THE SCHOOL FOR EXCELLENCE

UNITS 3&4 MATHEMATICAL METHODS 2007

COMPLIMENTARY WRITTEN EXAMINATION 2 - SOLUTIONS

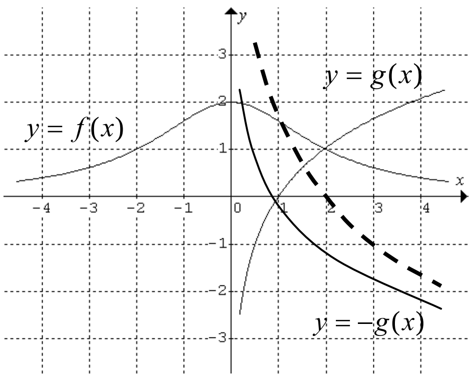
SECTION 1 – MULTIPLE CHOICE QUESTIONS

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
A	C	B	E	A	A	C	B	D	E	D

12	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**
B	A	D	D	A	C	D	C	B	C	E

QUESTION 1

Subtract ordinates across the common domain i.e. $(0, \infty)$. Alternatively, draw the graph of $y = -g(x)$ then add the ordinates of $y = f(x)$ and $y = -g(x)$ across the common domain.

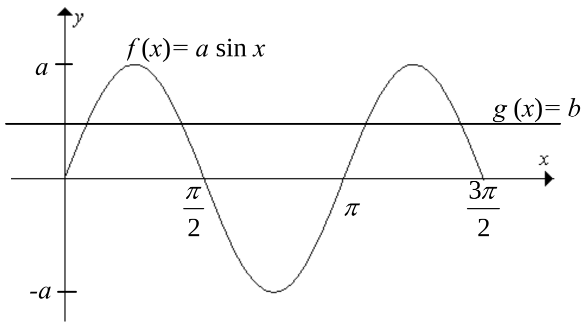


The answer is A.

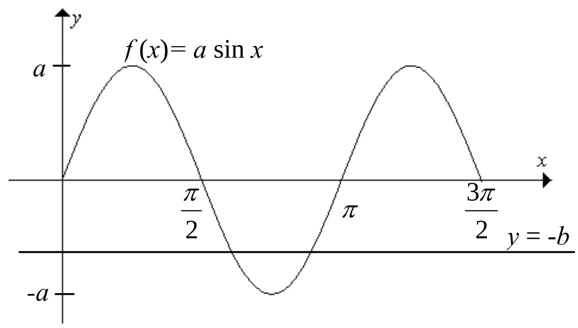
QUESTION 2

Option A: $b > a \Rightarrow \frac{b}{a} > 1$ therefore $\sin(2x) = \frac{b}{a}$ has no real solution (as $\frac{b}{a}$ must lie between -1 and 1 inclusive. Therefore option A is true.

Option B: If $0 < b < a$, it can be seen from the graph below that there are 4 solutions to $\sin(2x) = \frac{b}{a}$ for all possible values of a and b. Therefore option B is true.



Option C: If $0 < b < a$, it can be seen from the graph below that there are 2 solutions to $a\sin(2x) = -b$. Therefore option C is **not** true.



Option D: $f(x) = -g(x) \Rightarrow a\sin(2x) = -b$.

$a = b \Rightarrow \sin(2x) = -1 \qquad \therefore 2x = \frac{3\pi}{2} + 2n\pi, \quad n \in Z \qquad \therefore x = \frac{3\pi}{4} + n\pi$

There is only 1 solution for $0 < x < \frac{3\pi}{2}$ .Therefore option D is true.

Option E: $f(x) = g(x) \Rightarrow a\sin(2x) = b$

$a = b \Rightarrow \sin(2x) = 1 \qquad \therefore 2x = \frac{\pi}{2} + 2n\pi, \quad n \in Z \qquad \therefore x = \frac{\pi}{4} + n\pi$

There are 2 solutions for $0 < x < \frac{3\pi}{2}$. Therefore option E is true.

The answer is C.

QUESTION 3

To solve $|2x-3| \geq x^2 - 2$, first note that $|2x-3| = \begin{cases} 2x-3 & \text{when } 2x-3 \geq 0 \Rightarrow x \geq \frac{3}{2} \\ -(2x-3) & \text{when } 2x-3 < 0 \Rightarrow x < \frac{3}{2} \end{cases}$

Case 1:

$2x-3 \geq x^2 - 2, \quad x \geq \frac{3}{2}$,

$\therefore x^2 - 2x + 1 \leq 0$

$\therefore (x-1)^2 \leq 0$

$\therefore x = 1$.

