Neap:

Trial Examination 2007

VCE Mathematical Methods Units 3 & 4

Written Examination 1

Suggested Solutions

Question 1

$g(f(x)) = g((x-1)^2)$

$= \sqrt{(x-1)^2} + 1$ or $|x-1| + 1$ A1

Note: $|x-1| + 1 \neq x$

Question 2

a. The turning point of the graph of $y = f(x)$ is $(-1, -1)$.

Therefore the maximal domain for f to be a one-to-one function is $(-\infty, -1)$.

Therefore $a = -1$. A1

b. Originally, we have $y = x^2 + 2x$.

For the inverse, $x = y^2 + 2y$. M1

$x = y^2 + 2y + 1 - 1$

$= (y+1)^2 - 1$ M1

$x + 1 = (y+1)^2$

$y = -1 \pm \sqrt{x+1}$

However, ran f^{-1} = dom $f = (-\infty, -1)$

$f^{-1}(x) = -1 - \sqrt{x+1}$ A1

Question 3

a. **i.** period $= \dfrac{2\pi}{\frac{\pi}{6}} = 12$ A1

ii. range : $[-1, 3]$ A1

b. $g(x) = 1 - 2\sin\left(\frac{\pi}{6}(x-2)\right)$

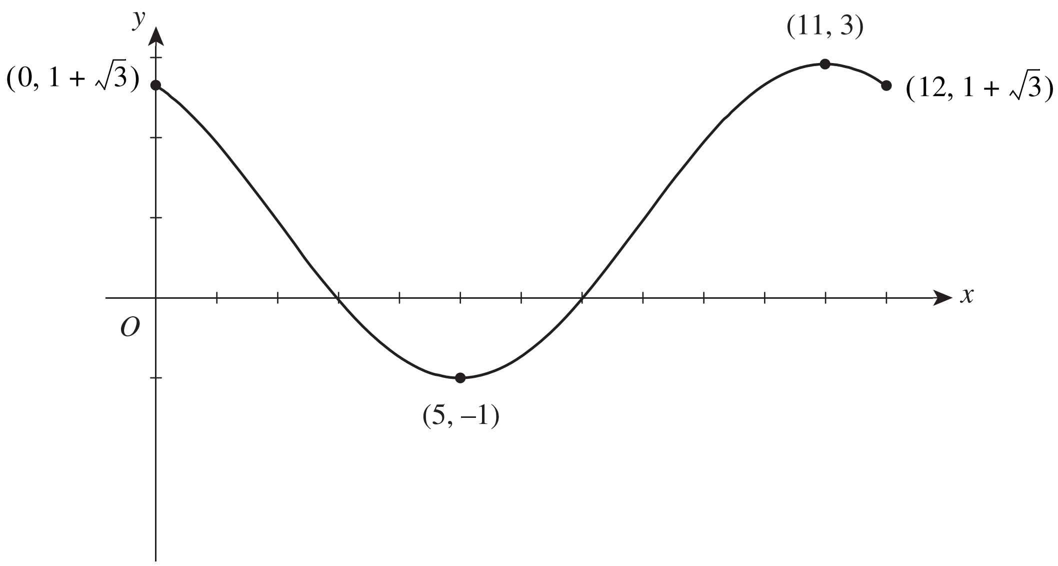
$g(x)$ has a minimum where $\sin\left(\frac{\pi}{6}(x-2)\right) = 1$ M1

$\therefore \frac{\pi}{6}(x-2) = \frac{\pi}{2}$

$x = 5$

So the minimum occurs at $(5, -1)$. A1

c.



Correct shape A1
Correct endpoints A1
Correct maximum and minimum A1

Question 4

$$f(x) = \log_e\left(\frac{x}{x-1}\right)$$

$$= \log_e\left(\frac{-x}{1-x}\right)$$

$$= \log_e(-x) - \log_e(1-x) \text{ (so that } f\left(-\tfrac{1}{2}\right) \text{ exists)}$$

$$f'(x) = \frac{-1}{-x} - \frac{-1}{1-x}$$

$$= \frac{1}{x} - \frac{1}{x-1}$$

$$= \frac{x-1-x}{x(x-1)}$$

$$= \frac{-1}{x(x-1)}$$ M1

$$= \frac{1}{x(1-x)}$$

$$f'\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}\left(1+\frac{1}{2}\right)}$$ M1

$$= -\frac{4}{3}$$ A1

Thus the gradient of the tangent is $-\frac{4}{3}$, so the gradient of the normal is $\frac{3}{4}$. A1

