INSIGHT

Trial Exam Paper

2008

MATHEMATICAL METHODS (CAS)

Written examination 2

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

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Question 1

The range of the function $f(x) = x^3 - 5x^2 + 3x + 9,\ x \in (0,5]$ is

A. $(0,5]$
B. $(9,24)$
C. $[9,24]$
D. $(0,24]$
E. $\mathbf{[0,24]}$

Answer is E.

Worked solution

Need to examine the graph of the function over the domain stated—it is not enough to just substitute in the endpoints. Graph has a minimum value of 0 and a maximum value of 24.

Question 2

A parabola of the form $y = ax^2 + bx + c$ passes through the points $(-1,0)$ $(0,2)$ $(1,2)$.

A matrix equation that can be used to solve the resulting system of simultaneous linear equations to find *a, b* and *c* is

A. $\begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{2} \\ \mathbf{2} \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Answer is B.

Worked solution

Substituting the points into the equation gives

$a - b + c = 0$

$0 + 0 + c = 2$

$a + b + c = 2$

Writing in matrix form gives option B.

Question 3

The number of solutions to the equation $(x^2 - a)(x^3 - b^3)(x + c) = 0$ where $a, b, c \in R^+$ is

A. 6
B. 5
C. **4**
D. 3
E. 2

Answer is C.

Worked solution

The solutions arise from the number of linear factors so we need to look to see whether the existing factors can be further reduced to linear factors.

$x^2 - a$ gives 2 linear factors
$x^3 - b^3$ gives 1 linear factor
$x + c$ already is a linear factor

so a total of 4 linear factors, therefore 4 solutions.

Question 4

Given that $\log_2 c + \log_2 5 = 3\log_2 3$, then c is equal to

A. 22
B. 1.8
C. $2^{5.4}$
D. **5.4**
E. 4

Answer is D.

Worked solution

Using log laws $\log_2 5c = \log_2 27$
so $5c = 27$
$c = \frac{27}{5} = 5.4$

Could also use the solve menu on CAS calculator.

Question 5

The curve with equation $h(x) = x^3 - bx^2 - 9x + 7$ has a stationary point when $x = -1$.
The value of b is

A. -3
B. **3**
C. $-\frac{1}{2}$
D. 2
E. 6

Answer is B.

Worked solution

Stationary point occurs when $f'(x) = 0$

So let $3x^2 - 2bx - 9 = 0 \ \ when\ x = -1$

this gives

$3 + 2b - 9 = 0$

$2b = 6$

$b = 3$

Question 6

The maximal domain, *D*, of the function $f : D \to R$ with the rule $f(x) = \cos(\sqrt{(2x-3)}\,)$ is

A. $R \setminus \left\{\frac{3}{2}\right\}$

B. $R \setminus \{3\}$

C. R

D. $(\frac{3}{2}, \infty)$

E. $\mathbf{[\frac{3}{2}, \infty)}$

Answer is E.

Worked Solution

For the function to be defined inside the square needs to be greater than or equal to 0.

So

$2x - 3 \geq 0$

$x \geq \frac{3}{2}$

Question 7

The period and amplitude of the function $f(x) = 1 - 2\sin(\frac{\pi}{4} - 2x)$ are respectively

A. $\frac{\pi}{2}, 2$

B. $\frac{\pi}{4}, -2$

C. **π, 2**

D. $2\pi, 2$

E. $\pi, -2$

Answer is C.

Worked Solution

amplitude is 2 (not negative!)

period is $\frac{2\pi}{2} = \pi$

Question 8

$\left|p^2 - 5p\right| = 6$ for

A. $p = -1$ only

B. $p = 6$ only

C. $p = -1$ or $p = 6$ only

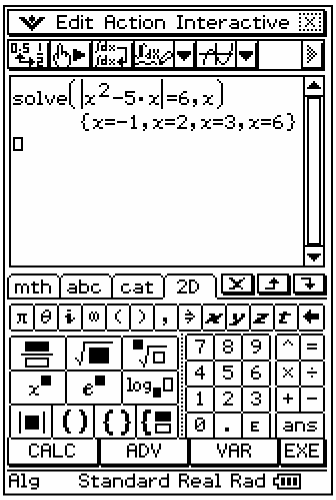
D. $p = 2$ or $p = 3$ only

E. **p = –1 or p = 6 or p = 2 or p = 3**

Answer is E.

Worked solution

Using a CAS calculator the equation can be solved algebraically as follows—



Question 9

The transformation $T: R^2 \to R^2$, which maps the curve with the equation $y = x^2 - 4$

to the curve with the equation $y = \left(\frac{x}{2}\right)^2 - 1$, could have the rule

A. $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

B. $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

C. $\mathbf{T}(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}) = \begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{3} \end{bmatrix}$

D. $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

E. $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

Answer is C.

Worked solution

To transform the equation from $y = x^2 - 4$ to $y = \left(\frac{x}{2}\right)^2 - 1$ the graph has undergone a dilation of factor 2 from the *x*-axis and a translation of +3 in the *y*-direction.

The transformation matrix that corresponds to this is C.

Question 10

Let $f: R \to R$ be a differentiable function.

Then for all $x \in R$ and *k* constant, the derivative of $e^{f(kx)}$ is equal to

A. $e^{f(kx)}$

B. $ke^{f(kx)}$

C. $kf(kx)e^{f(kx)}$

D. $f'(kx)e^{f(kx)}$

E. $\mathbf{kf'(kx)e^{f(kx)}}$

Answer is E.

Worked solution

Using the chain rule we differentiate the exponential and then differentiate the function attached to give E.

answer D is incorrect as f’(kx) is not the derivative of f’(kx)

Question 11

The average rate of change of the function with the rule $f(x)=\sqrt{x^2+2x}$ between $x=0$ and $x=4$ is

A. $2\sqrt{6}$

B. $\boldsymbol{\frac{\sqrt{6}}{2}}$

C. $\frac{2}{\sqrt{6}}$

D. $\sqrt{6}$

E. 4

Answer is B.

Worked solution

Average rate of change = $\frac{y_2-y_1}{x_2-x_1}$

$$=\frac{\sqrt{24}-0}{4-0}$$

$$=\frac{\sqrt{24}}{4}$$

$$=\frac{\sqrt{6}}{2}$$

Question 12

Which one of the following is **not** true about the function $f:R\rightarrow R,\ f(x)=(x-4)^{\frac{2}{3}}$?

A. The graph of *f* is continuous everywhere.

B. $f(x)\geq 0$ for all values of *x*.

C. $f'(x)>0$ for $x>4$

D. $f'(x)<0$ for $x<4$

E. **The graph of *f* is differentiable everywhere.**

Answer is E.

Worked solution

The graph is not differentiable at the cusp.

Question 13

The average value of the function $y = x^2$ over the interval $[0, 2]$ is

A. 4

B. 2

C. $\mathbf{\frac{4}{3}}$

D. $\frac{8}{3}$

E. 1

Answer is C.

Worked solution

$$\text{Average value} = \frac{1}{2-0}\int_0^2 f(x)\,dx$$

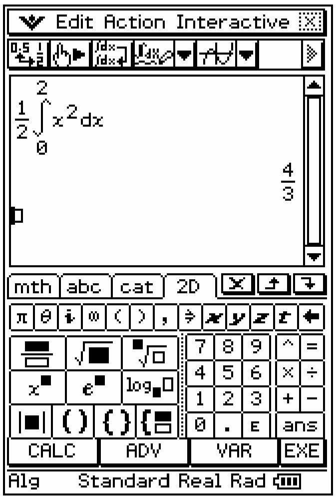
$$= \frac{1}{2}\int_0^2 x^2\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{8}{3}\right]$$

$$= \frac{4}{3}$$

Using a CAS calculator—



Question 14

The function f satisfies the functional equation $f(x-y)=\frac{f(x)}{f(y)}$ where *x* and *y* are any non-zero real numbers.