But $x \geq \frac{3}{2}$ is not satisfied.

Therefore there is no solution.

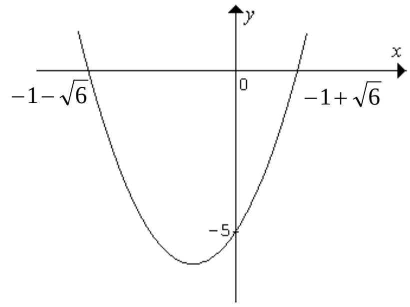
Case 2:

$-2x+3 \geq x^2 - 2, \quad x < \frac{3}{2}$,

$\therefore x^2 + 2x - 5 \leq 0$.

When $x^2 + 2x - 5 = 0$,

$x = -1 \pm \sqrt{6}$.



From the graph it can be seen that the solution to

$\therefore x^2 + 2x - 5 \leq 0$ is therefore

$\{x : -1+\sqrt{6} \leq x \leq -1-\sqrt{6}\}$.

Note that $x < \frac{3}{2}$ is satisfied.

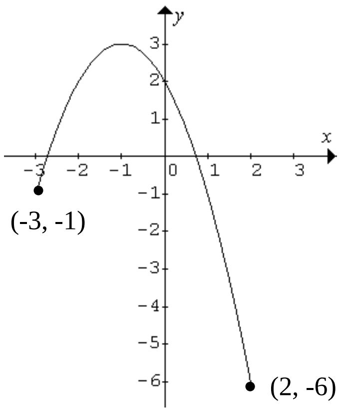
Therefore the solution to $|2x-3| \geq x^2 - 2$ is $\{x : -1+\sqrt{6} < x < -1-\sqrt{6}\}$.

The answer is B.

QUESTION 4

From the graph of $f : [-3, 2] \to R, f(x) = 3-(x+1)^2$

the range is $[-6, 3]$.



The answer is E.

QUESTION 5

A function must be one-to-one in order to have an inverse function.

$f : [0, a] \to R$, $f(x) = 3\cos(2x)$ is a one-to-one function for $a \leq \dfrac{\pi}{2}$.

The answer is A.

QUESTION 6

The graph of $y = a(x+b)^3(x+c)$ has:

- a stationary point of inflexion that is also an *x*-intercept at $x = -b$. This corresponds to $x = -1$ on the graph. Therefore $b = 1$.
- an x-intercept at $x = -c$. This corresponds to $x = 2$. Therefore $c = -2$.

Therefore $y = a(x+1)^3(x-2)$.

As $(0,1)$ is a point on the graph:

$\therefore 1 = a(0+1)^3(0-2)$

$\therefore 1 = -2a$

$\therefore a = -\frac{1}{2}$

$\therefore a = -\frac{1}{2},\ b = 1,\ c = -2$

The answer is A.

QUESTION 7

Linear approximation: $f(x+h) \approx f(x) + hf'(x)$.

Let $x = 1$ and $x + h = 0.9 \quad \therefore h = -0.1$.

$f(x) = \frac{x-1}{\sqrt{x}} = \sqrt{x} - \frac{1}{\sqrt{x}}$ and $f(1) = 0$.

$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}} + \frac{1}{2}\frac{1}{x\sqrt{x}} \qquad \therefore f'(1) = 1$.

$$\begin{aligned} f(0.9) &\approx f(1) - 0.1f'(1) \\ &\approx 0 - 0.1(1) \\ &\approx -0.1 \end{aligned}$$

$$\begin{aligned} \text{Change in } f &= f(0.9) - f(1) \\ &= -0.1 - 0 \\ &= -0.1 \end{aligned}$$

The answer is C.

QUESTION 8

Amplitude = 4

Period = $\frac{1}{8} = \frac{2\pi}{n}$

$\therefore n = 16\pi$

A. $f(t) = 2\sin\left(\frac{\pi t}{4}\right)$ is incorrect as the period = 8

B. $f(t) = 4\sin(16\pi t)$ is correct.

C. $f(t) = 4\sin(8\pi t)$ is incorrect as period = $\frac{1}{4}$

D. $f(t) = 4\sin\left(\frac{\pi t}{4}\right)$ is incorrect as period = 8

E. $f(t) = 2\sin\left(\frac{\pi t}{8}\right)$ is incorrect as the period = 16

The answer is B.