Question 5

$$\int_5^1 2(h(x)-1)dx = \int_5^1 (2h(x)-2)dx$$

$$= -2\int_1^5 h(x)dx - \int_5^1 2dx \qquad \text{M1}$$

$$= -2[4] - [2x]_5^1$$

$$= -8-(2-10)$$

$$= 0 \qquad \text{A1}$$

Question 6

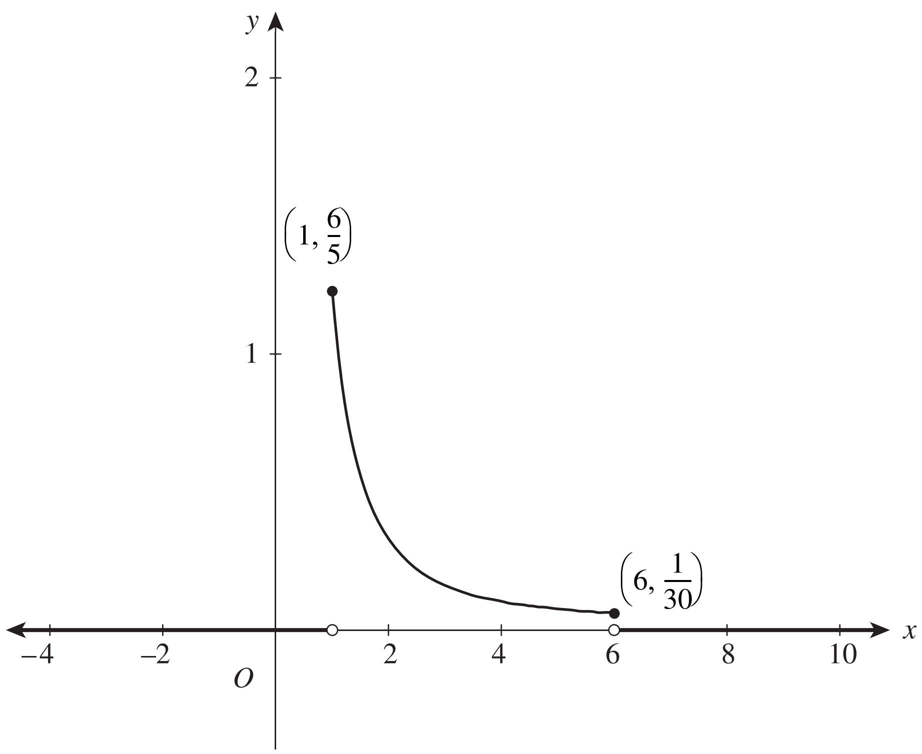
a. As $f(x)$ is a probability density function, $\int_1^6 \frac{k}{x^2}dx = 1$. M1

$$\left[-\frac{k}{x}\right]_1^6 = 1$$

$$-\frac{k}{6}+\frac{k}{1} = 1$$

$$k = \frac{6}{5} \qquad \text{A1}$$

b.

