of intersection.



Question 9

The inverse function f^{-1} of the function $f:(-\infty,2) \to R$, $f(x) = \log_e(1 - \frac{x}{2})$ has the rule given by

A. $f^{-1}(x) = 1 - e^{\frac{x}{2}}$

B. $f^{-1}(x) = 1 - e^{2x}$

C. $f^{-1}(x) = 2(1 - e^x)$

D. $f^{-1}(x) = \frac{1}{2}(e^x - 1)$

E. $f^{-1}(x) = 2(e^x - 1)$

Answer is C

Solution:

Rearranging *x* and *y* gives

$x = \log_e(1 - \frac{y}{2})$

$e^x = 1 - \frac{y}{2}$

$\frac{y}{2} = 1 - e^x$

$y = 2(1 - e^x)$

Question 10

Let $f: R \to R$ be a differentiable function.

Then for all $x \in R$, the derivative of $e^{f(kx)}$ is equal to

A. $e^{f(kx)}$

B. $ke^{f(kx)}$

C. $kf(kx)e^{f(kx)}$

D. $f'(kx)e^{f(kx)}$

E. $kf'(kx)e^{f(kx)}$

Answer is E.

Solution:

Using the chain rule we differentiate the exponential and then differentiate the function attached to give E.

Question 11

The average rate of change of the function with the rule $f(x) = \sqrt{x^2 + 2x}$ between $x = 0$ and $x = 4$ is

A. $2\sqrt{6}$

B. $\dfrac{\sqrt{6}}{2}$

C. $\dfrac{2}{\sqrt{6}}$

D. $\sqrt{6}$

E. 4

Answer is B.

Solution:

$$\text{Average rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sqrt{24} - 0}{4 - 0} = \frac{\sqrt{24}}{4} = \frac{\sqrt{6}}{2}$$

Section 1 continues
TURN OVER

Question 12

Which one of the following is **not** true about the function $f : R \to R,\ f(x) = (x-4)^{\frac{2}{3}}$?

A. The graph of *f* is continuous everywhere.
B. $f(x) \geq 0$ for all values of *x*.
C. $f'(x) > 0$ for $x > 4$
D. $f'(x) < 0$ for $x < 4$
E. The graph of *f* is differentiable everywhere.

Answer is E.

Solution:

The graph is not differentiable at the cusp.

Question 13

Using the approximation formula, $f(x+h) \approx f(x) + hf'(x)$ where $f(x) = \sqrt{x}$ with $x = 25$, an approximate value for $\sqrt{24.92}$ is given by

A. $f(25) + 0.08f'(25)$
B. $f(5) + 0.08f'(5)$
C. $f(25) - 0.08f'(25)$
D. $f(5) - 0.08f'(5)$
E. $f'(25)$

Answer is C.

Solution:

With $x = 25$ and $x + h = 24.92$ this means $h = -0.08$.

Question 14

The equation of the **normal** to the curve with equation $y = (x-2)e^{2x}$, at the point on the curve with $x = 2$, is

A. $y = e^4(x-2)$
B. $y = e^{2x}(x-2)$
C. $y = \frac{-1}{e}(x-2)$
D. $y = \frac{-1}{e^{2x}}(x-2)$
E. $y = -e^{-4}(x-2)$

Answer is E

Solution:

$\frac{dy}{dx} = 2e^{2x}(x-2) + e^{2x}$ using the product rule

$= e^{2x}(2x-4+1)$

$= e^{2x}(2x-3)$

when $x = 2$ $\frac{dy}{dx} = e^4$

so the gradient of the normal is $\frac{-1}{e^4} = -e^{-4}$

the only option that has this gradient is E, so answer is E

Note: could also use a program to generate the equation of the normal and then check the decimal approximations with the exact values.

Question 15

If $f(x) = \frac{(x-a)^2}{g(x)}$ then the derivative of $f(x)$ is

A. $\frac{2(x-a)}{g'(x)}$

B. $\frac{2(x-a)}{g(x)}$

C. $\frac{2(x-a)g'(x)}{(g(x))^2}$

D. $\frac{(x-a)[2g(x)-(x-a)g'(x)]}{(g(x))^2}$

E. $\frac{(x-a)[2g(x)-(x-a)g'(x)]}{(g'(x))^2}$

Answer is D

Solution:

Need to use the quotient rule to differentiate $f(x)$

$$f'(x) = \frac{g(x)\times 2(x-a) - (x-a)^2 g'(x)}{(g(x))^2}$$

$$= \frac{(x-a)[2g(x)-(x-a)g'(x)]}{[g(x)]^2}$$

Section 1 continues
TURN OVER

Question 16

$\int (\cos(3x-1)+12x^2)\ dx$ is equal to

A. $\frac{-1}{3}\sin(3x-1)+24x+c\,,c\in R$

B. $-3\sin(3x-1)+24x+c\,,c\in R$

C. $\frac{-1}{3}\sin(3x)+4x^3+c\,,c\in R$

D. $\frac{1}{3}\sin(3x-1)+4x^3+c\,,c\in R$

E. $-3\sin(3x-1)+4x^3+c$

Answer is D.

Solution:

Antidifferentiate both parts.

Question 17

If $\int_0^{\pi} f(x)dx = 2$ then $\int_0^{\pi}(2f(x)-\sin x)dx$ is equal to

A. 2

B. 6

C. 4

D. -2

E. 0

Answer is A.