QUESTION 9

$f(x) = x + \sqrt{x+a}$

$\therefore f'(x) = 1 + \frac{1}{2\sqrt{x+a}}$

For stationary points $f'(x) = 0$:

$0 = 1 + \frac{1}{2\sqrt{x+a}}$

$\therefore \frac{1}{2\sqrt{x+a}} = -1$

$\therefore \sqrt{x+a} = -\frac{1}{2}$.

This equation has no real solutions.

The answer is D.

QUESTION 10

$\log_e x = \log_e(x-1) + b$

$\log_e x - \log_e(x-1) = b$

$\log_e\left(\dfrac{x}{x-1}\right) = b$

$\dfrac{x}{x-1} = e^b$

$x = e^b(x-1)$

$x(1-e^b) = -e^b$

$x = \dfrac{e^b}{e^b - 1}$

The answer is E.

QUESTION 11

$f(x) = a\log_e(bx-c)$

$f(x) = a\log_e b\left(x - \dfrac{c}{b}\right)$

The function has a vertical asymptote at $x = \dfrac{c}{b}$, therefore the domain is $\left(\dfrac{c}{b}, \infty\right)$.

The answer is D.

QUESTION 12 ***The answer is B.***

QUESTION 13

The graph of f has a negative gradient throughout its domain. Therefore A and D are the only possibilities. As x increases the gradient approaches zero. Therefore A is most likely to be the correct graph.

The answer is A.

QUESTION 14

$y = x^n e^{3x-n}$

$\frac{dy}{dx} = nx^{n-1}e^{3x-n} + 3x^n e^{3x-n}$

$= x^{n-1}e^{3x-n}(n+3)$

At the point, $(1, e^{3-n})$, $\frac{dy}{dx} = 1^{n-1}e^{3-n}(n+3)$

$= (n+3)e^{3-n}$.

The answer is D.

QUESTION 15

Using the chain rule: $\frac{d}{dx}\left(\log_e \sqrt{2x^2+1}\right) = \frac{2x}{2x^2+1}$

The answer is D.

QUESTION 16

$\int_a^b 1 - 3f(x)\,dx = \int_a^b 1\,dx - 3\int_a^b f(x)\,dx$

$= [x]_a^b - 3(1)$

$= b - a - 3$

The answer is A.

QUESTION 17

Area = $-\int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx - \int_a^c f(x)\,dx$

$= 2\int_0^a f(x)\,dx - \int_a^c f(x)\,dx$

Note that, based on the graph, an assumption of symmetry is made.

$= 2\int_0^a f(x)\,dx + \int_c^a f(x)\,dx$

The answer is C.

QUESTION 18

$$\frac{d}{dx}\left(x\log_e(3x)\right)=1+\log_e(3x)$$

Integrate both sides:

$$x\log_e(3x)=x+c_1+\int\log_e(3x)dx$$

$$\therefore \int\log_e(3x)dx=x\log_e(3x)-x-c_1$$

$$\therefore \int 2\log_e(3x)dx=2(x\log_e(3x)-x-c_2)$$

$$=2x(\log_e(3x)-1)-c$$

The answer is D.

QUESTION 19

$$\Pr(X>k)=\frac{7}{8}$$

$$\therefore \int_k^1 \frac{3\sqrt{x}}{2}\,dx=\frac{7}{8} \qquad \therefore \left[x^{\frac{3}{2}}\right]_k^1=\frac{7}{8} \qquad \therefore 1-k^{\frac{3}{2}}=\frac{7}{8} \qquad \therefore k^{\frac{3}{2}}=\frac{1}{8} \qquad k=\frac{1}{4}.$$

The answer is C.

QUESTION 20

Let X be the number of defective MP3 players in a sample of n.

X is a binomial random variable where $p=1-0.98=0.02$ and the sample size is n.

Note: Pr(not defective) = 0.98

Pr(defective) = 0.02

$$\Pr(X>1)=1-\Pr(X=0)-\Pr(X=1)$$

$$=1-\binom{n}{0}(0.02)^0(0.98)^n-\binom{n}{1}(0.02)^1(0.98)^{n-1}$$

$$=1-(0.98)^n-n(0.02)(0.98)^{n-1}$$

The answer is B.

QUESTION 21

The standard deviation $\frac{\sigma}{2}$ will halve the spread so the graph will be narrow and taller. The mean will not affect the spread or height of the graph.

Option A is incorrect as the spread is too small.
Option B is incorrect as the spread is too large.
Option C is correct as its spread is half of the original graph and it is taller.
Option D is incorrect as the graph does not represent a normal distribution.
Option E is incorrect as the spread is too large.