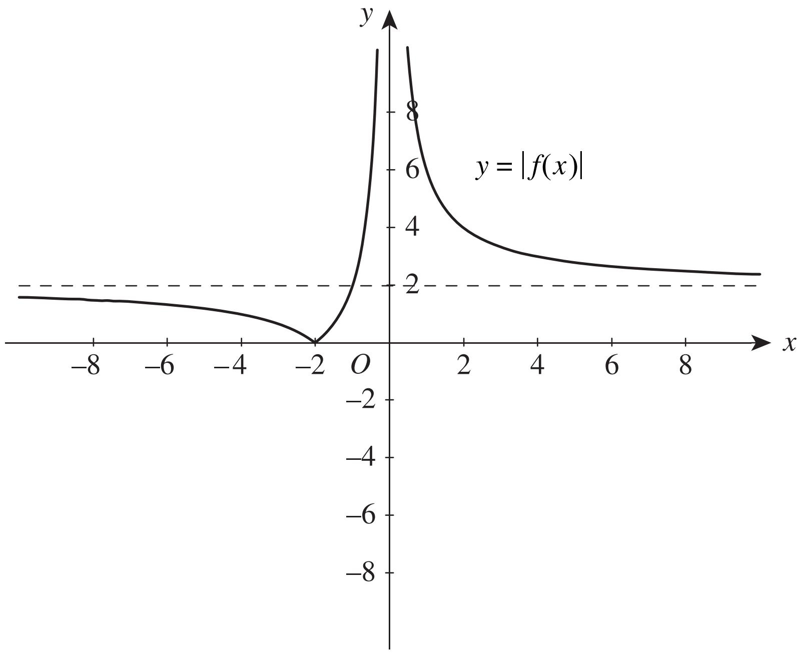


Correct endpoints A1

Correct graph on $(-\infty, \infty)$ A1

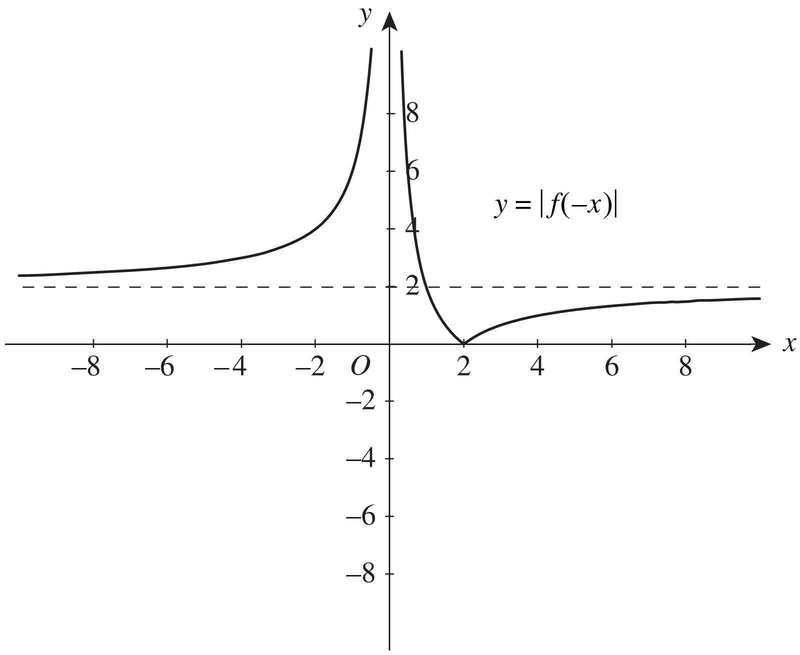
Question 7

a.



Correct graph of $|f(x)|$ A1

b.