A possible rule for the function is

A. $f(x)=\log_e(x)$

B. $\boldsymbol{f(x)=e^{x}}$

C. $f(x)=2x$

D. $f(x)=\sin(x)$

E. $f(x)=\frac{1}{x}$

Answer is B.

Worked solution

$f(x-y)=e^{(x-y)}$ *for* $f(x)=e^{x}$

and $\frac{f(x)}{f(y)}=\frac{e^{x}}{e^{y}}=e^{x-y}=f(x-y)$

Question 15

If $f(x)=\frac{(x-a)^2}{g(x)}$ then the derivative of $f(x)$ is

A. $\frac{2(x-a)}{g'(x)}$

B. $\frac{2(x-a)}{g(x)}$

C. $\frac{2(x-a)g'(x)}{(g(x))^2}$

D. $\boldsymbol{\frac{(x-a)[2g(x)-(x-a)g'(x)]}{(g(x))^2}}$

E. $\frac{(x-a)[2g(x)-(x-a)g'(x)]}{(g'(x))^2}$

Answer is D.

Worked solution

Need to use the quotient rule to differentiate $f(x)$

$$f'(x)=\frac{g(x)\times 2(x-a)-(x-a)^2g'(x)}{(g(x))^2}$$

$$=\frac{(x-a)[2g(x)-(x-a)g'(x)]}{[g(x)]^2}$$

Question 16

$\int (\cos(3x-1)+12x^2)\, dx$ is equal to

A. $\frac{1}{3}\sin(3x-1)+24x+c$

B. $-3\sin(3x-1)+24x+c$

C. $\frac{1}{3}\sin(3x)+4x^3+c$

D. $\mathbf{\frac{1}{3}\sin(3x-1)+4x^3+c}$

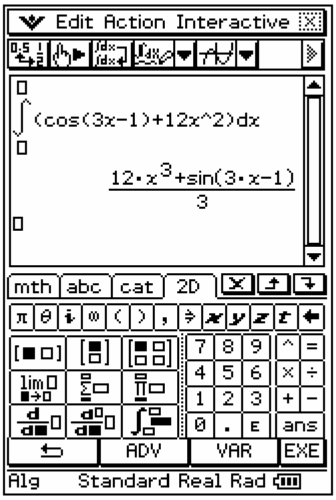
E. $3\sin(3x-1)+4x^3+c$

Answer is D.

Worked solution

Antidifferentiate both parts

Using a CAS calculator



Question 17

If $\int_0^{\pi} f(x)dx = 2$ then $\int_0^{\pi} (2f(x) - \sin x)dx$ is equal to

A. **2**
B. 3
C. 4
D. -2
E. 0

Answer is A.

Worked solution

Using the properties of integrals

$$\int_0^{\pi} (2f(x) - \sin x)dx$$

$$= 2\int_0^{\pi} f(x)\,dx - \int_0^{\pi} \sin x\ dx$$

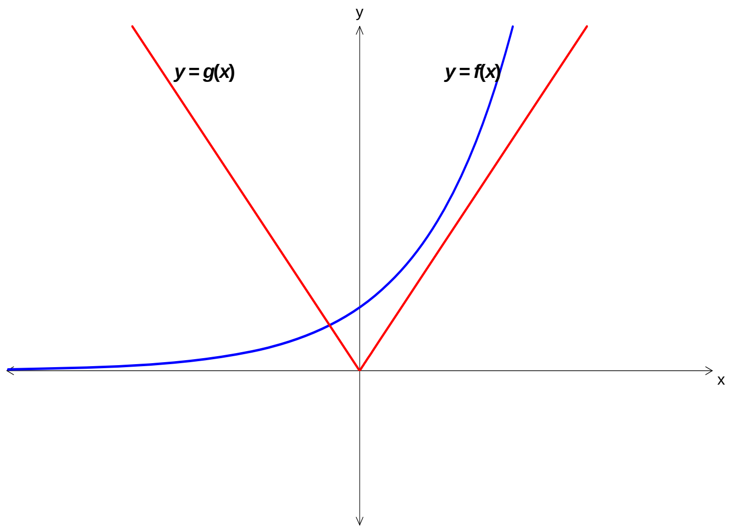
$$= 2\times 2 - [-\cos x]_0^{\pi} \text{ using result given and integrating } \sin x$$

$$= 4 - 2$$

$$= 2$$

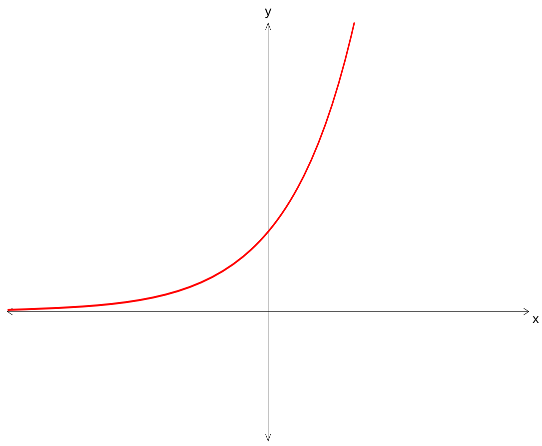
Question 18

The graphs of $y = f(x)$ and $y = g(x)$ are as shown.

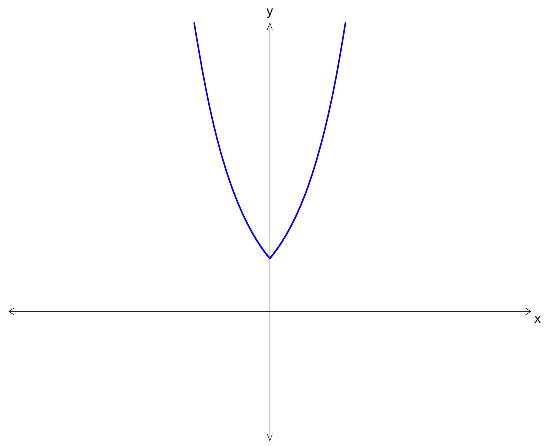


The graph of $y = f(g(x))$ is best represented by

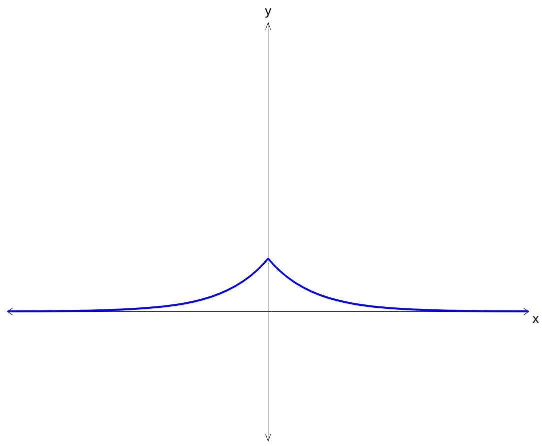
A.