The answer is C.

QUESTION 22

Let X = the number of goals in n trials.
X follows a binomial distribution with *p* = 0.4 and number of trials = *n*.

$$\Pr(X \geq 1) \geq 0.9$$

$$\therefore 1 - \Pr(X = 0) \geq 0.9$$

$$\therefore \Pr(X = 1) \leq 0.1$$

$$\therefore \binom{n}{0}(0.4)^0(0.6)^n \leq 0.1$$

$$\therefore (0.6)^n \leq 0.1$$

Using a graphics calculator, the solution to $(0.6)^n = 0.1$ is 4.508.
Therefore n must be 5 since $(0.6)^n \leq 0.1$ is required.

The answer is E.

SECTION 2 – EXTENDED ANSWER QUESTIONS

QUESTION 1

Note that $D_g = R \setminus \{1\}$.

a. (i) $f(g(x)) = f(\log_e |x-1|)$

$= e^{2\log_e |x-1|}$ A1

$= e^{\log_e |x-1|^2}$

$= (x-1)^2, \quad x \in R \setminus \{1\}$ since $D_g = R \setminus \{1\}$.

(ii) $(fog)'(x) = 2(x-1), \quad x \in R \setminus \{1\}$

$\therefore (fog)'(-1) = 2(-1-1) = -4$. A1

b. ran $f = (0, \infty)$ and dom $g = D_g = R \setminus \{1\}$.

$(0, \infty) \not\subset R \setminus \{1\}$ Note: $R \setminus \{1\} \equiv (-\infty, 1) \cup (1, +\infty)$

$\therefore \text{ran } f \not\subset \text{dom } g$

$\therefore f(g(x))$ does not exist. M1

c. (i) Require ran $f_1 = (1, \infty)$

$\therefore$ dom $f_1 = (0, \infty)$. A1

$g(f_1(x)) = g(e^{2x}) = \log_e |e^{2x} - 1|, \quad 0 < x < \infty$.

(ii) Require ran $f_2 = (0, 1)$

$\therefore$ dom $f_2 = (-\infty, 0)$ A1

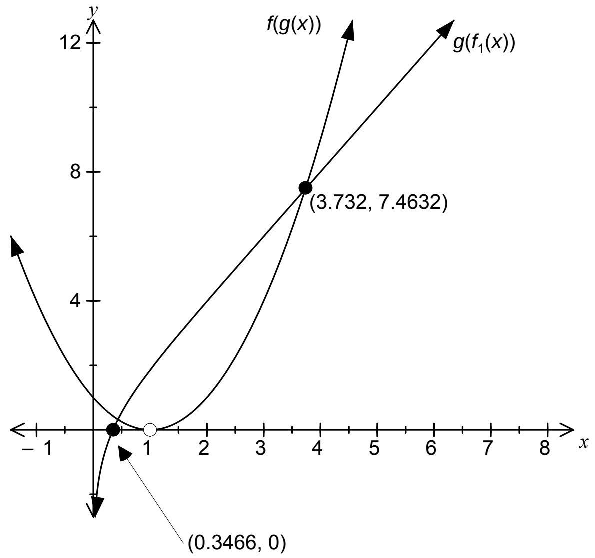
$g(f_2(x)) = g(e^{2x}) = \log_e |e^{2x} - 1|, \quad -\infty < x < 0$.

d. (i) $f(g(x)) = g(f_1(x))$

$\therefore (x-1)^2 = \log_e |e^{2x} - 1|, \quad x \neq 1$.

$\therefore$ The coordinates of the point of intersection that lies to the right of the line $x = 1$ are (3.7, 7.5). A1

(ii)



$$\text{Area} = \int_{1}^{3.732} \left(\log_e |e^{2x} - 1| - (x-1)^2\right) dx + \int_{3.732}^{5} \left((x-1)^2 - \log_e |e^{2x} - 1|\right) dx$$ M1

$\approx (12.858 - 6.797) + (14.536 - 11.072)$

$= 6.061 + 3.464$

$= 9.525$

$= 9.5$, correct to one decimal place. A1

Note 1: $\int_{1}^{3.732} \left(\log_e |e^{2x} - 1| - (x-1)^2\right) dx$ is an improper integral since the lower integral terminal lies outside the domain of $f(g(x)) = (x-1)^2$ (see the above graph). However, the integral can be shown to exist via a limiting process, which is why the calculator gives a finite value.