Solution:

Using the properties of integrals

$\int_0^{\pi}(2f(x)-\sin x)dx$

$= 2\int_0^{\pi} f(x)\,dx - \int_0^{\pi}\sin x\ dx$

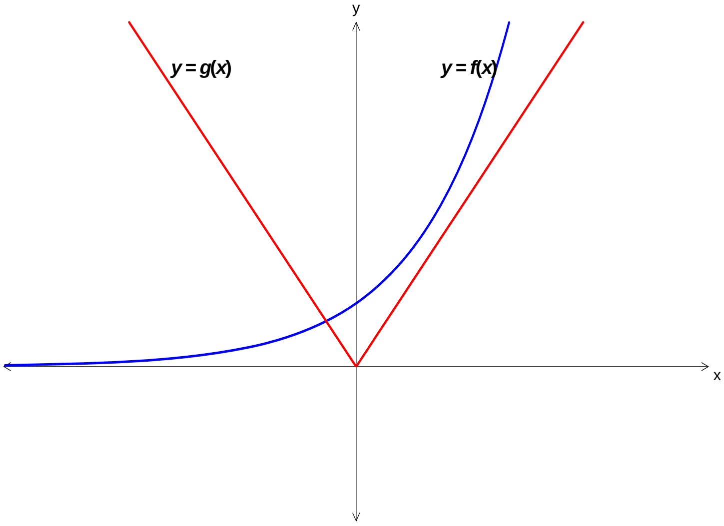
$= 2\times 2-[-\cos x]_0^{\pi}$ using result given and integrating $\sin x$

$= 4-2$

$= 2$

Question 18

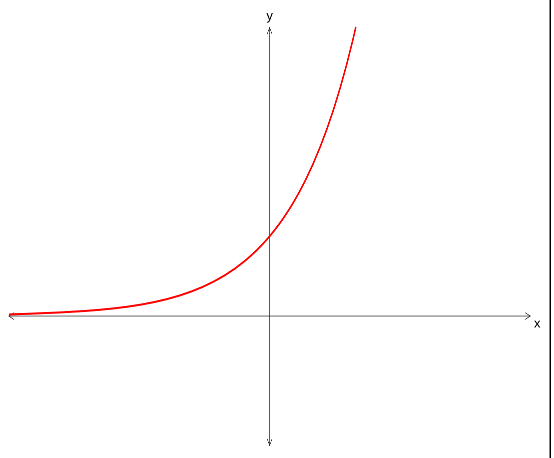
The graphs of $y = f(x)$ and $y = g(x)$ are as shown.



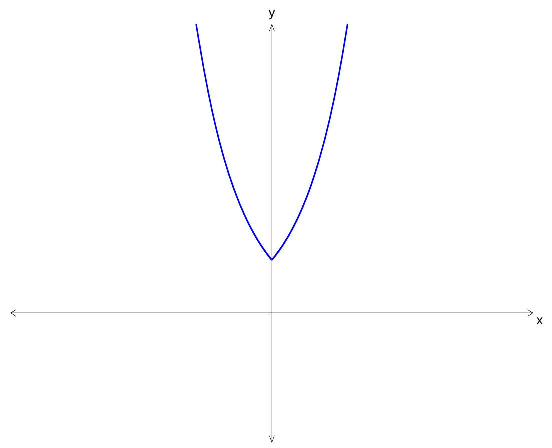
Section 1 continues
TURN OVER

The graph of $y = f(g(x))$ is best represented by

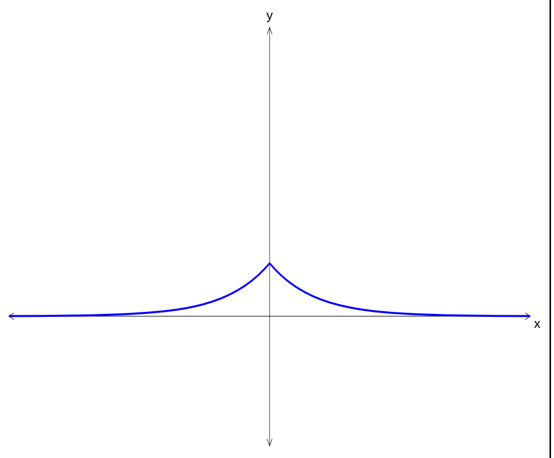
A.



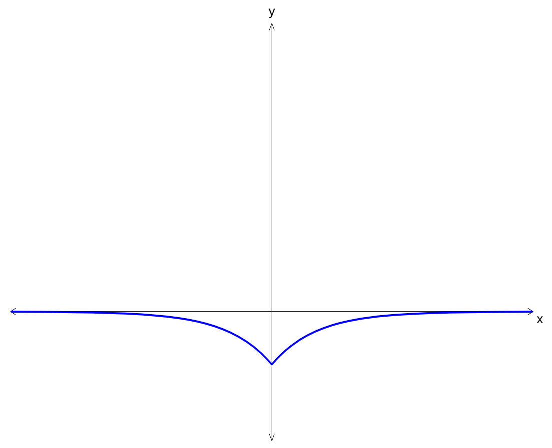
B.



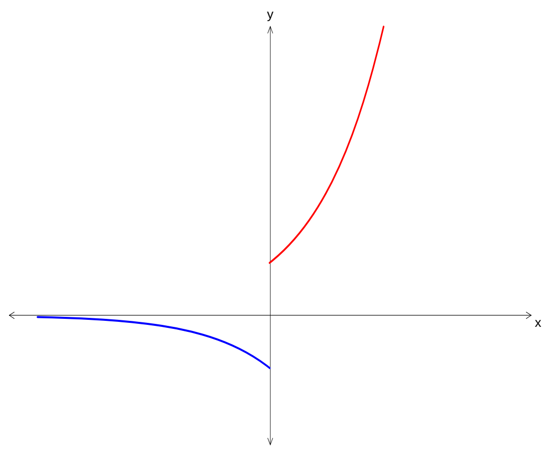
C.



D.



E.