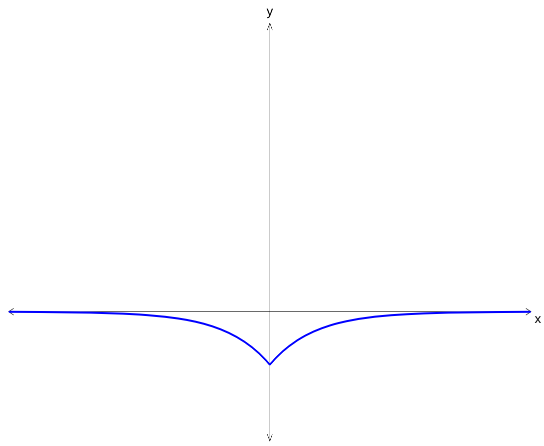
B.



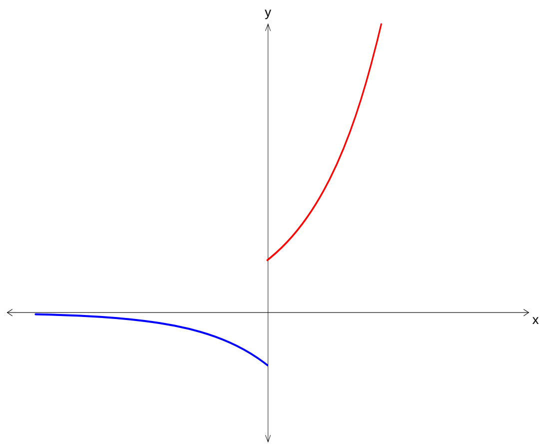
C.



D.



E.



SECTION 1 – continued
TURN OVER

Answer is B.

Worked solution

$f(x)$ looks like e^x and $g(x)$ looks like $|x|$

as an approximation $f(g(x))$ would be $e^{|x|}$ and B has the shape of $y = e^{|x|}$

Question 19

A bag contains 4 red, 3 white and 2 blue marbles. James selects a ball from the bag (and does not return it). After him, Kevin selects another ball from the bag. What is the probability that James does not select a blue ball and Kevin does not select a white ball?

A. $\frac{7}{12}$

B. $\frac{14}{27}$

C. $\frac{1}{12}$

D. $\mathbf{\frac{19}{36}}$

E. $\frac{41}{72}$

Answer is D.

Worked solution

The only options are Red, Red   Red, Blue   White, Red   White, Blue

This gives— $\frac{4}{9}\times\frac{3}{8}+\frac{4}{9}\times\frac{2}{8}+\frac{3}{9}\times\frac{4}{8}+\frac{3}{9}\times\frac{2}{8}=\frac{19}{36}$

Question 20

The speeds of vehicles travelling along a particular section of Citylink freeway are normally distributed with a mean of 95 km/h and a standard deviation of σ. 15% of drivers are found to be exceeding the 100 km/h speed limit.

The value of σ is closest to

A. 0.150
B. 1.036
C. **4.824**
D. 5.182
E. 0.207

Answer is C.

Worked solution

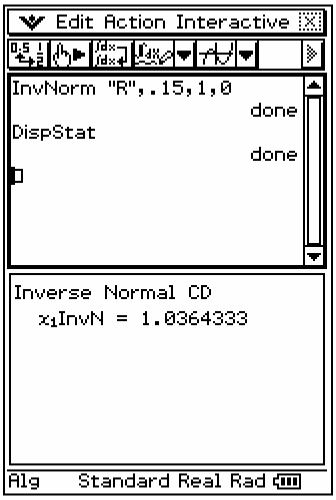
$\Pr(X > 100) = 0.15$

$\Pr(Z > \frac{100-95}{\sigma}) = 0.15,$ convert to standard normal

$1.036 = \frac{5}{\sigma}$, using inverse normal (see CAS screen below).

$\sigma = 4.824$

Using a CAS calculator the inverse normal value can be found as follows—



Question 21

Ben has constructed a spinner that will randomly display an integer between 0 and 4 with the following probabilities.

Number	*x*	0	1	2	3	4
Probability	*Pr(X=x)*	0.2	0.3	0.15	0.25	0.1

Ben spins the spinner 5 times. The probability of obtaining at least 3 odd numbers is

A. 0.55

B. 0.55^3

C. $(0.55)^3 + (0.55)^4 + (0.55)^5$

D. $(0.45)^2(0.55)^3 + (0.45)(0.55)^4 + (0.55)^5$

E. $\mathbf{10(0.45)^2(0.55)^3 + 5(0.45)(0.55)^4 + (0.55)^5}$

Answer is E.

Worked solution

The distribution is binomial with $X \sim Bi(n = 5,\ p = 0.55)$ and $\Pr(X \geq 3) =$ option E

Question 22

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The value of a such that $\Pr(X > a) = 0.875$ is

A. 0.875
B. 0.540
C. 0.204
D. 0.956
E. **0.500**

Answer is E.

Worked solution

$$\Pr(X > a) = \int_a^1 3x^2 \, dx = 0.875$$

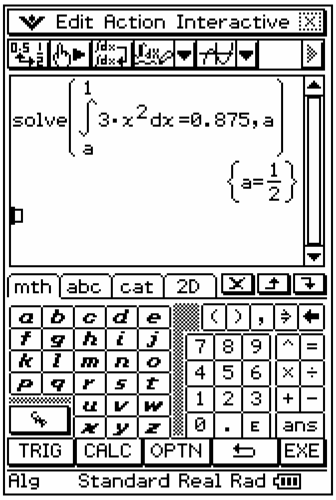
$$\left[x^3\right]_a^1 = 0.875$$

$$1 - a^3 = 0.875$$

$$a^3 = 0.125$$

$$a = 0.5$$

Using a CAS calculator and solving for a gives—



END OF SECTION 1
TURN OVER

SECTION 2

Question 1

The "peanut" spider, a rare South American spider, weaves a peanut-shaped web—hence the name.

An araneologist (person who studies spiders) observes the web-making process.

Initially the spider weaves a strand that has the shape that can be described by
$y = \frac{1}{35}x(x-7)(x^2 - 8x + 25)$ for $x \in [0,9]$ with all measurements in cm.

a. Show that $x^2 - 8x + 25 > 0$ for $x \in R$.

2 marks

Worked solution

Completing the square gives

$x^2 - 8x + 16 - 16 + 25$

$= (x-4)^2 + 9$

$(x-4)^2 \geq 0$ *so* $(x-4)^2 + 9 > 0$

Note: other techniques can also be used such as graphing or using the discriminant.

Mark allocation

- 1 mark for valid method
- 1 mark for valid conclusion

b. Find the *x*-intercepts of the graph of $y = \frac{1}{35}x(x-7)(x^2 - 8x + 25)$.