Note 2: During a calculation, accuracy greater than that specified for the answer should be used so as to avoid the accumulation of rounding error.

QUESTION 2

a. (i) $a =$ amplitude $= \frac{1}{2}|h_{\max} - h_{\min}| = \frac{1}{2}\left([18 + 2 + 100] - [18 + 2]\right)$

$= \frac{1}{2}(120 - 20) = 50$ metres. M1

(ii) Minimum height = 18 + 2 = 20 metres = (-Amplitude + Vertical Translation)

$\therefore 20 = b - a$

$\therefore 20 = b - 50$ M1

$\therefore b = 70$

b. Period = 80 seconds = $\frac{4}{3}$ minutes

$\therefore \frac{2\pi}{c} = \frac{4}{3}$

$\therefore c = 2\pi \times \frac{3}{4}$ M1

$= \frac{3\pi}{2}$

c. $h = 70 + 50\sin\frac{3\pi}{2}(t + d)$

When $t = 0,\ h = 20$

$\therefore 20 = 70 + 50\sin\left(\frac{3\pi\, d}{2}\right)$

$\therefore \sin\left(\frac{3\pi\, d}{2}\right) = -1$ M1

$\therefore \frac{3\pi\, d}{2} = \frac{3\pi}{2}$

$\therefore d = 1$ is a possible value. M1

d. Period = $\frac{4}{3}$ minutes.

Therefore the point *P* reaches the maximum height once every $\frac{4}{3}$ minutes:

$36 \div \frac{4}{3} = 36 \times \frac{3}{4} = 27$.

Therefore the point P reaches its maximum height 27 times during a 36 minute ride.
A1

e. $70 + 50\sin\frac{3\pi}{2}(t+1) \geq 95$.

First solve $70 + 50\sin\frac{3\pi}{2}(t+1) = 95$ where the smallest value of t satisfying $0 \leq t \leq 36$ is required:

$70 + 50\sin\frac{3\pi}{2}(t+1) = 95,\quad 0 \leq t \leq 36$

$\therefore \sin\frac{3\pi}{2}(t+1) = \frac{1}{2}$

$\therefore \frac{3\pi}{2}(t+1) = \frac{\pi}{6} + 2m\pi$ or $\frac{5\pi}{6} + 2m\pi$, where $m \in Z$. M1

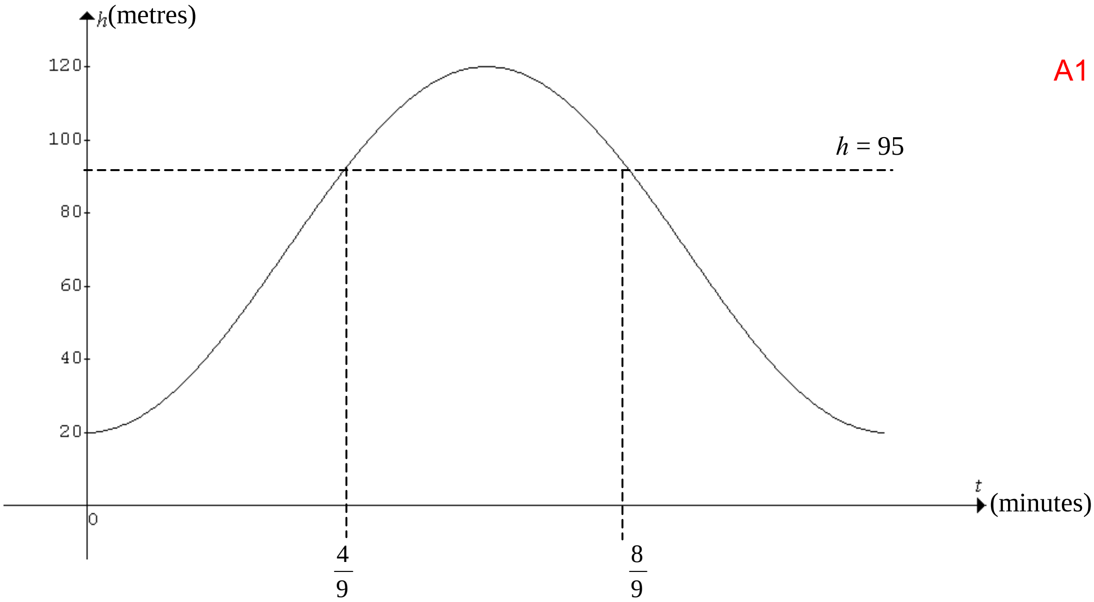
$\therefore t+1=\frac{1}{9}+\frac{4m}{3}$ or $\frac{5}{9}+\frac{4m}{3}$

$\therefore t=-\frac{8}{9}+\frac{4m}{3}=\frac{12m-8}{9}$ or $-\frac{4}{9}+\frac{4m}{3}=\frac{12m-4}{9}$.