Answer is B

Solution:

$f(x)$ looks like e^x and $g(x)$ looks like $|x|$

as an approximation $f(g(x))$ would be $e^{|x|}$ and B has the shape of $y = e^{|x|}$

Question 19

A bag contains 4 red, 3 white and 2 blue marbles. James selects a ball from the bag (and does not return it). After him, Kevin selects another ball from the bag. What is the probability that James does not select a blue ball and Kevin does not select a white ball?

A. $\frac{7}{12}$

B. $\frac{14}{27}$

C. $\frac{1}{12}$

D. $\frac{19}{36}$

E. $\frac{41}{72}$

Answer is D.

Solution:

The only options are Red, Red Red, Blue White, Red White, Blue

This gives— $\frac{4}{9}\times\frac{3}{8}+\frac{4}{9}\times\frac{2}{8}+\frac{3}{9}\times\frac{4}{8}+\frac{3}{9}\times\frac{2}{8}=\frac{19}{36}$

Question 20

The speeds of vehicles travelling along a particular section of Citylink freeway are normally distributed with a mean of 95 km/h and a standard deviation of σ. Fifteen percent of drivers are found to be exceeding the 100 km/h speed limit.

The value of σ is closest to

A. 0.150
B. 1.036
C. 4.824
D. 5.182
E. 0.207

Answer is C.

Solution:

$\Pr(X > 100) = 0.15$

$\Pr(Z > \frac{100-95}{\sigma}) = 0.15$, convert to standard normal

$1.036 = \frac{5}{\sigma}$, using inverse normal

$\sigma = 4.824$

Question 21

Ben has constructed a spinner that will randomly display an integer between 0 and 4 with the following probabilities.

Number	x	0	1	2	3	4
Probability	$Pr(X=x)$	0.2	0.3	0.15	0.25	0.1

Ben spins the spinner 5 times. The probability of obtaining at least 3 odd numbers is

A. 0.55
B. 0.55^3
C. $(0.55)^3 + (0.55)^4 + (0.55)^5$
D. $(0.45)^2(0.55)^3 + (0.45)(0.55)^4 + (0.55)^5$
E. $10(0.45)^2(0.55)^3 + 5(0.45)(0.55)^4 + (0.55)^5$

Answer is E

Solution:

The distribution is binomial with $X \sim Bi(n = 5,\ p = 0.55)$ and $\Pr(X \geq 3) = \text{option E}$

Question 22

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The value of a such that $\Pr(X > a) = 0.875$, correct to 3 decimal places is

A. 0.875
B. 0.540
C. 0.204
D. 0.956
E. 0.500

Answer is E

Solution:

$$\Pr(X > a) = \int_a^1 3x^2\,dx = 0.875$$

$$\left[x^3\right]_a^1 = 0.875$$

$$1 - a^3 = 0.875$$

$$a^3 = 0.125$$

$$a = 0.5$$

END OF SECTION 1
TURN OVER

SECTION 2

Question 1

The "peanut" spider, a rare South American spider weaves a peanut-shaped web—hence the name.

An araneologist (person who studies spiders) observes the web-making process.

Initially the spider weaves a strand that has the shape that can be described by $y = \frac{1}{35}x(x-7)(x^2-8x+25)$ for $x \in [0,9]$ with all measurements in cm.

a. Show that $x^2 - 8x + 25 > 0$.

2 marks

Solution:

Completing the square gives

$x^2 - 8x + 16 - 16 + 25$

$= (x-4)^2 + 9$

$(x-4)^2 \geq 0$ *so* $(x-4)^2 + 9 > 0$

Note: other techniques can also be used such as graphing or using the discriminant.

Mark allocation

- 1 mark for valid method
- 1 mark for valid conclusion

b. Find the *x*-intercepts of the graph of $y = \frac{1}{35}x(x-7)(x^2-8x+25)$.

1 mark

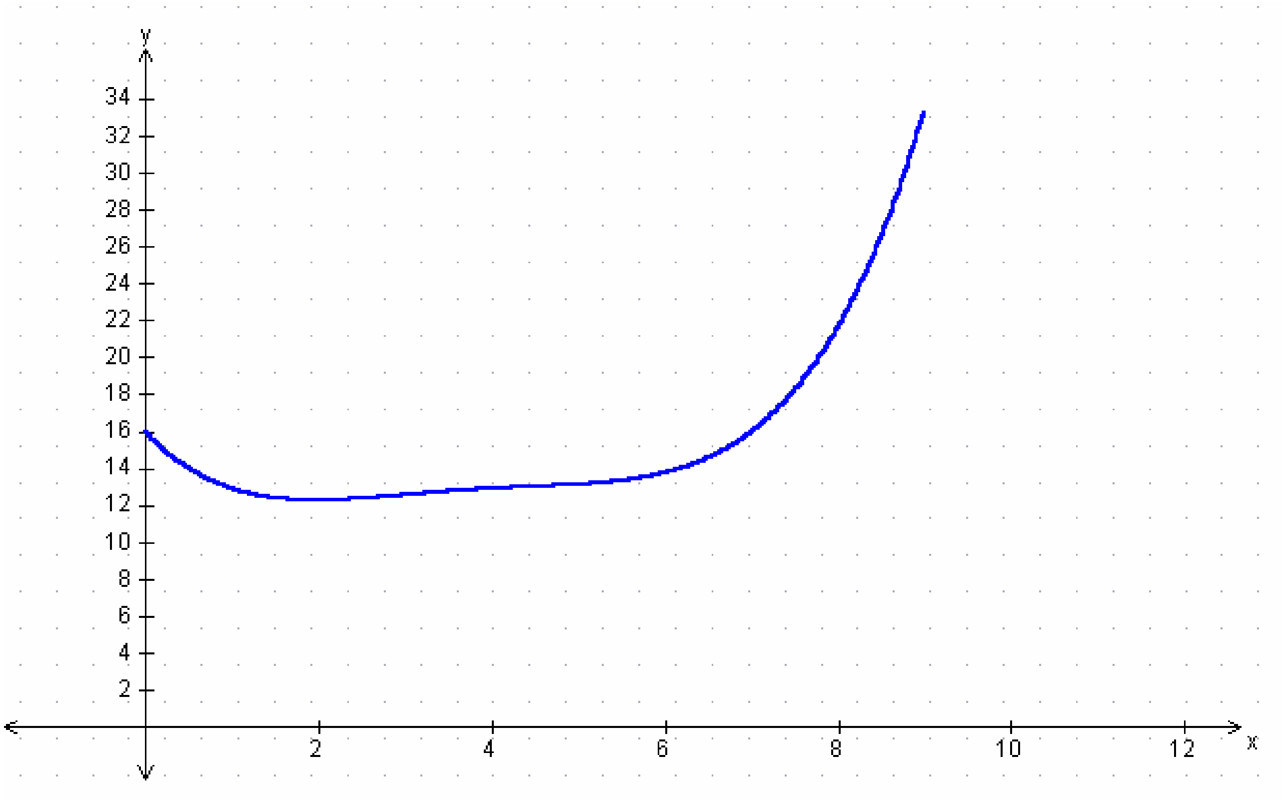
Solution:

As $x^2 - 8x + 25 > 0$ there are no solutions from this quadratic, therefore the only *x*-intercepts come from the two linear factors x and *(x-7)*. The *x*-intercepts are at $x = 0$ and $x = 7$.

Mark allocation

- 1 mark for correct intercepts.

This shape describes the bottom boundary line of the web which is the graph of *y* translated up 16 cm above the ground as shown in the graph below.



c. State the equation of the bottom boundary of the web, y_b, as shown in the diagram above.

1 mark

Solution:

y_b is the graph of *y* shifted up 16 units. This gives the equation

$$y_b = \frac{1}{35}x(x-7)(x^2-8x+25)+16$$

Mark allocation

- 1 mark for the correct equation.

The spider begins to weave the top boundary of the web. The symmetry of the web becomes apparent. The top boundary of the web is a reflection of the bottom boundary in the line $y = 18$. Its equation is given by $y_T = 20 - \frac{1}{35} x(x-7)(x^2 - 8x + 25)$.

d. State the transformations (in correct order) involved in producing y_T from *y*.

2 marks

Solution:

The transformations are—a reflection in the *x*-axis and a translation of 20 units up.

Mark allocation

- 1 mark for stating the reflection.
- 1 mark for stating the translation.

e. The two webs are attached to trees at their endpoints. State the coordinates of the endpoints. (correct to 2 decimal places).

2 marks

Solution:

Endpoints are at $(0, 16), (9, 33.49), (0, 20), (9, 2.51)$ and these can be obtained using the table facility on the calculator or by using value.

Mark allocation

- 1 mark for 2 correct endpoints.
- 2 marks for all endpoints correct.

f. The two boundary webs intersect at one point in the domain. Find the point of intersection. (correct to 2 decimal places).

1 mark

Solution:

Using a calculator and finding the point of intersection gives—(7.45, 18)

Mark allocation

- 1 mark for correct answer.

The spider then begins to weave vertical portions on the web joining the top boundary web with the bottom boundary web.

g. One vertical web is placed at $x=5cm$. Find the minimum length of this web. (correct to 2 decimal places).

2 marks

Solution:

At $x = 5$, $y_b = 13.143$ and $y_T = 22.857$ a difference of $9.71cm$

These values can be obtained from the table or by using value on the calculator.

Mark allocation

- 1 mark for the y values
- 1 mark for the correct length.

h. Three vertical webs of length $5cm$ are required. Where should these be positioned? Answers correct to 3 decimal places.

2 marks

Solution:

Set up an equation that is the difference between y_T and y_b, $(y = y_T - y_b)$, and use the graph to intersect with the lines $y = 5$ and $y = -5$

There are three points of intersection at $x = 0.105,\ 6.851,\ 7.844$

Mark allocation

- 1 mark for indicating an appropriate method—ie setting up the difference equation
- 1 mark for the correct x values.

i. Find the maximum length of a vertical web, correct to 2 decimal places.

2 marks
Total 15 marks

Solution:

Using the equation above $(y = y_T - y_b)$ find the maximum and minimum value of the function—

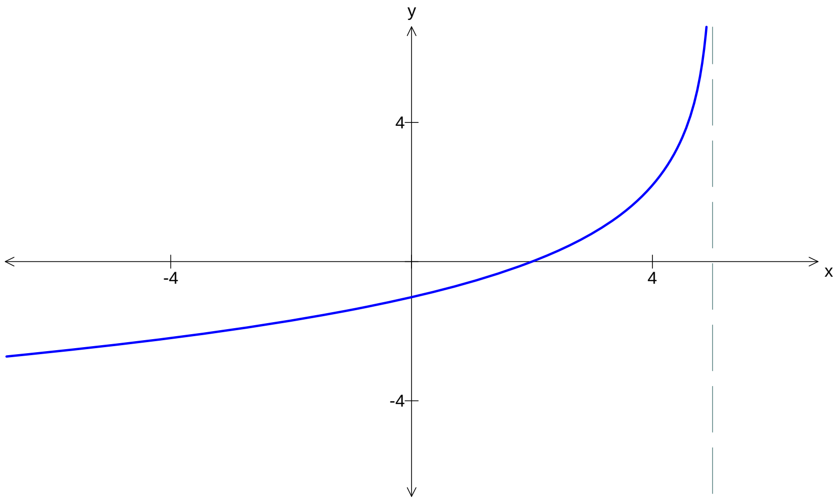
$y_{\max} = 11.43$ $y_{\min} = -30.97$. So the greatest distance between the graphs is 30.97 cm—therefore the maximum vertical web length is 30.97 cm.