1 mark

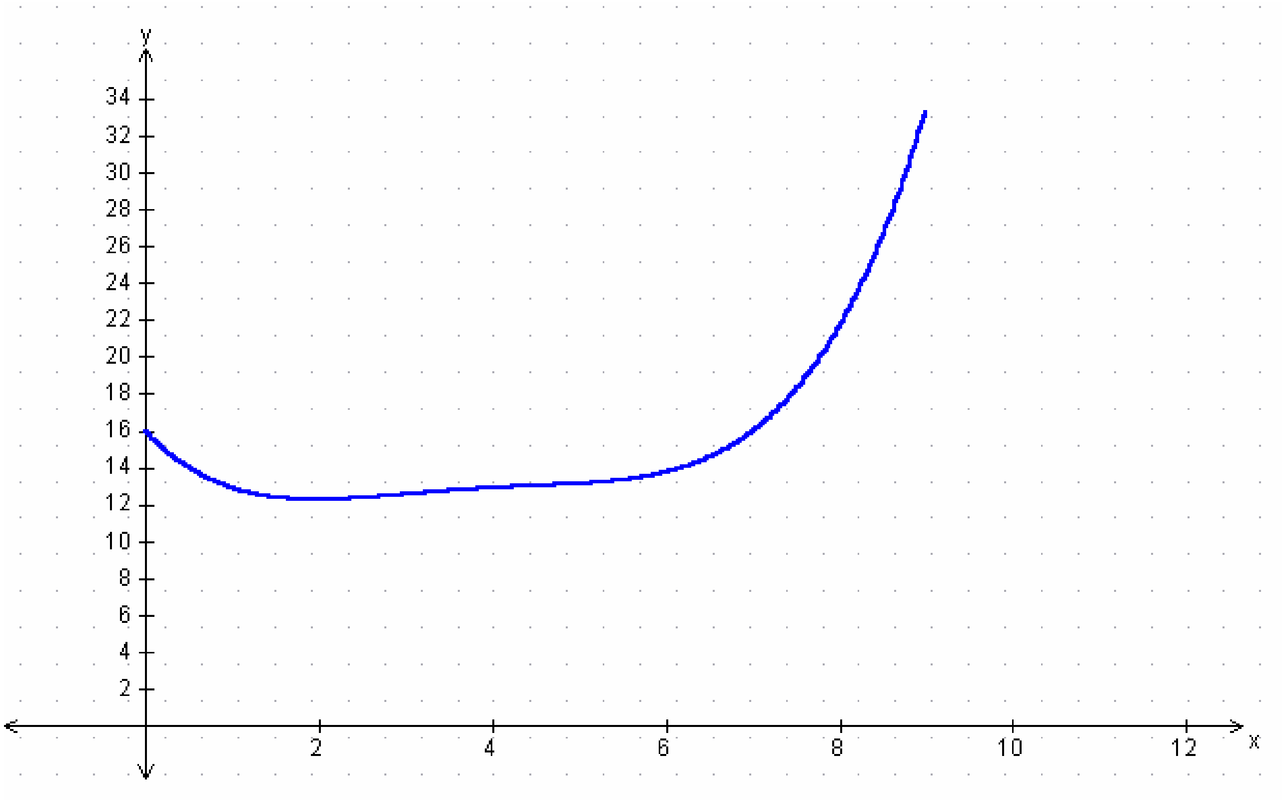
Worked solution

As $x^2 - 8x + 25 > 0$ there are no solutions from this quadratic, therefore the only *x*-intercepts come from the two linear factors *x* and *(x-7).* The *x*-intercepts are at $x = 0$ and $x = 7$.

Mark allocation

- 1 mark for correct intercepts.

The shape below describes the bottom boundary line of the web which is the graph of *y* translated up 16 cm above the ground as shown in the graph below.



c. State the equation of the bottom boundary of the web, y_b, as shown in the graph above.

1 mark

Worked solution

y_b is the graph of *y* shifted up 16 units. This gives the equation

$$y_b = \frac{1}{35}x(x-7)(x^2-8x+25)+16$$

Mark allocation

- 1 mark for the correct equation.

The spider begins to weave the top boundary of the web. The symmetry of the web becomes apparent. The top boundary of the web is a reflection of the bottom boundary in the line $y = 18$. Its equation is given by $y_T = 20 - \frac{1}{35}x(x-7)(x^2-8x+25)$.

d. State the transformations (in correct order) involved in producing y_T from *y*.

2 marks

Worked solution

The transformations are—a reflection in the *x*-axis and a translation of 20 units up.

Mark allocation

- 1 mark for stating one transformation correctly
- 1 mark for stating the second transformation correctly and in correct order.

e. The two webs are attached to trees at their endpoints. State the coordinates of the endpoints (correct to 2 decimal places).

2 marks

Worked solution

Endpoints are at $(0,\ 16),\ (9, 33.49),\ (0,\ 20),\ (9,\ 2.51)$ and these can be obtained using the table facility on the calculator or by using value.

Mark allocation

- 1 mark for 2 correct endpoints.
- 2 marks for all endpoints correct.

f. The two boundary webs intersect at one point in the domain. Find the point of intersection (correct to 2 decimal places).

1 mark

Worked solution

Using a calculator and finding the point of intersection gives—(7.45, 18)

Mark allocation

- 1 mark for correct answer.

The spider then begins to weave vertical portions on the web joining the top boundary web with the bottom boundary web.

g. One vertical web is placed at x = 5 cm. Find the minimum length of this web (correct to 2 decimal places).

2 marks

Worked solution

At $x = 5,\ y_b = 13.143$ and $y_T = 22.857$ a difference of 9.71cm

These values can be obtained from the table or by using value on the calculator.

Mark allocation

- 1 mark for the y values
- 1 mark for the correct length.

h. Three vertical webs of length 5 cm are required. Where should these be positioned? Answer correct to 3 decimal places.

2 marks

Worked solution

Set up an equation that is the difference between y_T and y_b, $(y = y_T - y_b)$, and use the graph to intersect with the lines $y = 5$ and $y = -5$

There are three points of intersection at $x = 0.105,\ 6.851,\ 7.844$

Mark allocation

- 1 mark for indicating an appropriate method—i.e. setting up the difference equation
- 1 mark for the correct *x* values.

i. Find the maximum length of a vertical web (correct to 2 decimal places).

2 marks
Total 15 marks

Worked solution

Using the equation above $(y = y_T - y_b)$ find the maximum and minimum value of the function—

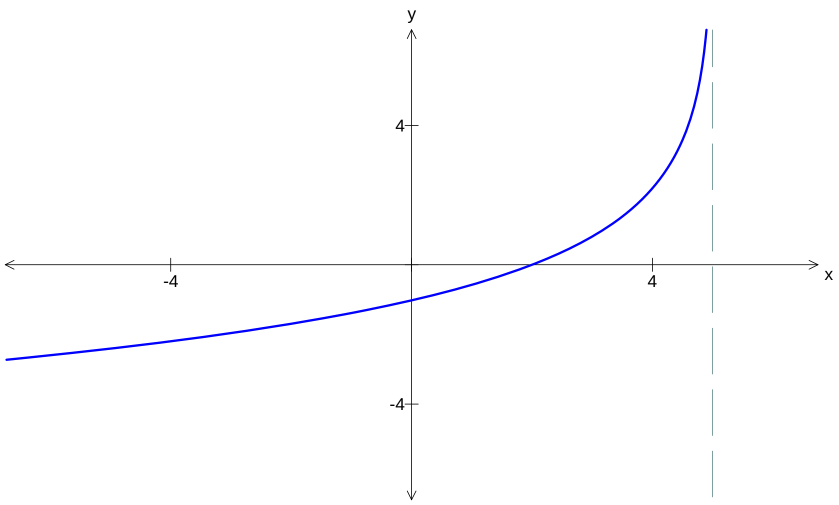
$y_{\max} = 11.43$ $y_{\min} = -30.97$. So the greatest distance between the graphs is 30.97 cm—therefore the maximum vertical web length is 30.97 cm.

Mark allocation

- 1 mark for indicating a valid method—(if student gives answer of 11.43 cm, then 1 mark)
- 1 mark for correct answer.

Question 2

Part of the graph of the function $f : R \to R,\ f(x) = -2\log_e(\frac{5-x}{3})$ is shown below.



a. State the equation of the asymptote.

1 mark

Worked solution

asymptote at $x = 5$

Mark allocation

- 1 mark for answer

b. Find the equation of the inverse function f^{-1}.