The smallest value of t satisfying $0 \le t \le 36$ is $t=\frac{4}{9}$ minutes. A1

Therefore the point P first reaches a height of at least 95 metres above ground level after $\frac{4}{9}$ minutes.

f. During the first rotation P is at a height of 95 metres above the ground a second time when $t=\frac{8}{9}$ minutes (see **part e**).



A1

From the graph it is clear that the number of minutes during one rotation that the point P is at least 95 metres above ground level is $\frac{8}{9}-\frac{4}{9}=\frac{4}{9}$ minutes:

$\frac{4}{9}$ minutes $=\frac{4}{9}\times 60$ seconds $=\frac{80}{3}$ seconds A1

g. $h = 70 + 50\sin\frac{3\pi}{2}(t+1)$

$\therefore \frac{dh}{dt} = 50 \times \frac{3\pi}{2}\cos\frac{3\pi}{2}(t+1)$

$= 75\pi\cos\frac{3\pi}{2}(t+1)$ M1

Let $\frac{dh}{dt} > 200$

$\therefore 75\pi\cos\frac{3\pi}{2}(t+1) > 200$

$\therefore \cos\frac{3\pi}{2}(t+1) > \frac{8}{3\pi}$

Enter $y = 75\pi\cos\frac{3\pi}{2}(t+1)$ and $y = \frac{8}{3\pi}$ in the ***y* = editor** of a graphics calculator.

Then use **2nd CALC 5: intersect** to find two consecutive solutions to $\cos\frac{3\pi}{2}(t+1) = \frac{8}{3\pi}$.

$\therefore$ $t = 0.2151$ minutes and $t = 0.4515$ minutes. M1

$0.452 - 0.215 = 0.2354$ minutes $= 14.18$ seconds.

Therefore $\frac{dh}{dt} > 200$ m/s for 14.18 seconds, which is less than 20 seconds. A1

The average person will not feel sick on the Southern Star Observation Wheel.

h. $h = 70 + 50\sin c(t+1)$

$\therefore \frac{dh}{dt} = 50c\cos(c(t+1))$

- The smallest positive value of c such that $\frac{dh}{dt} > 200$ for no more than 10 seconds at a time and the wheel turns as quickly as possible is required.
- 10 seconds = $\frac{1}{6}$ minutes.
- Therefore the positive value of c is required such that if t_1 and $t_2 > t_1$ are two consecutive solutions to $50c\cos(c(t+1)) = 200$, then $t_1 - t_2 = \frac{1}{6}$. M1

$$50c\cos(c(t+1)) = 200$$

$$\therefore \cos(c(t+1)) = \frac{4}{c}$$

$$\therefore c(t+1) = \pm\cos^{-1}\left(\frac{4}{c}\right)$$

$$\therefore t = \pm\frac{1}{c}\cos^{-1}\left(\frac{4}{c}\right) - 1$$

$$\therefore t_1 = -\frac{1}{c}\cos^{-1}\left(\frac{4}{c}\right) - 1 \text{ and } t_2 = \frac{1}{c}\cos^{-1}\left(\frac{4}{c}\right) - 1$$

$$\therefore t_2 - t_1 = \frac{2}{c}\cos^{-1}\left(\frac{4}{c}\right)$$ M1

$$\therefore \frac{1}{6} = \frac{2}{c}\cos^{-1}\left(\frac{4}{c}\right)$$

$$\therefore \cos\left(\frac{c}{12}\right) = \frac{4}{c}$$

- From the graphics calculator:

  $c = 4.267$, correct to three decimal places. A1

QUESTION 3

a. (i) When $t = 4, A = 9$:

$$\therefore 9 = \frac{a(4)}{3(4)+4}$$

$$\therefore 9 = \frac{4a}{16}$$

$$\therefore 9 = \frac{a}{4}$$

$$\therefore a = 36$$ M1

(ii) $A = \dfrac{36t}{3t+4}$

Using polynomial long division:

$$\begin{array}{r} 12 \\ 3t+4 \overline{\smash{\big)}\,36t} \\ 36t+48 \\ \hline -48 \end{array}$$

$$\therefore A = 12 - \frac{48}{3t+4}$$ M1

$$= 12\left(1 - \frac{4}{3t+4}\right)$$

b. $D = A - B$

$$\therefore D = 12\left(1 - \frac{4}{3t+4}\right) - 12\left(1 - \frac{3}{t+3}\right)$$

$$\therefore D = 12\left(\frac{3}{t+3} - \frac{4}{3t+4}\right)$$ M1

c. $D = 12\left(\dfrac{3}{t+3} - \dfrac{4}{3t+4}\right)$

$$\therefore D = 12(3(t+3)^{-1} - 4(3t+4)^{-1})$$

$$\therefore \frac{dD}{dt} = 12\left(-\frac{3}{(t+3)^2} + \frac{12}{(3t+4)^2}\right)$$ A2

d. Maximum value of D occurs when $\frac{dD}{dt} = 0$:

$$0 = 12\left(-\frac{3}{(t+3)^2} + \frac{12}{(3t+4)^2}\right), \quad 0 \le t \le 14.$$ M1

$$\therefore 0 = -\frac{3}{(t+3)^2} + \frac{12}{(3t+4)^2}$$

$$\therefore 0 = -\frac{1}{(t+3)^2} + \frac{4}{(3t+4)^2}$$

$$\therefore \frac{1}{(t+3)^2} = \frac{4}{(3t+4)^2}$$

$$\therefore (3t+4)^2 = 4(t+3)^2$$

$$\therefore 3t+4 = 2(t+3),$$

$$\therefore 3t+4 = 2t+6$$

$$\therefore t = 2$$ M1

When $t = 2$: $D = 12\left(\frac{3}{2+3} - \frac{4}{6+4}\right)$ = 2.4

Hence the maximum difference in new influenza cases between those previously unexposed and those exposed to the virus is $2.4 \times 100 = 240$. A1

Note: D is measured in hundreds.

e. $C(t) = 4(t^2+t)e^{-\frac{t}{k}}$

$$\therefore C'(t) = 4\left((2t+1)e^{-\frac{t}{k}} - \frac{1}{k}e^{-\frac{t}{k}}(t^2+t)\right)$$

$$= 4e^{-\frac{t}{k}}\left(2t+1-\frac{1}{k}(t^2+t)\right)$$ M1

For maximum concentration $C'(t) = 0$:

$\therefore 2t+1-\frac{1}{k}(t^2+t) = 0$ or $e^{-\frac{t}{k}} = 0$ M1

$\therefore 2tk + k - t^2 - t = 0$ No solution

When t = 4, $C'(t) = 0$: $8k + k - 16 - 4 = 0$

$$\therefore 9k = 20$$

$$\therefore k = \frac{20}{9}$$ A1

f. Let $A = C'(t)$

Substitute $k = \dfrac{20}{9}$:

$$\therefore A = 4e^{-\frac{9t}{20}}\left(2t + 1 - \frac{9}{20}(t^2 + t)\right)$$

$$= 4e^{-\frac{9t}{20}}\left(\frac{31t}{20} + 1 - \frac{9t^2}{20}\right)$$

$$= \frac{1}{5}e^{-\frac{9t}{20}}\left(31t + 20 - 9t^2\right)$$

$$\frac{dA}{dt} = \frac{1}{5}\left(-\frac{9}{20}e^{-\frac{9t}{20}}\left(31t + 20 - 9t^2\right) + e^{-\frac{9t}{20}}\left(31 - 18t\right)\right)$$ M1

For maximum absorption $\dfrac{dA}{dt} = 0$:

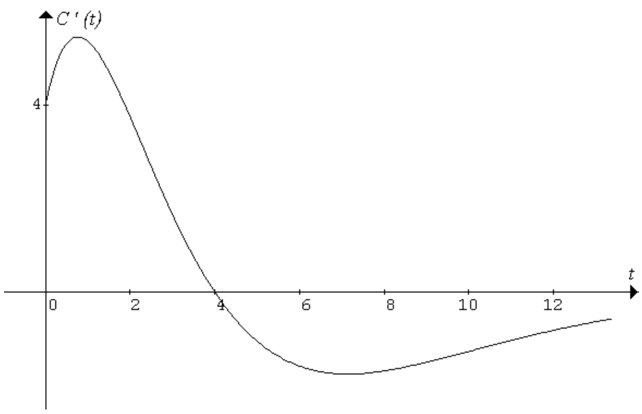
$$0 = \frac{1}{5}\left(-\frac{9}{20}e^{-\frac{9t}{20}}\left(31t + 20 - 9t^2\right) + e^{-\frac{9t}{20}}\left(31 - 18t\right)\right)$$

$$\therefore 0 = -\frac{9}{20}\left(31t + 20 - 9t^2\right) + 31 - 18t$$

$\therefore t = 0.76$ or 7.13 M1

From the graph of $C'(t)$ below it can be seen that the maximum rate of absorption occurs after 0.76 hours.