Mark allocation

- 1 mark for indicating a valid method—(if student gives answer of 11.43 cm, then 1 mark)
- 1 mark for correct answer.

Question 2

Part of the graph of the function $f: R \to R,\ f(x) = -2\log_e(\frac{5-x}{3})$ is shown below.



a. State the equation of the asymptote.

1 mark

Solution:

asymptote at $x = 5$

Mark allocation

- 1 mark for answer

b. Find the equation of the inverse function f^{-1}.

2 marks

Solution:

Interchange x *and* y

$x = -2\log_e(\frac{5-y}{3})$

$\frac{-x}{2} = \log_e(\frac{5-y}{3})$

$e^{\frac{-x}{2}} = \frac{5-y}{3}$

$3e^{\frac{-x}{2}} = 5 - y$

$y = 5 - 3e^{\frac{-x}{2}}$

Mark allocation

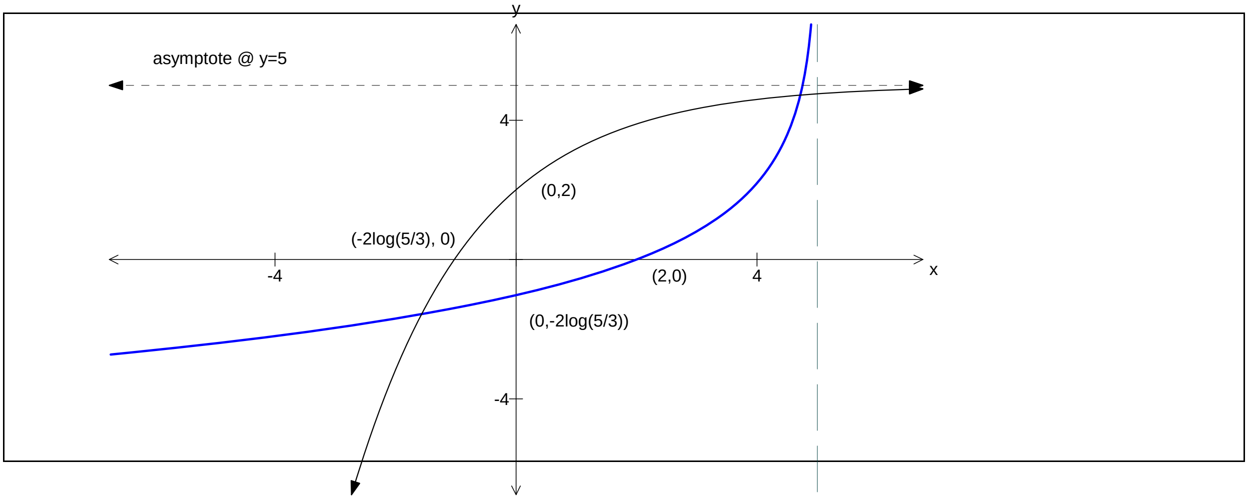
- 1 mark for valid method
- 1 mark for correct equation

Section 2 continues
TURN OVER

c. Sketch and label the inverse function f^{-1} on the axes above. Label axes intercepts as coordinates.

2 marks

Solution:



Mark allocation

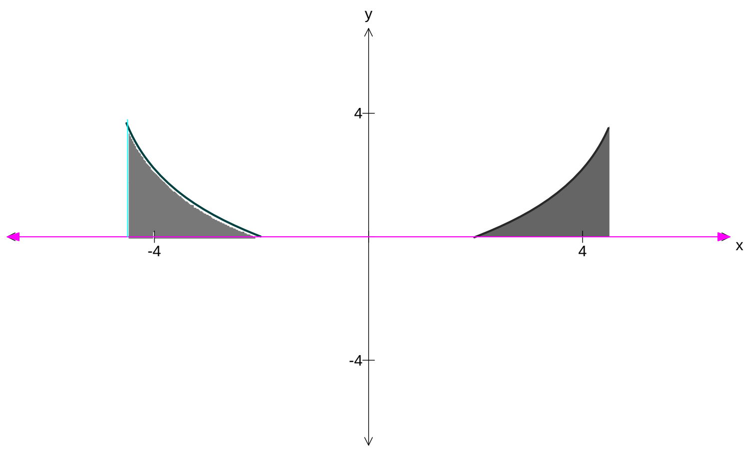
- 1 mark for correct shape
- 1 mark for asymptote correct and labelled and intercepts labelled

A skateboard ramp is made in the shape as shown below. The ramp consists of a horizontal section between two curved surfaces. The curved surfaces are described by the equations

$$g:[2,4.5]\rightarrow R,\ g(x)=f(x)$$

and $h:[-4.5,-2]\rightarrow R,\ h(x)=f(-x)$.

All measurements are in metres.



d. How many metres above the ground is the ramp when $x = 4.5$? Give your answer as an exact value.

1 mark

Solution:

Let $x = 4.5,\ to\ get\ y = -2\log_e(\frac{5-4.5}{3}) = -2\log_e(\frac{1}{6}) = 2\log_e 6$

Mark allocation

- 1 mark for correct (exact value) answer.

e. Use calculus to find the area of the shaded regions correct to two decimal places.

3 marks

Solution:

The area of the shaded regions equals

$A = 2\times\int_2^{4.5} -2\log_e(\frac{5-x}{3})\,dx$ which can't be determined,

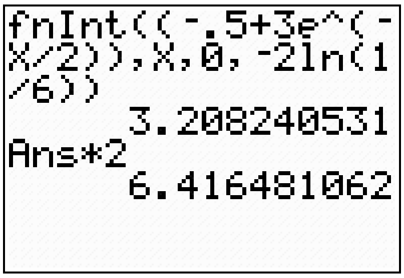
instead we use the inverse function from earlier

$= 2\times\int_0^{-2\log_e\frac{1}{6}} (4.5-(5-3e^{\frac{-x}{2}}))\ dx$

$= 2\times\left[-0.5x - 6e^{\frac{-x}{2}}\right]_0^{-2\log_e\frac{1}{6}}$

$= 6.42$ *square units*

A graphics calculator can be used to determine the numerical answer (see below), however the calculus step must be shown.