2 marks

Worked solution

Interchange x and y

$x = -2\log_e(\frac{5-y}{3})$

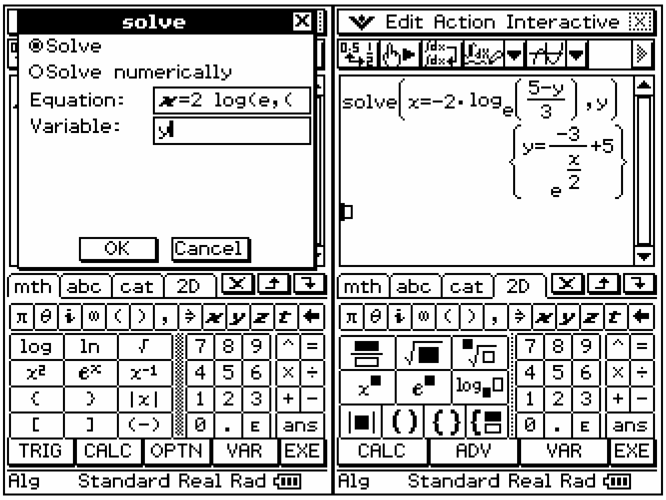
$\frac{-x}{2} = \log_e(\frac{5-y}{3})$

$e^{\frac{-x}{2}} = \frac{5-y}{3}$

$3e^{\frac{-x}{2}} = 5-y$

$y = 5-3e^{\frac{-x}{2}}$

Note: can be found by entering $x = -2\log_e(\frac{5-y}{3})$ and solving for y



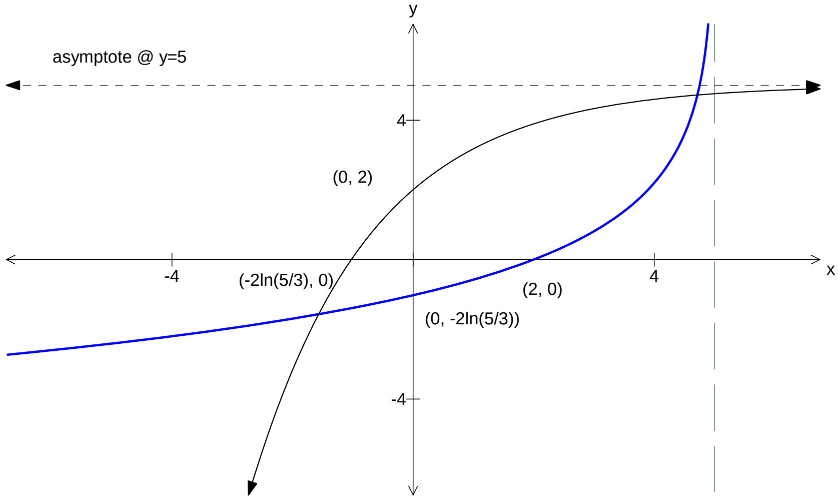
Mark allocation

- 1 mark for writing $x = -2\log_e(\frac{5-y}{3})$
- 1 mark for correct inverse equation

c. Sketch and label the inverse function f^{-1} on the axes above. Label any asymptotes with their equation. Label axes intercepts with coordinates.

2 marks

Worked solution



Mark allocation

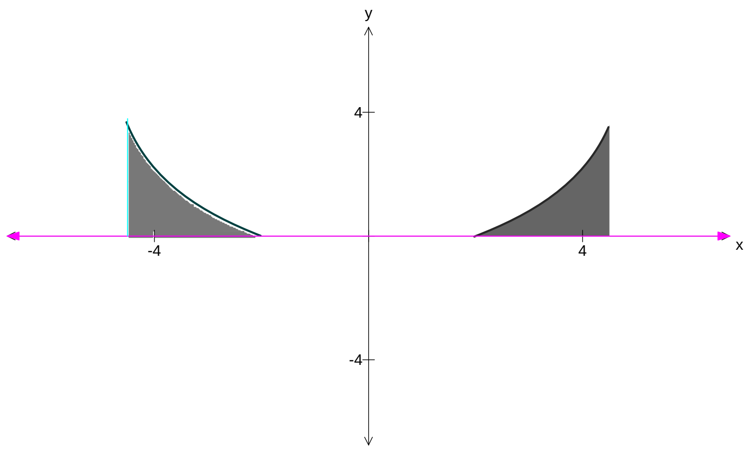
- 1 mark for correct shape
- 1 mark for asymptote correct and labelled

A skateboard ramp is made in the shape as shown below. The ramp consists of a horizontal section between two curved surfaces. The curved surfaces are described by the equations

$$g:[2,4.5]\to R,\ g(x)=f(x)$$

and $\quad h:[-4.5,-2]\to R,\ h(x)=f(-x)$.

All measurements are in metres.



d. How many metres above the ground is the ramp when $x = 4.5$? Give your answer as an exact value.

1 mark

Worked solution

Let $x = 4.5$, *to get* $y = -2\log_e\left(\frac{5-4.5}{3}\right) = -2\log_e\left(\frac{1}{6}\right) = 2\log_e(6)$

Mark allocation

- 1 mark for correct (exact value) answer.

e. Use calculus to find the area of the shaded regions correct to two decimal places.

3 marks

Worked solution

The area of the shaded regions equals

$A = 2 \times \int_2^{4.5} -2\log_e\left(\frac{5-x}{3}\right) dx$ which can't be determined,

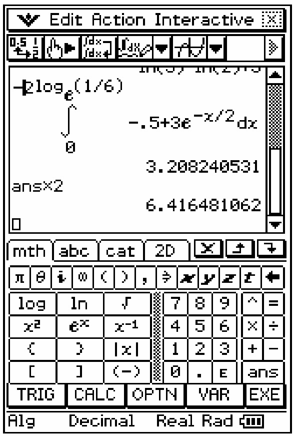
instead we use the inverse function from earlier

$= 2 \times \int_0^{-2\log_e \frac{1}{6}} \left(4.5 - \left(5 - 3e^{\frac{-x}{2}}\right)\right) dx$

$= 2 \times \left[-0.5x - 6e^{\frac{-x}{2}}\right]_0^{-2\log_e \frac{1}{6}}$

$= 6.42$ *square units*

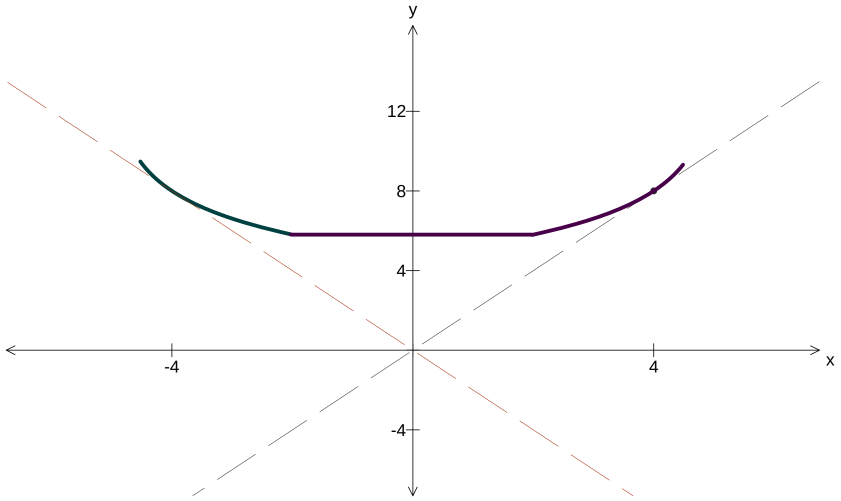
A CAS calculator can be used to determine the numerical answer (see below), however the calculus step showing the antidifferentiation must be shown.