A1



QUESTION 4

a. (i) $\Pr(5 < T < 15) = 0.98758 \approx 0.9876$. A1

(ii) $\Pr(T > 15) = 0.00621 \approx 0.0062$. A1

(iii) $\Pr(T > 5 | T < 15) = \dfrac{\Pr(T > 5 \cap T < 15)}{\Pr(T < 15)}$ M1

$= \dfrac{\Pr(5 < T < 15)}{\Pr(T < 15)}$

$= \dfrac{0.98758}{1 - 0.00621}$

$= 0.9938$ A1

Note: During a calculation, accuracy greater than that specified for the answer should be used so as to avoid the accumulation of rounding error.

b. $\Pr(T > k) = 0.0095$

$\therefore \Pr(T < k) = 1 - 0.0095 = 0.9905$

$\therefore k = \text{invnorm}(0.9915,\ 10,\ 2)$ M1

$= 14.69$

= 15 correct to the nearest minute. A1

c. (i) Let Y be the random variable "number of trains which arrive more than 15 minutes late".

$Y \sim \text{Binomial}(n = 10,\ p = \Pr(T > 15) = 0.00621)$. M1

$\Pr(Y = 2) = \binom{10}{2}(0.00621)^2(1 - 0.00621)^8$ = 0.0017, correct to four decimal places.

Alternatively: $\Pr(Y = 2) = \text{binompdf}(10,\ 0.00621,\ 2)$ = 0.0017. A1

(ii) Pr (first train is more than 15 minutes late and the last 9 are not)

$= p(1-p)^9$

$= 0.00621(1 - 0.00621)^9$

$= 0.0059$, correct to four decimal places. A1

d. $E(Y) = np = 200 \times 0.0062 = 1.24$

Therefore one train out of 200 will be more than 15 minutes late. A1

e. Let $X \sim Normal(\mu, \sigma^2)$

$\Pr(X > 5) = 0.0095$

$\therefore \Pr(X < 5) = 0.9905$

$\therefore \Pr(Z < k_1) = 0.9905$

Where $k_1 = \text{invnorm}(0.9905) = 2.34553$

$\therefore \dfrac{5-\mu}{\sigma} = 2.34553$ M1

$\therefore 2.34553\sigma = 5 - \mu$ ....(1)

$\Pr(X < 4) = 0.205$

$\therefore \Pr(Z < k_2) = 0.205$

Where $k_2 = \text{invnorm}(0.205) = -0.82389$

$\therefore \dfrac{4-\mu}{\sigma} = -0.82389$ M1

$\therefore -0.82389\sigma = 4 - \mu$ ....(2)

(1) – (2): $3.16942\sigma = 1$

$\therefore \sigma = 0.315515 \approx 0.3155$. A1

From (1): $\mu = 4.2600$, correct to four decimal places. A1

f. (i) $\Pr(X > 3) = 0.04\int_3^5 x\,dx + 0.04\int_5^{10} (10 - x)dx$ M1

$= 0.04\left[\dfrac{x^2}{2}\right]_3^5 + 0.04\left[10x - \dfrac{x^2}{2}\right]_5^{10}$

$= 0.04\left(\dfrac{25}{2} - \dfrac{9}{2}\right) + 0.04\left(100 - 50 - (50 - \dfrac{25}{2}\right)$

$= 0.8$ A1

(ii) $E(X) = 0.04\int_0^5 x^2 dx + 0.04\int_5^{10} (10x - x^2)dx$ M1

$= 0.04\left[\dfrac{x^3}{2}\right]_0^5 + 0.04\left[5x^2 - \dfrac{x^3}{3}\right]_5^{10}$

= 5 minutes. A1

(iii) $\Pr(X > k) = 0.8$

$\therefore \Pr(X < k) = 0.2$

$$\therefore 0.04\int_0^5 x\,dx = 0.2$$ M1

$$\therefore \left[\frac{x^2}{2}\right]_0^k = 5$$

$$\therefore \frac{k^2}{2} = 5$$

$\therefore k^2 = 10$

$\therefore k = \sqrt{10}$ A1