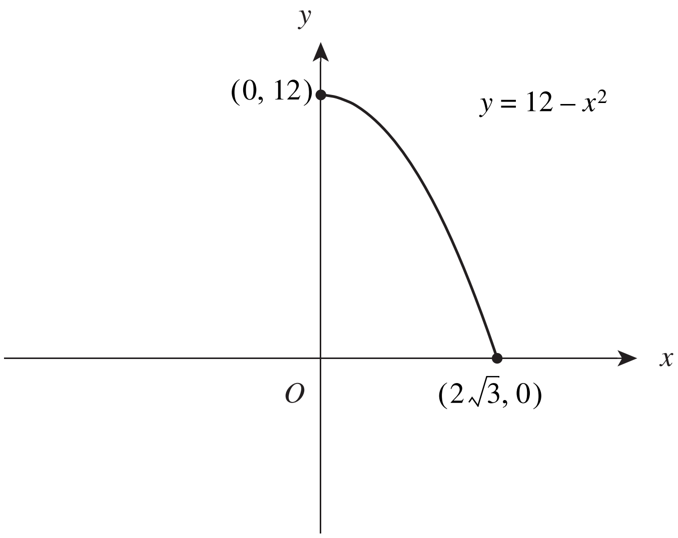


Correct graph of $|f(-x)|$ A1

c. range : $R^+ \cup \{0\}$ A1

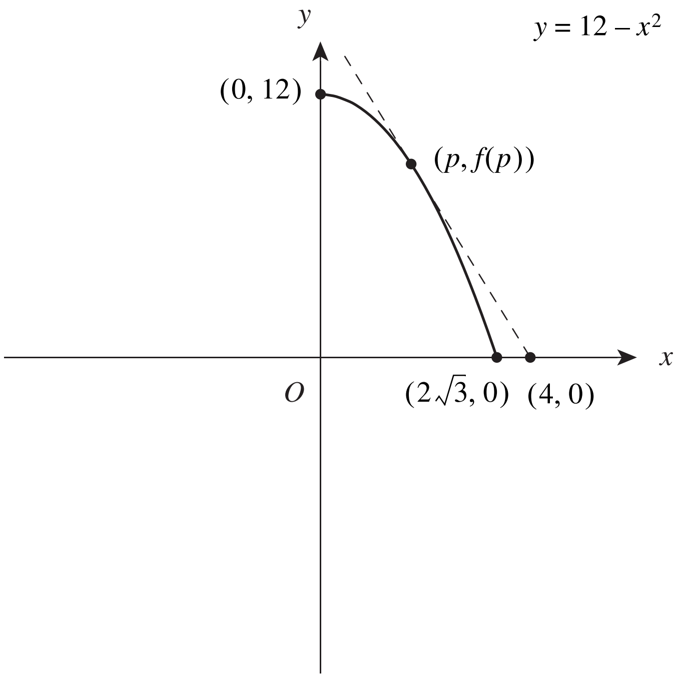
Question 8

a.



Correct graph A1

b.



The gradient of the tangent at $x = p$ is given by $f'(p)$.

$f'(x) = -2x$ so $f'(p) = -2p$

Equation of the tangent: $y - y_1 = m(x - x_1)$

$\therefore y - (12 - p^2) = -2p(x - p)$

$y = -2px + p^2 + 12$ M1 A1

As the tangent passes through the point $(4, 0)$

$0 = -8p + p^2 + 12$

$0 = (p - 6)(p - 2)$ M1

$p = 6$ or $p = 2$

Clearly $0 < p < 2\sqrt{3}$ so $p = 2$ A1

Question 9

a. $\frac{dr}{dt} = \frac{1}{25}, \; V = \frac{4}{3}\pi r^3, \; \frac{dV}{dr} = 4\pi r^2$

Using the chain rule,

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \times \frac{1}{25} = \frac{4\pi r^2}{25}$ M1

At $r = 10$,

$\frac{dV}{dt} = \frac{4\pi \times 100}{25} = 16\pi \text{ cm}^3/\text{s}$ A1

b. The area of a cross-section through the centre of the sphere is $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\frac{dA}{dt} = \frac{6\pi}{25}$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ M1

$\frac{6\pi}{25} = 2\pi r \times \frac{1}{25}$

$r = 3$

At $r = 3$,

$V = \frac{4}{3}\pi \times 3^3 = 36\pi \text{ cm}^3$ A1

Question 10

Point of intersection:

$\sin(x) = \cos(x)$

$\tan(x) = 1$

$x = \frac{\pi}{4}$ A1

$$\text{area} = \int_0^{\frac{\pi}{4}} (\cos(x) - \sin(x))dx$$

$= [\sin(x) + \cos(x)]_0^{\frac{\pi}{4}}$ M1

$= \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - (0 + 1)$

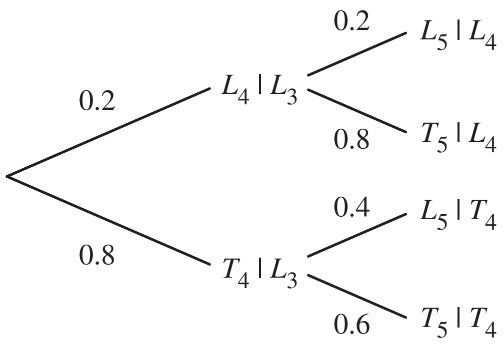
$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$

$= \frac{2 - \sqrt{2}}{\sqrt{2}}$

$= \sqrt{2} - 1$ A1

Question 11

Let T_i be the event 'on time for school on the ith day of the week', and let L_i be the event 'late for school on the ith day of the week'.



Tree diagram or table M1
Correct probabilities A1

$$\Pr(L_5) = 0.2 \times 0.2 + 0.8 \times 0.4$$
$$= 0.36$$

A1