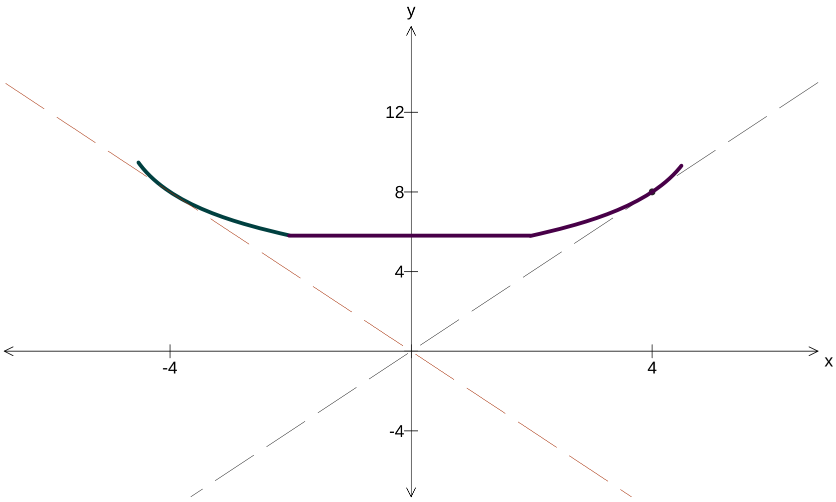


Mark allocation

- 1 mark for recognising to use the inverse function
- 1 mark for antidifferentiating
- 1 mark for the correct answer

Section 2 continues
TURN OVER

The ramp is to be used for a public exhibition by a group of international skateboarders. For the public display, the ramp is to be lifted and secured above the ground by a pair of diagonal supporting beams as shown in the diagram below. The equations of the supporting beams are described by the equations of the tangents to the ramp at the points $x = 4$ and $x = -4$.

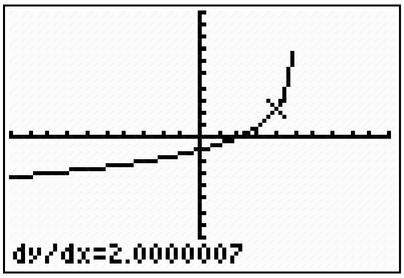


f. If the beams must pass through the origin, find how high the horizontal section of the ramp is lifted above the ground. Give your answer in exact form.

4 marks
Total 13 marks

Solution:

At $x = 4,\ \frac{dy}{dx} = 2$ (this can be found from graphics calculator)



At $x = 4,\ y = -2\log_e(\frac{1}{3})$

Equation of tangent—

$y - y_1 = m(x - x_1)$

$y + 2\log_e \frac{1}{3} = 2(x - 4)$

$y = 2x - 8 - 2\log_e \frac{1}{3}$

The tangent line has a *y*-intercept of $-8-2\log_e \frac{1}{3}$ so the ramp needs to be lifted up $8+2\log_e \frac{1}{3}$ units.

Mark allocation

- 1 mark for finding $\frac{dy}{dx}=2$
- 1 mark for finding $x=4,\ y=-2\log_e(\frac{1}{3})$
- 1 mark for finding equation of tangent
- 1 mark for exact value answer.

Question 3

A rare species of flower is grown in a hothouse. The temperature inside the hothouse is monitored and can be observed to go through two phases—an elevated temperature phase and a constant phase. These phases are cyclical and repeat regularly.

The temperature inside the hothouse is observed for 35 minutes and can be modelled by a continuous function of time described by

$$T(t)=\begin{cases}20\sqrt{\sin\frac{t}{2}+\cos\frac{t}{2}}+30 & for\ t\in[0,\frac{3\pi}{2})\cup(\frac{7\pi}{2},\frac{11\pi}{2})\cup(a,b)\\ m\ \ otherwise\end{cases}$$

where T is the temperature in ^0C and *t* is the time in minutes

a. What is the initial temperature?

1 mark

Solution:

When $t=0$, $T=50$ using the table facility on the calculator.

Mark allocation

- 1 mark for the answer

b. State the values of *a, b* and *m.*

3 marks

Solution:

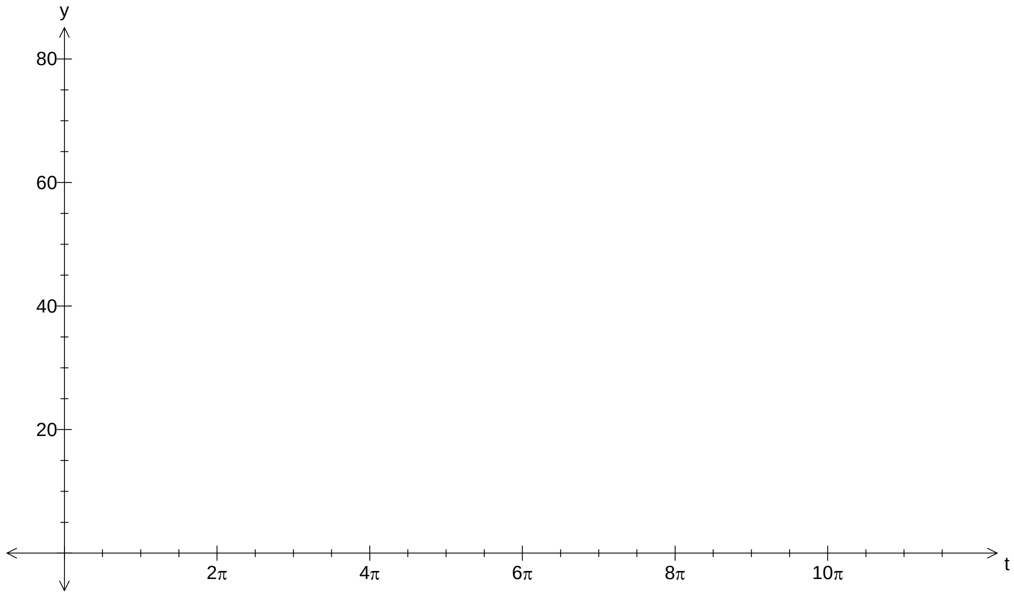
To be a continuous function $m=30$ this can be found by finding the value of *T* at $t=\frac{3\pi}{2}$.