Mark allocation

- 1 mark for recognising to use the inverse function
- 1 mark for antidifferentiating
- 1 mark for the correct answer

The ramp is to be used for a public exhibition by a group of international skateboarders. For the public display, the ramp is to be lifted and secured above the ground by a pair of diagonal supporting beams as shown in the diagram below. The equations of the supporting beams are described by the equations of the tangents to the ramp at the points $x = 4$ and $x = -4$.



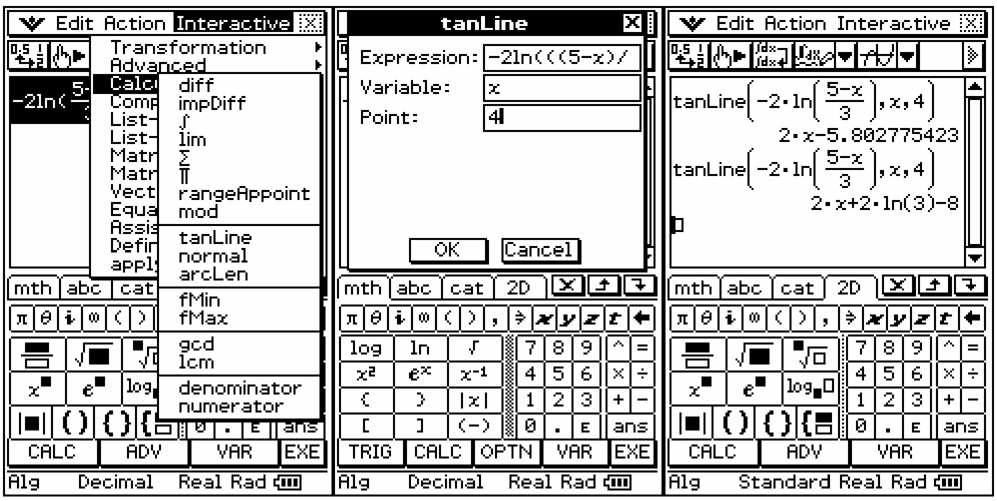
f. If the beams must pass through the origin, find how high the horizontal section of the ramp is lifted above the ground. Answer in exact form.

2 marks

Worked solution

Using a CAS calculator

Type in the expression and access tangent line from the interactive menu



Tangent line is

$y = 2x - 8 + 2\log_e 3$

The tangent line has a y-intercept of $-8 + 2\log_e 3$ so the ramp needs to be lifted up $8 - 2\log_e 3$ units.

Mark allocation

- 1 mark for finding equation of tangent
- 1 mark for exact value answer.

g. For safety reasons the horizontal section of the ramp cannot be lifted more than 4 metres above the ground—find where on the curve, correct to 3 decimal places, the tangent line supporting beams should be placed so that they still go through the origin.

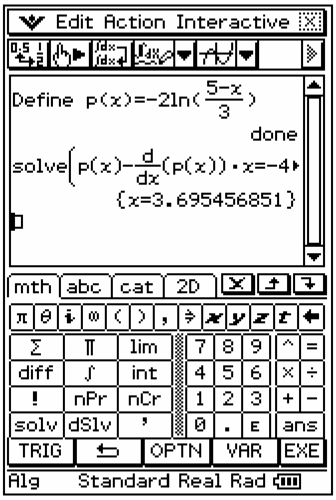
3 marks
Total 14 marks

Worked solution

We want the *y*-intercept of the tangent line to be $y = -4$

The tangent line has the equation $y = \frac{dy}{dx}x - 4$

So we solve $-4 = y - \frac{dy}{dx}x$



so the tangent needs to be drawn through the points at $x = 3.695$ and $x = -3.695$

Mark allocation

- 1 mark for knowing y = -4
- 1 mark for setting up equation
- 1 mark for answer

Question 3

A rare species of flower is grown in a hothouse. The temperature inside the hothouse is monitored and can be observed to go through two phases—an elevated temperature phase and a constant phase. These phases are cyclical and repeat regularly.

The temperature inside the hothouse is observed for 35 minutes and can be modelled by a continuous function of time described by

$$T(t) = \begin{cases} 20\sqrt{\sin\frac{t}{2} + \cos\frac{t}{2}} + 30 & for\; t \in [0, \frac{3\pi}{2}) \cup (\frac{7\pi}{2}, \frac{11\pi}{2}) \cup (a, b) \\ m \;\; \text{otherwise} \end{cases}$$

where T is the temperature inside the hothouse in ^{0}C and t is the time in minutes.

a. What is the initial temperature?

1 mark

Worked solution

When $t = 0$, $T = 50$ using the table facility on the calculator.

Mark allocation

- 1 mark for the answer

b. State the values of a, b and m.

3 marks

Worked solution

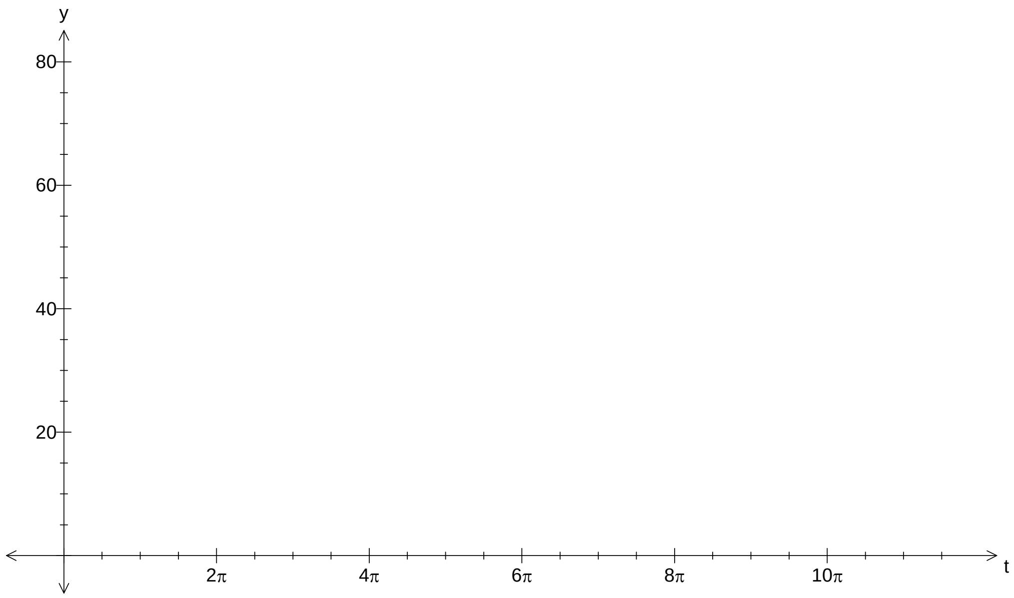
To be a continuous function $m = 30$ this can be found by finding the value of T at $t = \frac{3\pi}{2}$.

The length of each elevated phase is 2π and the gap between each elevated phase is also 2π so $a = \frac{15\pi}{2}$ and $b = \frac{19\pi}{2}$.