The length of each elevated phase is 2π and the gap between each elevated phase is also 2π so $a=\frac{15\pi}{2}$ and $b=\frac{19\pi}{2}$.

Mark allocation

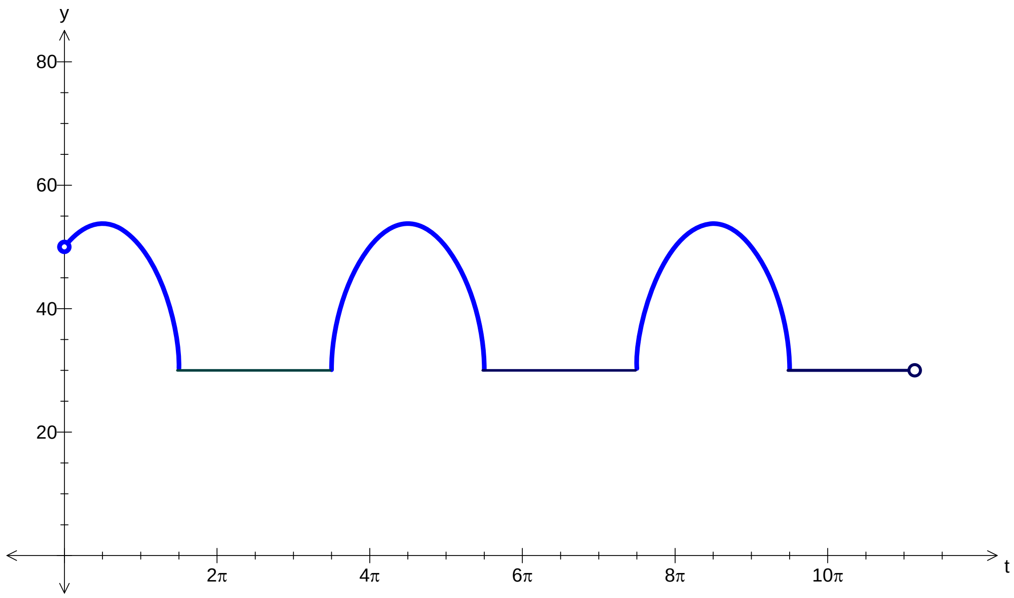
- 1 mark for each of the values *a, b, m.*

c. Sketch the graph of T for $0 < t < 35$



3 marks

Solution:



Mark allocation

- 1 mark for correct elevated phase
- 1 mark for correct constant phase
- 1 mark for endpoints

d. i. Find an expression for $\frac{dT}{dt}$ in the elevated phase.

2 marks

Solution:

Using the chain rule – –

$$\frac{dT}{dt} = 20 \times \frac{1}{2}(\sin\frac{t}{2} + \cos\frac{t}{2})^{\frac{-1}{2}}(\frac{1}{2}\cos\frac{t}{2} - \frac{1}{2}\sin\frac{t}{2})$$

$$= \frac{5(\cos\frac{t}{2} - \sin\frac{t}{2})}{\sqrt{\sin\frac{t}{2} + \cos\frac{t}{2}}}$$

Mark allocation

- 1 mark for evidence of chain rule
- 1 mark for correct derivative

ii. Hence write down an equation, the solution of which is the first value of t when the temperature is a maximum. Find this value of t, in exact form.

3 marks

Solution:

$$\frac{dT}{dt} = 0 \Rightarrow \frac{5(\cos\frac{t}{2} - \sin\frac{t}{2})}{\sqrt{\sin\frac{t}{2} + \cos\frac{t}{2}}} = 0$$

$$\Rightarrow (\cos\frac{t}{2} - \sin\frac{t}{2}) = 0$$

$$\Rightarrow \cos\frac{t}{2} = \sin\frac{t}{2}$$

$$\Rightarrow 1 = \tan\frac{t}{2}$$

first solution occurs at $\frac{\pi}{4}$

$$\frac{t}{2} = \frac{\pi}{4}$$

$$t = \frac{\pi}{2}$$

Mark allocation

- 1 mark for setting $\frac{dT}{dt} = 0$
- 1 mark for obtaining tan
- 1 mark for correct answer

iii. Find the maximum temperature correct to 2 decimal places.

1 mark

Solution:

Using calculator to find the maximum gives $T = 53.78^0$

Mark allocation

- 1 mark for answer

e. To ensure the flowers flourish and an adequate quantity is produced, the temperature must remain above a particular temperature R for a continuous period of exactly 3 minutes. Find the value of *R* for this hothouse. Give your answer correct to 3 decimal places.

3marks
Total 16 marks

Solution:

Need to centre the 3 minutes symmetrically over the peak of the elevated temperature phase—this means, for the second peak, we take 1.5 minutes either side of the maximum at $t = \frac{9\pi}{2}$.