Mark allocation

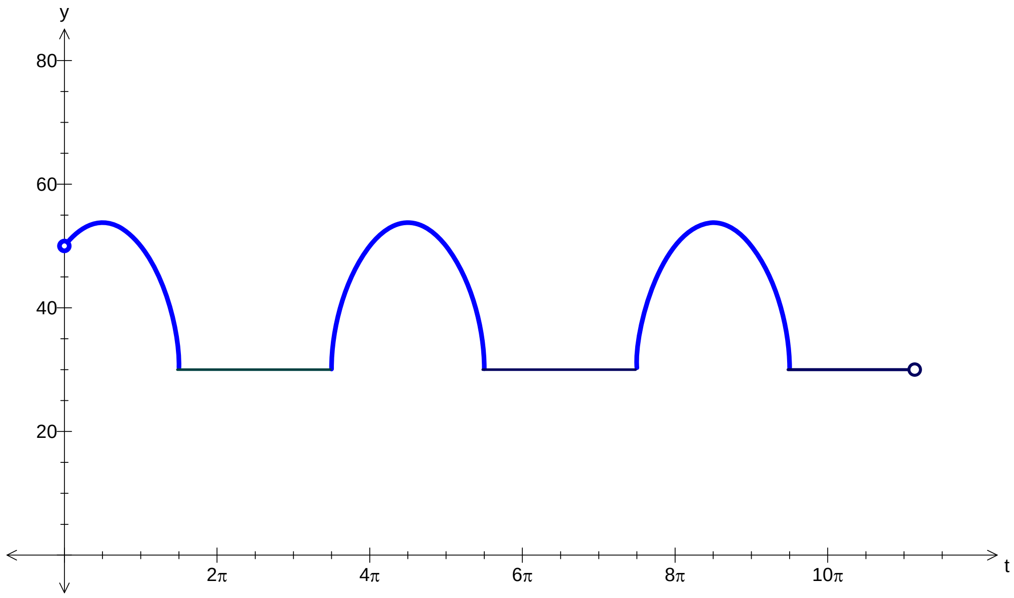
- 1 mark for each of the values a, b, m.

c. Sketch the graph of T for $0 < t < 35$ (Coordinates of points are not needed)



3 marks

Worked solution



Mark allocation

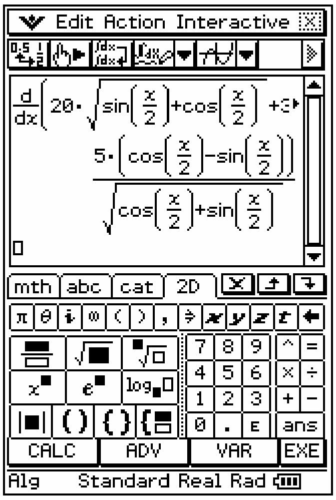
- 1 mark for correct elevated phase
- 1 mark for correct constant phase
- 1 mark for endpoints

d. **i** Find an expression for $\frac{dT}{dt}$ in the elevated phase.

1 mark

Worked solution

Using the CAS calculator—



Mark allocation

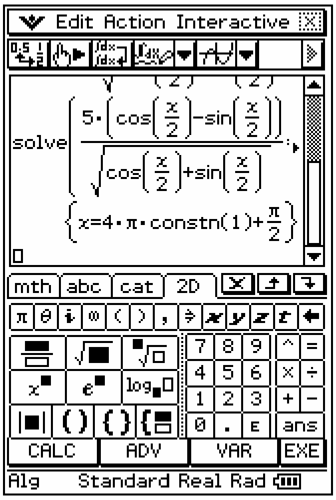
- 1 mark for correct derivative

ii. Hence write down an equation, the solution of which is the first value of t when the temperature is a maximum. Find this value of t in exact form.

2 marks

Worked solution

$$\frac{dT}{dt} = 0 \Rightarrow \frac{5(\cos\frac{t}{2} - \sin\frac{t}{2})}{\sqrt{\sin\frac{t}{2} + \cos\frac{t}{2}}} = 0 \text{ or simply } (\cos\frac{t}{2} - \sin\frac{t}{2}) = 0$$



so if the constant = 0 then $t = \frac{\pi}{2}$

Mark allocation

- 1 mark for setting $\frac{dT}{dt} = 0$
- 1 mark for correct answer

iii. Find the maximum temperature correct to 2 decimal places.

1 mark

Worked solution

Using calculator to find the maximum gives $T = 53.78^0$

Mark allocation

- 1 mark for answer

e. To ensure the flowers flourish and an adequate quantity is produced, the temperature must remain above a particular temperature R for a continuous period of exactly 3 minutes. Find the value of R for this hothouse. Give your answer correct to 3 decimal places.

2 marks
Total 13 marks

SECTION 2 – continued

Worked solution

Need to centre the 3 minutes symmetrically over the peak of the elevated temperature phase—this means, for the second peak, we take 1.5 minutes either side of the maximum at $t = \frac{9\pi}{2}$.

(Could also use the first peak at $t = \frac{\pi}{2}$)

So we need to find the temperature at $t = \frac{9\pi}{2} \pm 1.5$

Using a calculator we find the value of the function at $t = \frac{9\pi}{2} \pm 1.5$ as $T = 50.345^0$, so

$R = 50.345^0$

Mark allocation

- 1 mark for finding valid value of *t*.
- 1 mark for correct *R*.

Question 4

Type A butterflies are known to inhabit a remote Queensland island. On the Island there are 2 separate colonies of type A butterflies—the South Colony and the North Colony with 40% initially living in South colony and 60% initially living in the North Colony. Research has shown that each year 10% of the butterflies in the South Colony will move to the North Colony and 15% of the butterflies in the North Colony will move to the South Colony.

a. Determine the percentage of butterflies living in the North Colony at the end of the first year.

1 mark

Worked solution

Using a tree diagram or otherwise the options are SN and NN

$SN + NN$

$= 0.4 \times 0.1 + 0.6 \times 0.85$

$= 0.55$

$= 55\%$

Mark allocation

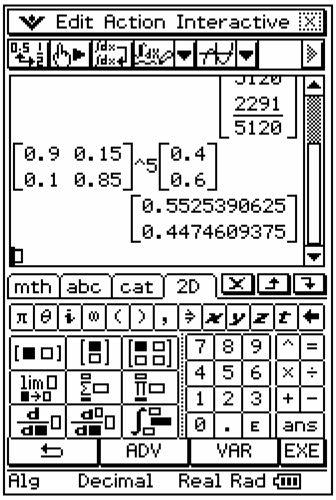
- 1 mark for the answer

b. Determine the percentage, correct to 2 decimal places, of butterflies living in the North Colony at the end of the fifth year.

2 marks

Solution:

Using matrices gives—



So there are 44.75% of the butterflies in the north colony after 5 years

Mark allocation

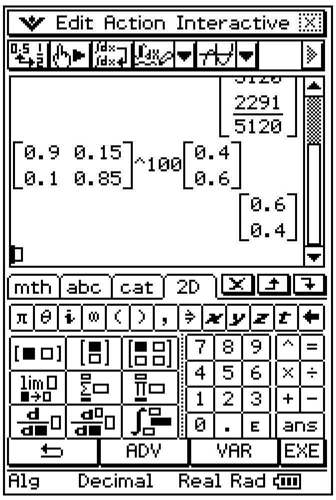
- 1 mark for setting up the matrices
- 1 mark for answer

c. Find the percentage of butterflies that will eventually be living in the North Colony in the long term.

2 marks

Worked solution

Long term means as $a \to \infty$ in the expression $\begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix}^a \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$. This can be determined by evaluating a limit expression or by choosing a suitably large value for *a* (say 100) and using the CAS calculator to determine.