(Could also use the first peak at $t = \frac{\pi}{2}$)

So we need to find the temperature at $t = \frac{9\pi}{2} \pm 1.5$

Using a calculator we find the value of the function at $t = \frac{9\pi}{2} \pm 1.5$ as $T = 50.345^0$, so $R = 50.345^0$

Mark allocation

- 1 mark for using method of symmetry (or other valid method)
- 1 mark for finding valid value of *t*.
- 1 mark for correct *R*.

Question 4

Type A butterflies are known to inhabit a remote Queensland island. On the Island there are 2 separate colonies of type A butterflies—the South Colony and the North Colony with 40% initially living in South colony and 60% initially living in the North Colony. Research has shown that each year 10% of the butterflies in the South Colony will move to the North Colony and 15% of the butterflies in the North Colony will move to the South Colony.

a. Determine the percentage of butterflies living in the North Colony at the end of the first year.

1 mark

Solution:

Using a tree diagram or otherwise the options are SN and NN

$SN + NN$

$= 0.4 \times 0.1 + 0.6 \times 0.85$

$= 0.55$

$= 55\%$

Mark allocation

- 1 mark for the answer

b. Determine the percentage of butterflies living in the North Colony at the end of the second year. Answer correct to two decimal places.

2 marks

Solution:

Again using a tree diagram or otherwise the options are SSN, SNN, NSN, NNN

$SSN + SNN + NSN + NNN$

$= 0.4 \times 0.9 \times 0.1 + 0.4 \times 0.1 \times 0.85 + 0.6 \times 0.15 \times 0.1 + 0.6 \times 0.85 \times 0.85$

$= 0.5125$

$= 51.25\%$

Mark allocation

- 1 mark for determining the options
- 1 mark for answer

Section 2 continues
TURN OVER

The length, X centimetres, of the wings of the type A butterfly has been found to have a probability density function

$$f(x) = \begin{cases} 0.05e^{-0.05x} & for \quad x > 0 \\ 0 & for \quad x \le 0 \end{cases}$$

c. i. Find the mean length of the wings of the type A butterfly.

2 marks

Solution:

$$mean = \int_0^{\infty} x \times f(x)\ dx$$

$$= \int_0^{\infty} x.0.05e^{-.05x}\ dx$$

$$= 20$$

Note the integral can be evaluated from the calculator—no need to show a calculus step—for ∞ use a sufficiently large value of x say 1000.

Mark allocation

- 1 mark for writing down the integral
- 1 mark for the answer.

ii. What percentage, correct to two decimal places, of type A Butterflies have wings of length more than 30 centimetres?

2 marks

Solution:

$$\Pr(X > 30) = \int_{30}^{\infty} 0.05e^{-0.05x}\ dx$$

$$= 1 - \int_0^{30} 0.05e^{-0.05x}\ dx$$

$$= 1 - 0.776870$$

$$= 0.2231$$

$$= 22.31\%$$

Mark allocation

- 1 mark for integral (again no need to use calculus)
- 1 mark for answer

d. Four type A butterflies are captured. What is the probability, correct to 2 decimal places, that exactly two of the four type A butterflies have a wing of length more than 30 centimetres?

2 marks

Solution:

Let Y= number of type A butterflies from the four captured whose wing length is greater than 30 cm

Distribution is now a binomial with p=0.2231 and n=4. Using a calculator program
$\Pr(Y = 2) = 0.180252 = 0.18$

Mark allocation

- 1 mark for recognising binomial and stating the parameters n and p
- 1 mark for the answer

Type B butterflies also inhabit the island. The two butterflies are nearly identical in shape, colour and size. The length X_B centimetres of the wings of a type B butterfly is normally distributed with a mean of 22 and a standard deviation of 2.

A rough approach to determining whether a butterfly is Type A or Type B is to measure the length of the wings. The butterfly is classified as type A if the length is less than a specified value *c*, and as type B otherwise.

e. If $c = 20$ calculate the probability that a type A butterfly is misclassified, and the probability that a type B butterfly is misclassified.

2 marks

Solution:

The probability that type A is misclassified is

$$\Pr(X_A > 20) = \int_{20}^{\infty} 0.05e^{-0.05x}\, dx$$
$$= 0.3679$$

The probability that Type B is misclassified is

$$\Pr(X_B < 20) = normcdf(-\infty, 20, 22, 2)$$
$$= 0.1587$$

Mark allocation

- 1 mark for Type A value
- 1 mark for Type B value

f. Find the value of *c*, correct to 3 decimal places, for which the two probabilities of misclassification are equal. (**Please note**: your calculator may take some time to complete this problem. Don't panic.)

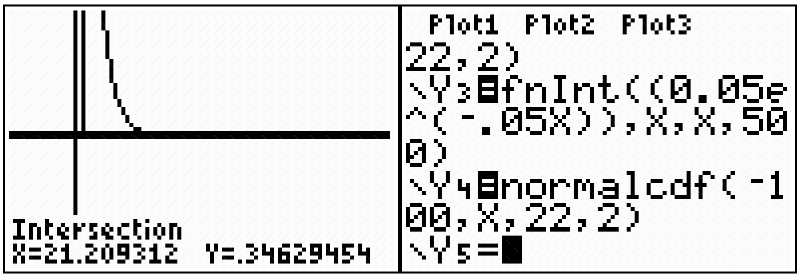
3 marks
Total 14 marks

Solution:

We need to set up two equations involving unknown c values and the equate the two to find when the areas under the curves are equal

i.e. when $\Pr(X_A > c) = \Pr(X_B < c)$ so we want $\int_c^{\infty} 0.05e^{-0.05x}\ dx = normcdf(-\infty, c, 22, 2)$.

This can only be determined using a calculator. We graph the curves that describe the areas and find the point of intersection of the two area graphs—see the screen dump below.



The point of intersection says that the two areas are equal if $c = 21.209$.

Mark allocation

- 1 mark for identifying the probabilities
- 1 mark for writing the areas
- 1 mark for answer

END OF SOLUTION BOOK