Or alternatively, the steady state probability is given by $\frac{a}{a+b}$ for the matrix $\begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$

so in our case—

$\frac{a}{a+b} = \frac{0.1}{0.1+0.15} = 0.4$ so 40% of the population are in the north colony in the long term.

Mark allocation

- 1 mark for method of finding value
- 1 mark for correct value.

The length, X_A centimetres, of the wings of the type A butterfly has been found to have a probability density function

$$f(x) = \begin{cases} 0.05e^{-0.05x} & for \quad x > 0 \\ 0 & for \quad x \le 0 \end{cases}$$

d. **i.** Find the mean length of the wings of the type A butterfly.

2 marks

Worked solution

$$mean = \int_0^{\infty} x \times f(x)\ dx$$

$$= \int_0^{\infty} x.0.05e^{-.05x}\ dx$$

$$= 20$$

Note the integral can be evaluated from the calculator—no need to show a calculus step—for ∞ use a sufficiently large value of x say 1000.

Mark allocation

- 1 mark for writing down the integral
- 1 mark for the answer.

ii. What percentage, correct to 2 decimal places, of type A Butterflies have wings of length more than 30 centimetres?

2 marks

Worked solution

$$\Pr(X > 30) = \int_{30}^{\infty} 0.05e^{-0.05x}\ dx$$

$$= 1 - \int_0^{30} 0.05e^{-0.05x}\ dx$$

$$= 1 - 0.776870$$

$$= 0.2231$$

$$= 22.31\%$$

Mark allocation

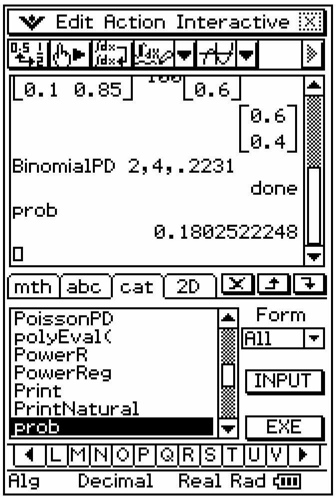
- 1 mark for integral (again no need to use calculus)
- 1 mark for answer

e. Four type A butterflies are captured. What is the probability, correct to 2 decimal places, that exactly two of the four type A butterflies have a wing of length more than 30 centimetres?

2 marks

Worked solution

Distribution is now a binomial with p = 0.2231 and n = 4. Using a CAS calculator



$\Pr(Y = 2) = 0.180252 = 18.02\%$

Mark allocation

- 1 mark for recognising binomial and stating the parameters n and p
- 1 mark for the answer

Type B butterflies also inhabit the island. The two butterflies are nearly identical in shape, colour and size. The length X_B centimetres of the wings of a type B butterfly is given by the probability density function defined by

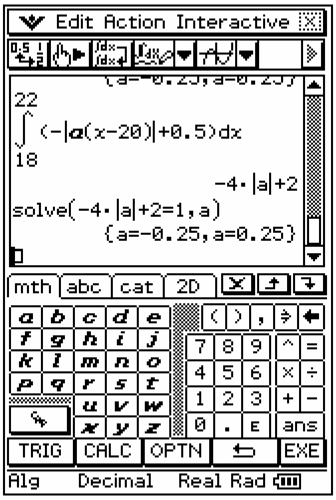
$$g(x)=\begin{cases}\frac{1}{2}-|a(x-20)|, & 18<x<22\\ 0 & \text{otherwise}\end{cases}$$

f. Find the value of a, if $a>0$

1 mark

Worked solution

Using a CAS calculator set up $\int_{18}^{22}\frac{1}{2}-|a(x-20)|\,dx=1$ and solve for *a*.



therefore a = 0.25

Mark allocation

- 1 mark for answer

A rough approach to determining whether a butterfly is Type A or Type B is to measure the length of the wings. The butterfly is classified as type A if the length is less than a specified value *c*, and as type B otherwise.

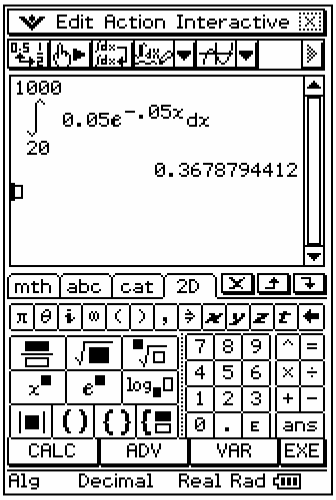
g. If $c=20$ calculate the probability, correct to 3 decimal places, that a type A butterfly is misclassified, and the probability, correct to 3 decimal places, that a type B butterfly is misclassified.

2 marks

Worked solution

The probability that type A is misclassified is

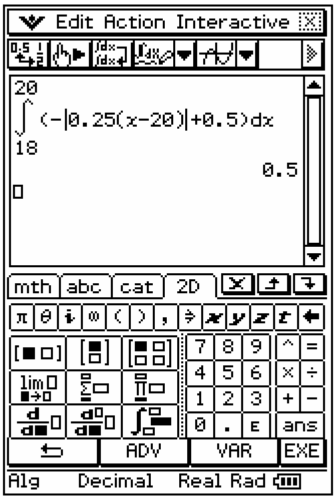
$$\Pr(X_A>20)=\int_{20}^{\infty}0.05e^{-0.05x}\,dx$$
$$=0.368$$



The probability that Type B is misclassified is

$$\Pr(X_B < 20) = \int_{18}^{20} 0.5 - |0.25(x-20)|\,dx$$ (or could recognise symmetry to obtain probability)

$$= 0.500$$



Mark allocation:

- 1 mark for Type A value
- 1 mark for Type B value

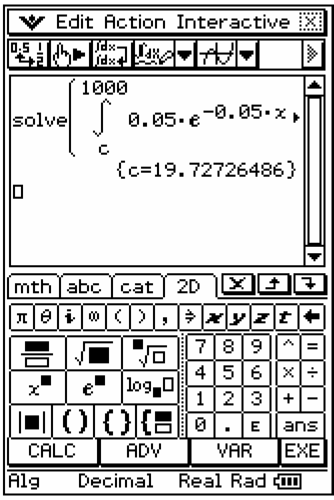
h. Find the value of c, correct to 3 decimal places, for which the two probabilities of misclassification are equal.

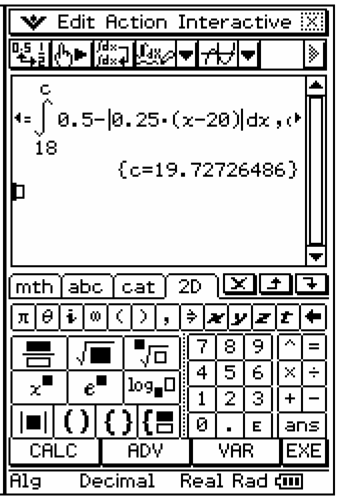
2 marks
Total 16 marks

Worked solution

We need to set up two equations involving unknown c values and the equate the two to find when the areas under the curves are equal

i.e. when $\Pr(X_A > c) = \Pr(X_B < c)$ so we want $\int_c^{\infty} 0.05e^{-0.05x}\ dx = \int_{18}^{c} 0.5 - |0.25(x-20)|\ dx$





therefore c = 19.727

Note: This can be found by solving numerically on CAS—instead of ∞ use a sufficiently large number such as 1000.

Mark allocation

- 1 mark for writing the areas
- 1 mark for answer

END OF WORKED SOLUTIONS